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A discrete-ordinates solution for radiative-transfer models that include polarization effects

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Abstract

A recently developed version of the discrete-ordinates method is used along with elementary numerical linear-algebra techniques to establish an accurate solution for all components in a Fourier representation of the Stokes vector basic to the scattering of polarized light. Computational aspects of the solution are discussed, and numerical results for each of the four Stokes parameters are given for a test case based on an atmosphere of randomly orientated oblate spheroids. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

One of the more challenging problems in the basic theory of radiative transfer in plane-parallel media is the one initially formulated by Kuščer and Ribarič [1] so as to include what we consider to be a (rather) general model for the scattering of polarized light. The original work of Kuščer and Ribarič was reported in terms of complex parameters, but in 1981 the Kuščer–Ribarič theory was reformulated [2] in terms of real parameters and thus was made considerably more convenient for analytical and computational work. And so following Ref. [2]. we let $\boldsymbol{I}(\tau, \mu, \phi)$ denote a column vector, with the four Stokes parameters $I(\tau, \mu, \phi)$, $Q(\tau, \mu, \phi)$, $U(\tau, \mu, \phi)$ and $V(\tau, \mu, \phi)$ as components, and consider the (vector) equation of transfer written as

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}(\tau, \mu, \phi) + \boldsymbol{I}(\tau, \mu, \phi) = \frac{\varpi}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \boldsymbol{P}(\mu, \mu', \phi - \phi') \boldsymbol{I}(\tau, \mu', \phi') \, \mathrm{d}\phi' \, \mathrm{d}\mu' \tag{1}$$

for $\tau \in (0, \tau_0)$, $\mu \in [-1, 1]$ and $\phi \in [0, 2\pi]$. Here ϖ is the albedo for single scattering, $\tau \in [0, \tau_0]$ is the optical variable, τ_0 is the optical thickness of the plane-parallel medium, $\mu \in [-1, 1]$ is the cosine of the polar angle (as measured from the *positive* τ-axis) and ϕ is the azimuthal angle. Together the

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polar and azimuthal angles define the direction of propagation of the radiation. Here, as in Refs. [2,3], the phase matrix has the analytical representation

$$\boldsymbol{P}(\mu,\mu',\phi-\phi') = \sum_{m=0}^{L} \Delta_{0,m}\{\boldsymbol{C}^m(\mu,\mu')\cos[m(\phi-\phi')] + \boldsymbol{S}^m(\mu,\mu')\sin[m(\phi-\phi')]\}, \tag{2}$$

where

$$\Delta_{0,m} = \tfrac{1}{2}(2-\delta_{0,m}), \tag{3a}$$

$$\boldsymbol{C}^m(\mu,\mu') = \boldsymbol{A}^m(\mu,\mu') + \boldsymbol{D}\boldsymbol{A}^m(\mu,\mu')\boldsymbol{D} \tag{3b}$$

and

$$\boldsymbol{S}^m(\mu,\mu') = \boldsymbol{A}^m(\mu,\mu')\boldsymbol{D} - \boldsymbol{D}\boldsymbol{A}^m(\mu,\mu'). \tag{3c}$$

We note that

$$\boldsymbol{A}^m(\mu,\mu') = \sum_{l=m}^{L} \boldsymbol{P}_l^m(\mu)\boldsymbol{B}_l\boldsymbol{P}_l^m(\mu') \tag{4}$$

and

$$\boldsymbol{D} = \mathrm{diag}\{1, 1, -1, -1\}. \tag{5}$$

In addition, using a slightly different notation from Refs. [2,3], we write

$$\boldsymbol{P}_l^m(\mu) = \begin{bmatrix} P_l^m(\mu) & 0 & 0 & 0 \\ 0 & R_l^m(\mu) & -T_l^m(\mu) & 0 \\ 0 & -T_l^m(\mu) & R_l^m(\mu) & 0 \\ 0 & 0 & 0 & P_l^m(\mu) \end{bmatrix} \tag{6}$$

where

$$P_l^m(\mu) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} (1-\mu^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_l(\mu) \tag{7}$$

is used to denote the *normalized* Legendre function and where the *normalized* R and T functions are defined as [2,3]

$$R_l^m(\mu) = -\tfrac{1}{2}(i)^m[P_{m,2}^l(\mu) + P_{m,-2}^l(\mu)] \tag{8}$$

and

$$T_l^m(\mu) = -\tfrac{1}{2}(i)^m[P_{m,2}^l(\mu) - P_{m,-2}^l(\mu)] \tag{9}$$

where, for $l \geq \sup(|m|,|n|)$,

$$P^l_{m,n}(\mu) = A^l_{m,n}(1-\mu)^{-(n-m)/2}(1+\mu)^{-(n+m)/2} \frac{\mathrm{d}^{l-n}}{\mathrm{d}\mu^{l-n}} [(1-\mu)^{l-m}(1+\mu)^{l+m}] \tag{10}$$

with

$$A^l_{m,n} = \frac{(-1)^{l-m}(i)^{n-m}}{2^l(l-m)!}\left[\frac{(l-m)!(l+n)!}{(l+m)!(l-n)!}\right]^{1/2}. \tag{11}$$

Continuing to follow Refs. [2,3], we note that the scattering law is defined by the collection of *Greek constants* $\{\alpha_l, \beta_l, \gamma_l, \delta_l, \varepsilon_l, \zeta_l\}$ so that

$$\boldsymbol{B}_l = \begin{bmatrix} \beta_l & \gamma_l & 0 & 0 \\ \gamma_l & \alpha_l & 0 & 0 \\ 0 & 0 & \zeta_l & -\varepsilon_l \\ 0 & 0 & \varepsilon_l & \delta_l \end{bmatrix} \tag{12}$$

for $0 \leq l \leq L$.

In this work we seek a solution of Eq. (1), for all $\tau \in (0, \tau_0)$, $\mu \in [-1, 1]$ and $\phi \in [0, 2\pi]$, subject to the boundary conditions

$$\boldsymbol{I}(0, \mu, \phi) = \pi\delta(\mu - \mu_0)\delta(\phi - \phi_0)\boldsymbol{F} \tag{13a}$$

and

$$\boldsymbol{I}(\tau_0, -\mu, \phi) = \frac{\lambda_0}{\pi}\boldsymbol{L} \int_0^1 \int_0^{2\pi} \boldsymbol{I}(\tau_0, \mu', \phi')\mu' \,\mathrm{d}\phi' \,\mathrm{d}\mu' \tag{13b}$$

for $\mu \in (0, 1]$ and $\phi \in [0, 2\pi]$. Here, λ_0 is the coefficient for Lambert reflection,

$$\boldsymbol{L} = \mathrm{diag}\{1, 0, 0, 0\} \tag{14}$$

and

$$\boldsymbol{F} = [F_I \quad F_Q \quad F_U \quad F_V]^{\mathrm{T}} \tag{15}$$

is the flux vector. Note that we use the superscript T to denote the transpose operation. Naturally, we consider that the flux vector $\boldsymbol{F}$ and the direction, defined by $\{\mu_0, \phi_0\}$, of the incident beam are specified along with the optical thickness of the layer τ_0, the albedo for single scattering ϖ and the scattering law, defined by $\boldsymbol{B}_l$ for $0 \leq l \leq L$.

2. The reduced problem

Since the boundary condition given by Eq. (13a) introduces into $\boldsymbol{I}(\tau, \mu, \phi)$ a component that is a generalized function, we express the complete solution to the problem defined by Eqs. (1) and (13) in the form

$$\boldsymbol{I}(\tau,\mu,\phi)=\boldsymbol{I}_*(\tau,\mu,\phi)+\pi\delta(\mu-\mu_0)\delta(\phi-\phi_0)\boldsymbol{F}\mathrm{e}^{-\tau/\mu} \tag{16}$$

where $\boldsymbol{I}_*(\tau,\mu,\phi)$ is the reduced or diffuse field. Continuing, we introduce

$$\boldsymbol{\Phi}_1^m(\phi)=(2-\delta_{0,m})\mathrm{diag}\{\cos m\phi,\cos m\phi,\sin m\phi,\sin m\phi\} \tag{17a}$$

and

$$\boldsymbol{\Phi}_2^m(\phi)=(2-\delta_{0,m})\mathrm{diag}\{-\sin m\phi,-\sin m\phi,\cos m\phi,\cos m\phi\} \tag{17b}$$

and note that we can write the phase matrix as

$$\boldsymbol{P}(\mu,\mu',\phi-\phi')=\sum_{m=0}^{L}\sum_{k=1}^{2}\boldsymbol{\Phi}_k^m(\phi-\phi')\boldsymbol{A}^m(\mu,\mu')\boldsymbol{D}_k \tag{18}$$

where $\boldsymbol{A}^m(\mu,\mu')$ is given by Eq. (4) and where

$$\boldsymbol{D}_1=\mathrm{diag}\{1,1,0,0\} \tag{19a}$$

and

$$\boldsymbol{D}_2=\mathrm{diag}\{0,0,1,1\}. \tag{19b}$$

Now, to accomplish the desired Fourier decomposition, we let

$$\boldsymbol{I}_*(\tau,\mu,\phi)=\frac{1}{2}\sum_{m=0}^{L}\sum_{k=1}^{2}\boldsymbol{\Phi}_k^m(\phi-\phi_0)\boldsymbol{I}_k^m(\tau,\mu) \tag{20}$$

and substitute Eq. (20) into Eq. (16) and the resulting equation into Eqs. (1) and (13) to find that the Fourier components, for $m=0,1,\dots,L$ and $k=1$ and 2, must satisfy the equation of transfer

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{I}_k^m(\tau,\mu)+\boldsymbol{I}_k^m(\tau,\mu)=\frac{\varpi}{2}\sum_{l=m}^{L}\boldsymbol{P}_l^m(\mu)\boldsymbol{B}_l\int_{-1}^{1}\boldsymbol{P}_l^m(\mu')\boldsymbol{I}_k^m(\tau,\mu')\,\mathrm{d}\mu'+\boldsymbol{Q}_k^m(\tau,\mu) \tag{21}$$

for $\tau\in(0,\tau_0)$ and $\mu\in[-1,1]$ and the boundary conditions

$$\boldsymbol{I}_k^m(0,\mu)=\boldsymbol{0} \tag{22a}$$

and

$$\boldsymbol{I}_k^m(\tau_0,-\mu)=2\lambda_0\delta_{0,m}\delta_{1,k}\boldsymbol{L}\left[\mu_0\boldsymbol{D}_1\boldsymbol{F}\mathrm{e}^{-\tau_0/\mu_0}+\int_0^1\boldsymbol{I}_k^m(\tau_0,\mu')\mu'\,\mathrm{d}\mu'\right] \tag{22b}$$

for $\mu\in(0,1]$. Here the inhomogeneous source term is

$$\boldsymbol{Q}_k^m(\tau,\mu)=\frac{\varpi}{2}\sum_{l=m}^{L}\boldsymbol{P}_l^m(\mu)\boldsymbol{B}_l\boldsymbol{P}_l^m(\mu_0)\boldsymbol{D}_k\boldsymbol{F}\mathrm{e}^{-\tau/\mu_0}. \tag{23}$$

To obtain the complete solution we seek, we now must solve the collection of problems defined by Eqs. (21)–(23).

3. A discrete-ordinates solution

In a recent paper [4] concerning a radiative-transfer problem based on the scalar version (the intensity is the only unknown) of the model considered in this work a variation of the discrete-ordinates method was developed, evaluated and found to be very effective. And so here we wish to generalize the solution reported in Ref. [4] in order to solve efficiently and accurately the collection of problems defined by Eqs. (21)–(23). Before starting with our solution, we point out that while there are many features in the work we now report that are common to the solution developed in Ref. [4], there are significant differences (to be noted) as well.

We would like also to note that we exclude in our development here the conservative case as defined and discussed in Section 8.

As a matter of strategy, we note that, as in Ref. [4], we intend to use the discrete-ordinates method only to find approximate values for the integral terms in Eq. (21), and once that is done we will solve Eq. (21), with the integral terms replaced by discrete-ordinates approximations to those terms, to find the desired Fourier component $\boldsymbol{I}_k^m(\tau, \mu)$ for all τ and μ. This second aspect of our approach is what we refer to as a "post-processing" step [5]. And so, we suppress some of the explicit notation of the Fourier indices m and k and start with our discrete-ordinates equations, relevant to the homogeneous version of Eq. (21), written as

$$\mu_i \frac{\mathrm{d}}{\mathrm{d}\tau} \boldsymbol{I}(\tau, \mu_i) + \boldsymbol{I}(\tau, \mu_i) = \frac{\varpi}{2} \sum_{l=m}^{L} \boldsymbol{P}_l^m(\mu_i) \boldsymbol{B}_l \sum_{\alpha=1}^{N} w_\alpha \boldsymbol{I}_{l,\alpha}(\tau) \tag{24a}$$

and

$$-\mu_i \frac{\mathrm{d}}{\mathrm{d}\tau} \boldsymbol{I}(\tau, -\mu_i) + \boldsymbol{I}(\tau, -\mu_i) = \frac{\varpi}{2} \sum_{l=m}^{L} \boldsymbol{P}_l^m(-\mu_i) \boldsymbol{B}_l \sum_{\alpha=1}^{N} w_\alpha \boldsymbol{I}_{l,\alpha}(\tau) \tag{24b}$$

for $i = 1, 2, \ldots, N$. Here, to compact our notation we have introduced

$$\boldsymbol{I}_{l,\alpha}(\tau) = \boldsymbol{P}_l^m(\mu_\alpha) \boldsymbol{I}(\tau, \mu_\alpha) + \boldsymbol{P}_l^m(-\mu_\alpha) \boldsymbol{I}(\tau, -\mu_\alpha). \tag{25}$$

In writing Eqs. (24) as we have, we clearly are considering that the N quadrature points $\{\mu_\alpha\}$ and the N weights $\{w_\alpha\}$ are defined for use on the integration interval [0,1]. We note also that [6]

$$\boldsymbol{P}_l^m(-\mu) = (-1)^{l-m} \boldsymbol{D} \boldsymbol{P}_l^m(\mu) \boldsymbol{D}. \tag{26}$$

Eqs. (24) clearly have exponential solutions, so we substitute

$$\boldsymbol{I}(\tau, \pm\mu_i) = \boldsymbol{\Phi}(\nu, \pm\mu_i) \mathrm{e}^{-\tau/\nu} \tag{27}$$

into those equations to find

$$\left(1 - \frac{\mu_i}{\nu}\right) \boldsymbol{\Phi}(\nu, \mu_i) = \frac{\varpi}{2} \sum_{l=m}^{L} \boldsymbol{P}_l^m(\mu_i) \boldsymbol{B}_l \sum_{\alpha=1}^{N} w_\alpha \boldsymbol{\Phi}_{l,\alpha}(\nu) \tag{28a}$$

and

$$\left(1 + \frac{\mu_i}{\nu}\right) \boldsymbol{D} \boldsymbol{\Phi}(\nu, -\mu_i) = \frac{\varpi}{2} \sum_{l=m}^{L} (-1)^{l-m} \boldsymbol{P}_l^m(\mu_i) \boldsymbol{D} \boldsymbol{B}_l \sum_{\alpha=1}^{N} w_\alpha \boldsymbol{\Phi}_{l,\alpha}(\nu) \tag{28b}$$

for $i = 1, 2, \ldots, N$ and where

$$\boldsymbol{\Phi}_{l,\alpha}(v) = \boldsymbol{P}_l^m(\mu_\alpha)\boldsymbol{\Phi}(v, \mu_\alpha) + (-1)^{l-m}\boldsymbol{D}\boldsymbol{P}_l^m(\mu_\alpha)\boldsymbol{D}\boldsymbol{\Phi}(v, -\mu_\alpha). \tag{29}$$

In order to compact our formulation we make use of more matrix notation. We introduce the $(4N \times 1)$ vectors

$$\boldsymbol{\Phi}_+(v) = [\boldsymbol{\Phi}^{\mathrm{T}}(v, \mu_1), \boldsymbol{\Phi}^{\mathrm{T}}(v, \mu_2), \ldots, \boldsymbol{\Phi}^{\mathrm{T}}(v, \mu_N)]^{\mathrm{T}} \tag{30a}$$

and

$$\boldsymbol{\Phi}_-(v) = [\boldsymbol{\Phi}^{\mathrm{T}}(v, -\mu_1)\boldsymbol{D}, \boldsymbol{\Phi}^{\mathrm{T}}(v, -\mu_2)\boldsymbol{D}, \ldots, \boldsymbol{\Phi}^{\mathrm{T}}(v, -\mu_N)\boldsymbol{D}]^{\mathrm{T}}, \tag{30b}$$

the $(4N \times 4N)$ matrices

$$\boldsymbol{W} = \mathrm{diag}\{w_1\boldsymbol{I}, w_2\boldsymbol{I}, \ldots, w_N\boldsymbol{I}\} \tag{31a}$$

and

$$\boldsymbol{M} = \mathrm{diag}\{\mu_1\boldsymbol{I}, \mu_2\boldsymbol{I}, \ldots, \mu_N\boldsymbol{I}\}, \tag{31b}$$

where $\boldsymbol{I}$ is the 4×4 identity matrix, and the $(4N \times 4)$ matrices

$$\boldsymbol{\Pi}(l, m) = [\boldsymbol{P}_l^m(\mu_1), \boldsymbol{P}_l^m(\mu_2), \ldots, \boldsymbol{P}_l^m(\mu_N)]^{\mathrm{T}} \tag{32}$$

so that we can rewrite Eqs. (28) as

$$\left(\boldsymbol{I} - \frac{1}{v}\boldsymbol{M}\right)\boldsymbol{\Phi}_+(v) = \frac{\varpi}{2}\sum_{l=m}^{L}\boldsymbol{\Pi}(l, m)\boldsymbol{B}_l\boldsymbol{G}_l^m(v) \tag{33a}$$

and

$$\left(\boldsymbol{I} + \frac{1}{v}\boldsymbol{M}\right)\boldsymbol{\Phi}_-(v) = \frac{\varpi}{2}\sum_{l=m}^{L}(-1)^{l-m}\boldsymbol{\Pi}(l, m)\boldsymbol{D}\boldsymbol{B}_l\boldsymbol{G}_l^m(v) \tag{33b}$$

where (now) $\boldsymbol{I}$ is the $4N \times 4N$ identity matrix and

$$\boldsymbol{G}_l^m(v) = \boldsymbol{\Pi}^{\mathrm{T}}(l, m)\boldsymbol{W}\boldsymbol{\Phi}_+(v) + (-1)^{l-m}\boldsymbol{D}\boldsymbol{\Pi}^{\mathrm{T}}(l, m)\boldsymbol{W}\boldsymbol{\Phi}_-(v). \tag{34}$$

We follow Ref. [4] and let

$$\boldsymbol{U} = \boldsymbol{\Phi}_+(v) + \boldsymbol{\Phi}_-(v) \tag{35a}$$

and

$$\boldsymbol{V} = \boldsymbol{\Phi}_+(v) - \boldsymbol{\Phi}_-(v) \tag{35b}$$

so that we can take the sum and the difference of Eqs. (33) to obtain

$$\boldsymbol{E}\boldsymbol{X} = \frac{1}{v}\boldsymbol{Y} \tag{36a}$$

and

$$\boldsymbol{F}\boldsymbol{Y} = \frac{1}{v}\boldsymbol{X} \tag{36b}$$

where

$$\boldsymbol{E} = \left(\boldsymbol{I} - \frac{\varpi}{2}\sum_{l=m}^{L}\boldsymbol{\Pi}(l,m)\boldsymbol{B}_l[\boldsymbol{I} + (-1)^{l-m}\boldsymbol{D}]\boldsymbol{\Pi}^{\mathrm{T}}(l,m)\boldsymbol{W}\right)\boldsymbol{M}^{-1}, \tag{37a}$$

$$\boldsymbol{F} = \left(\boldsymbol{I} - \frac{\varpi}{2}\sum_{l=m}^{L}\boldsymbol{\Pi}(l,m)\boldsymbol{B}_l[\boldsymbol{I} - (-1)^{l-m}\boldsymbol{D}]\boldsymbol{\Pi}^{\mathrm{T}}(l,m)\boldsymbol{W}\right)\boldsymbol{M}^{-1}, \tag{37b}$$

$$\boldsymbol{X} = \boldsymbol{M}\boldsymbol{U} \tag{38a}$$

and

$$\boldsymbol{Y} = \boldsymbol{M}\boldsymbol{V}. \tag{38b}$$

Clearly we can eliminate between Eqs. (36) to obtain the eigenvalue problems

$$(\boldsymbol{F}\boldsymbol{E})\boldsymbol{X} = \lambda\boldsymbol{X} \tag{39a}$$

and

$$(\boldsymbol{E}\boldsymbol{F})\boldsymbol{Y} = \lambda\boldsymbol{Y} \tag{39b}$$

where $\lambda = 1/v^2$. We note that the required separation constants $\{v_j\}$ are readily available once we find the eigenvalues $\{\lambda_j\}$ defined by either of Eqs. (39). We choose to express our results in terms of the eigenvalues and eigenvectors defined by Eq. (39a).

Continuing, we have seen that the eigenvalues defined by Eq. (39a) can be complex numbers, and so here we assume only that Eq. (39a) defines a full set of eigenvectors. We therefore let λ_j and $\boldsymbol{X}(\lambda_j)$, for $j = 1,2,\dots,4N$, denote the collection of eigenvalues and eigenvectors of Eq. (39a). The separation constants we require clearly occur in plus–minus pairs, and so letting v_j, for the $j = 1,2,\dots,4N$, denote the reciprocal of the square root of λ_j, we can use Eqs. (35), (36) and (38) to obtain

$$\boldsymbol{\Phi}_+(v_j) = \tfrac{1}{2}\boldsymbol{M}^{-1}(\boldsymbol{I} + v_j\boldsymbol{E})\boldsymbol{X}(\lambda_j) \tag{40a}$$

and

$$\boldsymbol{\Phi}_-(v_j) = \tfrac{1}{2}\boldsymbol{M}^{-1}(\boldsymbol{I} - v_j\boldsymbol{E})\boldsymbol{X}(\lambda_j) \tag{40b}$$

for $j = 1,2,\dots,4N$. We note that

$$\boldsymbol{\Phi}_+(-v_j) = \boldsymbol{\Phi}_-(v_j), \tag{41}$$

and so at this point we have available all we require for defining our solution to Eqs. (24). To be specific we consider that v_j has a real part that is positive or zero (in our computations we have not found a case where v_j has a real part equal to zero). We let

$$\boldsymbol{I}_+(\tau) = [\boldsymbol{I}^{\mathrm{T}}(\tau,\mu_1), \boldsymbol{I}^{\mathrm{T}}(\tau,\mu_2), \dots, \boldsymbol{I}^{\mathrm{T}}(\tau,\mu_N)]^{\mathrm{T}} \tag{42a}$$

and

$$\boldsymbol{I}_-(\tau) = [\boldsymbol{I}^{\mathrm{T}}(\tau,-\mu_1), \boldsymbol{I}^{\mathrm{T}}(\tau,-\mu_2), \dots, \boldsymbol{I}^{\mathrm{T}}(\tau,-\mu_N)]^{\mathrm{T}} \tag{42b}$$

so we can express our discrete-ordinates solution to the homogeneous version of Eq. (21) as

$$I^h_+(\tau) = \sum_{j=1}^{4N} [A_j \boldsymbol{\Phi}_+(\nu_j) e^{-\tau/\nu_j} + B_j \boldsymbol{\Phi}_-(\nu_j) e^{-(\tau_0-\tau)/\nu_j}] \tag{43a}$$

and

$$I^h_-(\tau) = \boldsymbol{\Delta} \sum_{j=1}^{4N} [A_j \boldsymbol{\Phi}_-(\nu_j) e^{-\tau/\nu_j} + B_j \boldsymbol{\Phi}_+(\nu_j) e^{-(\tau_0-\tau)/\nu_j}] \tag{43b}$$

where the $4N \times 4N$ matrix $\boldsymbol{\Delta}$ is given by

$$\boldsymbol{\Delta} = \text{diag}\{\boldsymbol{D}, \boldsymbol{D}, \dots, \boldsymbol{D}\} \tag{44}$$

and where the constants $\{A_j\}$ and $\{B_j\}$ are at this point arbitrary. Note that in Eqs. (43) we have added the superscript h to remind us that these solutions refer to the homogeneous version of Eq. (21). For emphasis, we state here that we have found that both the separation constants ν_j and the vectors $\boldsymbol{\Phi}_\pm(\nu_j)$ can be complex, and this is a very different observation from what we have found for the scalar case [4].

In order to rewrite Eqs. (43) in terms of real quantities we wish to distinguish between real and complex separation constants $\{\nu_j\}$. And so letting J_r denote the number of real separation constants and J_c denote the number of complex conjugate pairs of separation constants, we can write Eqs. (43) as

$$I^h_+(\tau) = \boldsymbol{R}_+(\tau) + \boldsymbol{C}_+(\tau) \tag{45a}$$

and

$$I^h_-(\tau) = \boldsymbol{R}_-(\tau) + \boldsymbol{C}_-(\tau) \tag{45b}$$

where

$$\boldsymbol{R}_+(\tau) = \sum_{j=1}^{J_r} [A_j \boldsymbol{\Phi}_+(\nu_j) e^{-\tau/\nu_j} + B_j \boldsymbol{\Phi}_-(\nu_j) e^{-(\tau_0-\tau)/\nu_j}], \tag{46a}$$

$$\boldsymbol{C}_+(\tau) = \sum_{j=1}^{J_c} \sum_{\alpha=1}^{2} [A_j^{(\alpha)} \boldsymbol{F}_+^{(\alpha)}(\tau, \nu_j) + B_j^{(\alpha)} \boldsymbol{F}_-^{(\alpha)}(\tau_0 - \tau, \nu_j)], \tag{46b}$$

$$\boldsymbol{R}_-(\tau) = \boldsymbol{\Delta} \sum_{j=1}^{J_r} [A_j \boldsymbol{\Phi}_-(\nu_j) e^{-\tau/\nu_j} + B_j \boldsymbol{\Phi}_+(\nu_j) e^{-(\tau_0-\tau)/\nu_j}], \tag{46c}$$

$$\boldsymbol{C}_-(\tau) = \boldsymbol{\Delta} \sum_{j=1}^{J_c} \sum_{\alpha=1}^{2} [A_j^{(\alpha)} \boldsymbol{F}_-^{(\alpha)}(\tau, \nu_j) + B_j^{(\alpha)} \boldsymbol{F}_+^{(\alpha)}(\tau_0 - \tau, \nu_j)], \tag{46d}$$

$$\boldsymbol{F}_\pm^{(1)}(\tau, \nu_j) = \text{Re}\{e^{-\tau/\nu_j}\}\text{Re}\{\boldsymbol{\Phi}_\pm(\nu_j)\} - \text{Im}\{e^{-\tau/\nu_j}\}\text{Im}\{\boldsymbol{\Phi}_\pm(\nu_j)\} \tag{46e}$$

and

$$\boldsymbol{F}_{\pm}^{(2)}(\tau, \nu_j) = \mathrm{Im}\{\mathrm{e}^{-\tau/\nu_j}\}\mathrm{Re}\{\boldsymbol{\Phi}_{\pm}(\nu_j)\} + \mathrm{Re}\{\mathrm{e}^{-\tau/\nu_j}\}\mathrm{Im}\{\boldsymbol{\Phi}_{\pm}(\nu_j)\}. \tag{46f}$$

To be complete, we note that all of the $8N = 2(J_{\mathrm{r}} + 2J_{\mathrm{c}})$ constants (A's and B's) in Eqs. (46) are to be determined from the boundary conditions of our problem.

4. The infinite-medium Green's functions

Having developed our discrete-ordinates solution to the homogeneous version of Eq. (21), we now require a particular solution to account for the inhomogeneous term $\boldsymbol{Q}_k^m(\tau, \mu)$ that appears there. Our way [7] of constructing the desired particular solution is based on expressing the particular solution in terms of the infinite-medium Green's function, and so we now follow Ref. [7] and consider the following *two* problems: in regard to $\boldsymbol{G}(\tau, \pm\mu_i : x, \mu_\alpha)$ we have

$$\left(\mu_i \frac{\mathrm{d}}{\mathrm{d}\tau} + 1\right)\boldsymbol{G}(\tau, \mu_i : x, \mu_\alpha) = \frac{\varpi}{2} \sum_{l=m}^{L} \boldsymbol{S}_l^m(\tau, \mu_i : x, \mu_\alpha) + \boldsymbol{I}\delta(\tau - x)\delta_{i,\alpha} \tag{47a}$$

and

$$\left(-\mu_i \frac{\mathrm{d}}{\mathrm{d}\tau} + 1\right)\boldsymbol{G}(\tau, -\mu_i : x, \mu_\alpha) = \frac{\varpi}{2} \sum_{l=m}^{L} \boldsymbol{S}_l^m(\tau, -\mu_i : x, \mu_\alpha) \tag{47b}$$

and in regard to $\boldsymbol{G}(\tau, \pm\mu_i : x, -\mu_\alpha)$ we have

$$\left(\mu_i \frac{\mathrm{d}}{\mathrm{d}\tau} + 1\right)\boldsymbol{G}(\tau, \mu_i : x, -\mu_\alpha) = \frac{\varpi}{2} \sum_{l=m}^{L} \boldsymbol{S}_l^m(\tau, \mu_i : x, -\mu_\alpha) \tag{48a}$$

and

$$\left(-\mu_i \frac{\mathrm{d}}{\mathrm{d}\tau} + 1\right)\boldsymbol{G}(\tau, -\mu_i : x, -\mu_\alpha) = \frac{\varpi}{2} \sum_{l=m}^{L} \boldsymbol{S}_l^m(\tau, -\mu_i : x, -\mu_\alpha) + \boldsymbol{I}\delta(\tau - x)\delta_{i,\alpha} \tag{48b}$$

for i, $\alpha = 1, 2, \ldots, N$. In Eqs. (47) and (48), $\boldsymbol{I}$ is the (4×4) identity matrix,

$$\boldsymbol{S}_l^m(\tau, \pm\mu_i : x, \pm\mu_\alpha) = \boldsymbol{P}_l^m(\pm\mu_i)\boldsymbol{B}_l \sum_{\beta=1}^{N} w_\beta \boldsymbol{G}_{l,\beta}(\tau, x, \pm\mu_\alpha) \tag{49a}$$

and

$$\boldsymbol{G}_{l,\beta}(\tau : x, \pm\mu_\alpha) = \boldsymbol{P}_l^m(\mu_\beta)\boldsymbol{G}(\tau, \mu_\beta : x, \pm\mu_\alpha) + \boldsymbol{P}_l^m(-\mu_\beta)\boldsymbol{G}(\tau, -\mu_\beta : x, \pm\mu_\alpha). \tag{49b}$$

Here we take the source location to be $x \in (0, \tau_0)$. Further, we consider that the source direction is defined by $\mu_\alpha \in \{\mu_i\}$. We note that $\delta(\tau - x)$ is the Dirac delta "function" and that $\delta_{i,\alpha}$ is the Kronecker delta. In addition, we point out that by including the identity matrix in the source term of Eqs. (47a) and (48b) we clearly are considering that each of the two Green's functions we seek is a (4×4) matrix.

To develop the desired solution for $\boldsymbol{G}(\tau, \xi_\beta : x, \pm\mu_\alpha)$ we can, as discussed for example by Case and Zweifel [8], write one solution (of the homogeneous equation) valid for $\tau > x$ and another solution valid for $\tau < x$; we can then match-up these two solutions with the "jump" conditions, for $i, \alpha = 1, 2, \dots, N$,

$$\mu_i \lim_{\varepsilon \to 0} [\boldsymbol{G}(x+\varepsilon, \mu_i : x, \mu_\alpha) - \boldsymbol{G}(x-\varepsilon, \mu_i : x, \mu_\alpha)] = \boldsymbol{I}\delta_{i,\alpha} \tag{50a}$$

and

$$-\mu_i \lim_{\varepsilon \to 0} [\boldsymbol{G}(x+\varepsilon, -\mu_i : x, \mu_\alpha) - \boldsymbol{G}(x-\varepsilon, -\mu_i : x, \mu_\alpha)] = \boldsymbol{0} \tag{50b}$$

for $\boldsymbol{G}(\tau, \pm\mu_i : x, \mu_\alpha)$ and

$$\mu_i \lim_{\varepsilon \to 0} [\boldsymbol{G}(x+\varepsilon, \mu_i : x, -\mu_\alpha) - \boldsymbol{G}(x-\varepsilon, \mu_i : x, -\mu_\alpha)] = \boldsymbol{0} \tag{51a}$$

along with

$$-\mu_i \lim_{\varepsilon \to 0} [\boldsymbol{G}(x+\varepsilon, -\mu_i : x, -\mu_\alpha) - \boldsymbol{G}(x-\varepsilon, -\mu_i : x, -\mu_\alpha)] = \boldsymbol{I}\delta_{i,\alpha} \tag{51b}$$

for $\boldsymbol{G}(\tau, \pm\mu_i : x, -\mu_\alpha)$. Looking back to Eqs. (42), we now write our discrete-ordinates solution for the two Green's functions

$$\boldsymbol{G}_\pm(\tau : x, \mu_\alpha) = [\boldsymbol{G}^{\mathrm{T}}(\tau, \pm\mu_1 : x, \mu_\alpha), \dots, \boldsymbol{G}^{\mathrm{T}}(\tau, \pm\mu_N : x, \mu_\alpha)]^{\mathrm{T}} \tag{52a}$$

and

$$\boldsymbol{G}_\pm(\tau : x, -\mu_\alpha) = [\boldsymbol{G}^{\mathrm{T}}(\tau, \pm\mu_1 : x, -\mu_\alpha), \dots, \boldsymbol{G}^{\mathrm{T}}(\tau, \pm\mu_N : x, -\mu_\alpha)]^{\mathrm{T}} \tag{52b}$$

as

$$\boldsymbol{G}_+(\tau : x, \pm\mu_\alpha) = \sum_{j=1}^{4N} \boldsymbol{\Phi}_+(\nu_j)\boldsymbol{A}_j(\pm\mu_\alpha)\mathrm{e}^{-(\tau-x)/\nu_j}, \quad \tau > x, \tag{53a}$$

$$\boldsymbol{G}_-(\tau : x, \pm\mu_\alpha) = \boldsymbol{\Delta}\sum_{j=1}^{4N} \boldsymbol{\Phi}_-(\nu_j)\boldsymbol{A}_j(\pm\mu_\alpha)\mathrm{e}^{-(\tau-x)/\nu_j}, \quad \tau > x, \tag{53b}$$

$$\boldsymbol{G}_+(\tau : x, \pm\mu_\alpha) = -\sum_{j=1}^{4N} \boldsymbol{\Phi}_-(\nu_j)\boldsymbol{B}_j(\pm\mu_\alpha)\mathrm{e}^{-(x-\tau)/\nu_j}, \quad \tau < x, \tag{54a}$$

and

$$\boldsymbol{G}_-(\tau : x, \pm\mu_\alpha) = -\boldsymbol{\Delta}\sum_{j=1}^{4N} \boldsymbol{\Phi}_+(\nu_j)\boldsymbol{B}_j(\pm\mu_\alpha)\mathrm{e}^{-(x-\tau)/\nu_j}, \quad \tau < x, \tag{54b}$$

where now the arbitrary constants in Eqs. (53) and (54) are the (1×4) vectors

$$\boldsymbol{A}_j(\pm\mu_\alpha) = [A_{1,j}(\pm\mu_\alpha)A_{2,j}(\pm\mu_\alpha)A_{3,j}(\pm\mu_\alpha)A_{4,j}(\pm\mu_\alpha)] \tag{55a}$$

and

$$\boldsymbol{B}_j(\pm\mu_\alpha) = [B_{1,j}(\pm\mu_\alpha)B_{2,j}(\pm\mu_\alpha)B_{3,j}(\pm\mu_\alpha)B_{4,j}(\pm\mu_\alpha)]. \tag{55b}$$

To find the constants $\{\boldsymbol{A}_j\}$ and $\{\boldsymbol{B}_j\}$ required to complete the Green's functions we substitute Eqs. (53) and (54) into Eqs. (50) and (51) to find the systems of linear algebraic equations

$$\boldsymbol{M}\sum_{j=1}^{4N}[\boldsymbol{\Phi}_+(v_j)\boldsymbol{A}_j(\mu_\alpha)+\boldsymbol{\Phi}_-(v_j)\boldsymbol{B}_j(\mu_\alpha)]=\boldsymbol{R}_\alpha \tag{56a}$$

and

$$-\boldsymbol{M}\boldsymbol{\Delta}\sum_{j=1}^{4N}[\boldsymbol{\Phi}_-(v_j)\boldsymbol{A}_j(\mu_\alpha)+\boldsymbol{\Phi}_+(v_j)\boldsymbol{B}_j(\mu_\alpha)]=\boldsymbol{0} \tag{56b}$$

for the first problem and

$$\boldsymbol{M}\sum_{j=1}^{4N}[\boldsymbol{\Phi}_+(v_j)\boldsymbol{A}_j(-\mu_\alpha)+\boldsymbol{\Phi}_-(v_j)\boldsymbol{B}_j(-\mu_\alpha)]=\boldsymbol{0} \tag{57a}$$

and

$$-\boldsymbol{M}\boldsymbol{\Delta}\sum_{j=1}^{4N}[\boldsymbol{\Phi}_-(v_j)\boldsymbol{A}_j(-\mu_\alpha)+\boldsymbol{\Phi}_+(v_j)\boldsymbol{B}_j(-\mu_\alpha)]=\boldsymbol{R}_\alpha \tag{57b}$$

for the second problem. Here we have introduced the $(4N\times 4)$ matrix

$$\boldsymbol{R}_\alpha=[\boldsymbol{I}\delta_{1,\alpha},\boldsymbol{I}\delta_{2,\alpha},\dots,\boldsymbol{I}\delta_{N,\alpha}]^{\mathrm{T}} \tag{58}$$

where we continue to use $\boldsymbol{I}$ to denote the (4×4) identity matrix.

To solve Eqs. (56) and (57) we wish to make use of some basic orthogonality properties of the vectors $\boldsymbol{\Phi}_\pm(v_j)$. To establish these properties we first consider an adjoint problem defined by replacing $\boldsymbol{B}_l$ in Eqs. (33) with $\boldsymbol{B}_l^{\mathrm{T}}$ to obtain

$$\left(\boldsymbol{I}-\frac{1}{v}\boldsymbol{M}\right)\boldsymbol{\Psi}_+(v)=\frac{\varpi}{2}\sum_{l=m}^{L}\boldsymbol{\Pi}(l,m)\boldsymbol{B}_l^{\mathrm{T}}\boldsymbol{\Gamma}_l^m(v) \tag{59a}$$

and

$$\left(\boldsymbol{I}+\frac{1}{v}\boldsymbol{M}\right)\boldsymbol{\Psi}_-(v)=\frac{\varpi}{2}\sum_{l=m}^{L}(-1)^{l-m}\boldsymbol{\Pi}(l,m)\boldsymbol{D}\boldsymbol{B}_l^{\mathrm{T}}\boldsymbol{\Gamma}_l^m(v) \tag{59b}$$

where

$$\boldsymbol{\Gamma}_l^m(v)=\boldsymbol{\Pi}^{\mathrm{T}}(l,m)\boldsymbol{W}\boldsymbol{\Psi}_+(v)+(-1)^{l-m}\boldsymbol{D}\boldsymbol{\Pi}^{\mathrm{T}}(l,m)\boldsymbol{W}\boldsymbol{\Psi}_-(v). \tag{60}$$

It is not difficult to show that the eigenvalues defined by Eqs. (39), with $\boldsymbol{E}$ and $\boldsymbol{F}$ given by Eqs. (37), will not be changed if $\boldsymbol{B}_l$ in Eqs. (37) is replaced by $\boldsymbol{B}_l^{\mathrm{T}}$, and so the adjoint vectors $\boldsymbol{\Psi}_\pm(v_k)$ are defined over the same spectrum as the vectors $\boldsymbol{\Phi}_\pm(v_j)$. We now consider Eq. (33) evaluated at $v=v_j$, then we premultiply Eq. (33a) by $\boldsymbol{\Psi}_+^{\mathrm{T}}(v_k)\boldsymbol{W}$ and Eq. (33b) by $\boldsymbol{\Psi}_-^{\mathrm{T}}(v_k)\boldsymbol{W}$ and add the resulting two equations. At this point we form a second equation (from the one just obtained) by interchanging the indices j and k, interchanging the direct and adjoint vectors, changing $\boldsymbol{B}_l$ to $\boldsymbol{B}_l^{\mathrm{T}}$ and taking the transpose of the resulting equation. Now we subtract this second equation from the first to find

$$\boldsymbol{\Psi}^{\mathrm{T}}_{+}(\nu_k)\boldsymbol{W}\boldsymbol{M}\boldsymbol{\Phi}_{+}(\nu_j) - \boldsymbol{\Psi}^{\mathrm{T}}_{-}(\nu_k)\boldsymbol{W}\boldsymbol{M}\boldsymbol{\Phi}_{-}(\nu_j) = 0, \quad \nu_k \neq \nu_j. \tag{61}$$

In a similar way we can show that

$$\boldsymbol{\Psi}^{\mathrm{T}}_{+}(\nu_k)\boldsymbol{W}\boldsymbol{M}\boldsymbol{\Phi}_{-}(\nu_j) - \boldsymbol{\Psi}^{\mathrm{T}}_{-}(\nu_k)\boldsymbol{W}\boldsymbol{M}\boldsymbol{\Phi}_{+}(\nu_j) = 0. \tag{62}$$

Now we can premultiply Eq. (56a) by $\boldsymbol{\Psi}^{\mathrm{T}}_{+}(\nu_k)\boldsymbol{W}$ and Eq. (56b) by $\boldsymbol{\Psi}^{\mathrm{T}}_{-}(\nu_k)\boldsymbol{W}\boldsymbol{\Delta}$ and add the two resulting equations to find, after noting Eqs. (61) and (62),

$$\boldsymbol{A}_j(\mu_\alpha) = \frac{1}{N(\nu_j)}\boldsymbol{\Psi}^{\mathrm{T}}_{+}(\nu_j)\boldsymbol{W}\boldsymbol{R}_\alpha \tag{63}$$

where

$$N(\nu_j) = \boldsymbol{\Psi}^{\mathrm{T}}_{+}(\nu_j)\boldsymbol{W}\boldsymbol{M}\boldsymbol{\Phi}_{+}(\nu_j) - \boldsymbol{\Psi}^{\mathrm{T}}_{-}(\nu_j)\boldsymbol{W}\boldsymbol{M}\boldsymbol{\Phi}_{-}(\nu_j). \tag{64}$$

Continuing, we premultiply Eq. (56a) by $\boldsymbol{\Psi}^{\mathrm{T}}_{-}(\nu_k)\boldsymbol{W}$ and Eq. (56b) by $\boldsymbol{\Psi}^{\mathrm{T}}_{+}(\nu_k)\boldsymbol{W}\boldsymbol{\Delta}$, add the two resulting equations and note Eqs. (61) and (62) to obtain

$$\boldsymbol{B}_j(\mu_\alpha) = -\frac{1}{N(\nu_j)}\boldsymbol{\Psi}^{\mathrm{T}}_{-}(\nu_j)\boldsymbol{W}\boldsymbol{R}_\alpha. \tag{65}$$

In a similar way we find from Eqs. (57)

$$\boldsymbol{A}_j(-\mu_\alpha) = \frac{1}{N(\nu_j)}\boldsymbol{\Psi}^{\mathrm{T}}_{-}(\nu_j)\boldsymbol{W}\boldsymbol{\Delta}\boldsymbol{R}_\alpha \tag{66}$$

and

$$\boldsymbol{B}_j(-\mu_\alpha) = -\frac{1}{N(\nu_j)}\boldsymbol{\Psi}^{\mathrm{T}}_{+}(\nu_j)\boldsymbol{W}\boldsymbol{\Delta}\boldsymbol{R}_\alpha. \tag{67}$$

Since Eqs. (63) and (65)–(67) define the constants A and B required in Eqs. (53) and (54), we consider that our two Green's functions are established, and so we are ready to use them to find the particular solution we seek.

5. A particular solution

We now consider the discrete-ordinates version of Eq. (21) written as

$$\mu_i \frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{I}(\tau,\mu_i) + \boldsymbol{I}(\tau,\mu_i) = \frac{\varpi}{2}\sum_{l=m}^{L}\boldsymbol{P}_l^m(\mu_i)\boldsymbol{B}_l\sum_{\beta=1}^{N} w_\beta \boldsymbol{I}_{l,\beta}(\tau) + \boldsymbol{Q}(\tau,\mu_i) \tag{68a}$$

and

$$-\mu_i \frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{I}(\tau,-\mu_i) + \boldsymbol{I}(\tau,-\mu_i) = \frac{\varpi}{2}\sum_{l=m}^{L}\boldsymbol{P}_l^m(-\mu_i)\boldsymbol{B}_l\sum_{\beta=1}^{N} w_\beta \boldsymbol{I}_{l,\beta}(\tau) + \boldsymbol{Q}(\tau,-\mu_i) \tag{68b}$$

for $i = 1, 2, \dots, N$. In writing Eqs. (68), we have used

$$\boldsymbol{I}_{l,\beta}(\tau) = \boldsymbol{P}_l^m(\mu_\beta)\boldsymbol{I}(\tau, \mu_\beta) + \boldsymbol{P}_l^m(-\mu_\beta)\boldsymbol{I}(\tau, -\mu_\beta). \tag{69}$$

Since in Section 3 we have already developed our general solution to the homogeneous version of Eqs. (68), we lack only a particular solution of Eqs. (68) to account for the inhomogeneous source terms. In fact, a particular solution is immediately available since we can express our particular solution in terms of the two Green's functions from Section 4. Our general result is

$$\boldsymbol{I}^p(\tau, \pm\mu_i) = \sum_{\alpha=1}^{N} \int_0^{\tau_0} [\boldsymbol{G}(\tau, \pm\mu_i : x, \mu_\alpha)\boldsymbol{Q}(x, \mu_\alpha) + \boldsymbol{G}(\tau, \pm\mu_i : x, -\mu_\alpha)\boldsymbol{Q}(x, -\mu_\alpha)]\,\mathrm{d}x \tag{70}$$

or

$$\boldsymbol{I}^p_\pm(\tau) = \sum_{\alpha=1}^{N} \int_0^{\tau_0} [\boldsymbol{G}_\pm(\tau : x, \mu_\alpha)\boldsymbol{Q}(x, \mu_\alpha) + \boldsymbol{G}_\pm(\tau : x, -\mu_\alpha)\boldsymbol{Q}(x, -\mu_\alpha)]\,\mathrm{d}x. \tag{71}$$

Now using Eqs. (53) and (54), we can rewrite Eq. (71) as

$$\boldsymbol{I}^p_+(\tau) = \sum_{j=1}^{4N} [\mathscr{A}_j(\tau)\boldsymbol{\Phi}_+(v_j) + \mathscr{B}_j(\tau)\boldsymbol{\Phi}_-(v_j)] \tag{72a}$$

and

$$\boldsymbol{I}^p_-(\tau) = \boldsymbol{\Delta} \sum_{j=1}^{4N} [\mathscr{A}_j(\tau)\boldsymbol{\Phi}_-(v_j) + \mathscr{B}_j(\tau)\boldsymbol{\Phi}_+(v_j)] \tag{72b}$$

where

$$\mathscr{A}_j(\tau) = \int_0^{\tau} \sum_{\alpha=1}^{N} [\boldsymbol{A}_j(\mu_\alpha)\boldsymbol{Q}(x, \mu_\alpha) + \boldsymbol{A}_j(-\mu_\alpha)\boldsymbol{Q}(x, -\mu_\alpha)]\,\mathrm{e}^{-(\tau - x)/v_j}\,\mathrm{d}x \tag{73a}$$

and

$$\mathscr{B}_j(\tau) = -\int_\tau^{\tau_0} \sum_{\alpha=1}^{N} [\boldsymbol{B}_j(\mu_\alpha)\boldsymbol{Q}(x, \mu_\alpha) + \boldsymbol{B}_j(-\mu_\alpha)\boldsymbol{Q}(x, -\mu_\alpha)]\,\mathrm{e}^{-(x - \tau)/v_j}\,\mathrm{d}x. \tag{73b}$$

To complete our general result we now use Eqs. (63) and(65)–(67) in Eqs. (73) to obtain

$$\mathscr{A}_j(\tau) = \int_0^{\tau} a_j(x)\,\mathrm{e}^{-(\tau - x)/v_j}\,\mathrm{d}x \tag{74a}$$

and

$$\mathscr{B}_j(\tau) = \int_\tau^{\tau_0} b_j(x)\,\mathrm{e}^{-(x - \tau)/v_j}\,\mathrm{d}x \tag{74b}$$

where

$$a_j(x) = \frac{1}{N(v_j)}\,[\boldsymbol{\Psi}_+^{\mathrm{T}}(v_j)\boldsymbol{W}\boldsymbol{Q}_+(x) + \boldsymbol{\Psi}_-^{\mathrm{T}}(v_j)\boldsymbol{W}\boldsymbol{\Delta}\boldsymbol{Q}_-(x)] \tag{75a}$$

and

$$b_j(x) = \frac{1}{N(v_j)}\,[\boldsymbol{\Psi}_-^{\mathrm{T}}(v_j)\boldsymbol{W}\boldsymbol{Q}_+(x) + \boldsymbol{\Psi}_+^{\mathrm{T}}(v_j)\boldsymbol{W}\boldsymbol{\Delta}\boldsymbol{Q}_-(x)] \tag{75b}$$

and where

$$\boldsymbol{Q}_+(x) = [\boldsymbol{Q}^{\mathrm{T}}(x, \mu_1), \boldsymbol{Q}^{\mathrm{T}}(x, \mu_2), \ldots, \boldsymbol{Q}^{\mathrm{T}}(x, \mu_N)]^{\mathrm{T}} \tag{76a}$$

and

$$\boldsymbol{Q}_-(x) = [\boldsymbol{Q}^{\mathrm{T}}(x, -\mu_1), \boldsymbol{Q}^{\mathrm{T}}(x, -\mu_2), \ldots, \boldsymbol{Q}^{\mathrm{T}}(x, -\mu_N)]^{\mathrm{T}}. \tag{76b}$$

While Eqs. (74) and (75) are our general results (and so cannot be simplified without specifying the source terms) we now see from Eq. (23) that for the problem considered in this work we can write

$$\boldsymbol{Q}_+(x) = \boldsymbol{Q}_+\,\mathrm{e}^{-x/\mu_0} \tag{77a}$$

and

$$\boldsymbol{Q}_-(x) = \boldsymbol{Q}_-\mathrm{e}^{-x/\mu_0} \tag{77b}$$

where $\boldsymbol{Q}_\pm$ are constants. If we now use Eqs. (74), (75) and (77) we find the special results we need here, namely

$$\mathscr{A}_j(\tau) = \mu_0 v_j a_j C(\tau : v_j, \mu_0) \tag{78a}$$

and

$$\mathscr{B}_j(\tau) = \mu_0 v_j b_j\,\mathrm{e}^{-\tau/\mu_0} S(\tau_0 - \tau : v_j, \mu_0) \tag{78b}$$

where now

$$a_j = \frac{1}{N(v_j)}\,[\boldsymbol{\Psi}_+^{\mathrm{T}}(v_j)\boldsymbol{W}\boldsymbol{Q}_+ + \boldsymbol{\Psi}_-^{\mathrm{T}}(v_j)\boldsymbol{W}\boldsymbol{\Delta}\boldsymbol{Q}_-] \tag{79a}$$

and

$$b_j = \frac{1}{N(v_j)}\,[\boldsymbol{\Psi}_-^{\mathrm{T}}(v_j)\boldsymbol{W}\boldsymbol{Q}_+ + \boldsymbol{\Psi}_+^{\mathrm{T}}(v_j)\boldsymbol{W}\boldsymbol{\Delta}\boldsymbol{Q}_-]. \tag{79b}$$

In addition, the S and C functions are given by

$$S(\tau : x, y) = \frac{1 - \mathrm{e}^{-\tau/x}\mathrm{e}^{-\tau/y}}{x + y} \tag{80a}$$

and

$$C(\tau : x, y) = \frac{\mathrm{e}^{-\tau/x} - \mathrm{e}^{-\tau/y}}{x - y}. \tag{80b}$$

Even though some of the separation constants can be complex, we can express our particular solution in terms of real quantities. If we let

$$\boldsymbol{A}_{\pm}(\tau, \nu_j) = \mathscr{A}_j(\tau)\boldsymbol{\Phi}_{\pm}(\nu_j) + \mathscr{A}_j^*(\tau)\boldsymbol{\Phi}_{\pm}^*(\nu_j) \tag{81a}$$

and

$$\boldsymbol{B}_{\pm}(\tau, \nu_j) = \mathscr{B}_j(\tau)\boldsymbol{\Phi}_{\pm}(\nu_j) + \mathscr{B}_j^*(\tau)\boldsymbol{\Phi}_{\pm}^*(\nu_j), \tag{81b}$$

where we use the superscript asterisk to denote the operation of complex conjugation, then we can write

$$\boldsymbol{I}_{+}^{p}(\tau) = \sum_{j=1}^{J_r} [\mathscr{A}_j(\tau)\boldsymbol{\Phi}_{+}(\nu_j) + \mathscr{B}_j(\tau)\boldsymbol{\Phi}_{-}(\nu_j)] + \sum_{j=1}^{J_c} [\boldsymbol{A}_{+}(\tau, \nu_j) + \boldsymbol{B}_{-}(\tau, \nu_j)] \tag{82a}$$

and

$$\boldsymbol{I}_{-}^{p}(\tau) = \boldsymbol{\Delta}\sum_{j=1}^{J_r} [\mathscr{A}_j(\tau)\boldsymbol{\Phi}_{-}(\nu_j) + \mathscr{B}_j(\tau)\boldsymbol{\Phi}_{+}(\nu_j)] + \boldsymbol{\Delta}\sum_{j=1}^{J_c} [\boldsymbol{A}_{-}(\tau, \nu_j) + \boldsymbol{B}_{+}(\tau, \nu_j)]. \tag{82b}$$

6. A first solution and post-processing

Having found a particular solution, we are ready to construct the complete solution we seek. We note Eqs. (45) and write

$$\boldsymbol{I}_{+}(\tau) = \boldsymbol{R}_{+}(\tau) + \boldsymbol{C}_{+}(\tau) + \boldsymbol{I}_{+}^{p}(\tau) \tag{83a}$$

and

$$\boldsymbol{I}_{-}(\tau) = \boldsymbol{R}_{-}(\tau) + \boldsymbol{C}_{-}(\tau) + \boldsymbol{I}_{-}^{p}(\tau) \tag{83b}$$

where the $\boldsymbol{R}$ and $\boldsymbol{C}$ components are given by Eqs. (46) and where the known particular solution is given by Eqs. (82). So all we have to do now is to substitute Eqs. (83) into the boundary conditions, listed as Eqs. (22), to find the A and B coefficients required in Eqs. (83). Continuing to suppress some of the explicit notation (m and k) we find the linear systems

$$\sum_{j=1}^{J_r} \{\boldsymbol{M}_j A_j + \boldsymbol{N}_j B_j\} + \sum_{j=1}^{J_c} \sum_{\alpha=1}^{2} \{\boldsymbol{M}_j^{(\alpha)} A_j^{(\alpha)} + \boldsymbol{N}_j^{(\alpha)} B_j^{(\alpha)}\} = \boldsymbol{K}_{+} \tag{84a}$$

and

$$\sum_{j=1}^{J_r} \{\hat{\boldsymbol{N}}_j A_j + \hat{\boldsymbol{M}}_j B_j\} + \sum_{j=1}^{J_c} \sum_{\alpha=1}^{2} \{\hat{\boldsymbol{N}}_j^{(\alpha)} A_j^{(\alpha)} + \hat{\boldsymbol{M}}_j^{(\alpha)} B_j^{(\alpha)}\} = \boldsymbol{K}_{-} \tag{84b}$$

where

$$\boldsymbol{M}_j = \boldsymbol{\Phi}_{+}(\nu_j), \tag{85a}$$

$$\boldsymbol{N}_j = \boldsymbol{\Phi}_{-}(\nu_j)\,\mathrm{e}^{-\tau_0/\nu_j}, \tag{85b}$$

$$\boldsymbol{M}_j^{(\alpha)} = \boldsymbol{F}_+^{(\alpha)}(0, \nu_j), \tag{85c}$$

$$\boldsymbol{N}_j^{(\alpha)} = \boldsymbol{F}_-^{(\alpha)}(\tau_0, \nu_j), \tag{85d}$$

$$\hat{\boldsymbol{M}}_j = \boldsymbol{M}_j - \boldsymbol{R}_{b,j}, \tag{85e}$$

$$\hat{\boldsymbol{N}}_j = \boldsymbol{N}_j - \boldsymbol{R}_{a,j}, \tag{85f}$$

$$\hat{\boldsymbol{M}}_j^{(\alpha)} = \boldsymbol{M}_j^{(\alpha)} - \boldsymbol{R}_{b,j}^{(\alpha)} \tag{85g}$$

and

$$\hat{\boldsymbol{N}}_j^{(\alpha)} = \boldsymbol{N}_j^{(\alpha)} - \boldsymbol{R}_{a,j}^{(\alpha)}. \tag{85h}$$

In addition

$$\boldsymbol{K}_+ = -\boldsymbol{I}_+^p(0) \tag{86a}$$

and

$$\boldsymbol{K}_- = -\boldsymbol{I}_-^p(\tau_0) + \Lambda \boldsymbol{E}_1. \tag{86b}$$

Here

$$\Lambda = 2\lambda_0 \delta_{0,m} \delta_{1,k} [\mu_0 F_{\boldsymbol{I}}\, \mathrm{e}^{-\tau_0/\mu_0} + \boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{W} \boldsymbol{M} \boldsymbol{I}_+^p(\tau_0)]. \tag{86c}$$

Also, in writing Eqs. (86b) and (86c) we have introduced the $(4N \times 1)$ vector

$$\boldsymbol{E}_1 = [\boldsymbol{e}_1, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_1]^{\mathrm{T}} \tag{87}$$

with

$$\boldsymbol{e}_1 = [1 \quad 0 \quad 0 \quad 0]. \tag{88}$$

To complete our definitions of Eqs. (84) and (85) we note that the $\boldsymbol{R}$ terms are present to account for the Lambertian reflection. To be explicit, we can write

$$\boldsymbol{R}_{a,j} = 2\lambda_0 \delta_{0,m} \delta_{1,k} [\boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{W} \boldsymbol{M} \boldsymbol{\Phi}_+(\nu_j)] \boldsymbol{E}_1\, \mathrm{e}^{-\tau_0/\nu_j}, \tag{89a}$$

$$\boldsymbol{R}_{b,j} = 2\lambda_0 \delta_{0,m} \delta_{1,k} [\boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{W} \boldsymbol{M} \boldsymbol{\Phi}_-(\nu_j)] \boldsymbol{E}_1, \tag{89b}$$

$$\boldsymbol{R}_{a,j}^{(\alpha)} = 2\lambda_0 \delta_{0,m} \delta_{1,k} [\boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{W} \boldsymbol{M} \boldsymbol{F}_+^{(\alpha)}(\tau_0, \nu_j)] \boldsymbol{E}_1 \tag{89c}$$

and

$$\boldsymbol{R}_{b,j}^{(\alpha)} = 2\lambda_0 \delta_{0,m} \delta_{1,k} [\boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{W} \boldsymbol{M} \boldsymbol{F}_-^{(\alpha)}(0, \nu_j)] \boldsymbol{E}_1\,. \tag{89d}$$

Clearly once we have solved Eqs. (84) to find the A and B constants we have available, by way of Eqs. (82) and (83), a first version of the desired solution. However, we wish to improve on this first solution, which clearly in not even defined for all μ, by following a post-processing procedure [5]. And so to obtain our final results we substitute Eqs. (83) into the right-hand side of

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}(\tau, \mu) + \boldsymbol{I}(\tau, \mu) = \frac{\varpi}{2} \sum_{l=m}^{L} \boldsymbol{P}_l^m(\mu) \boldsymbol{B}_l \sum_{\beta=1}^{N} w_\beta \boldsymbol{I}_{l,\beta}(\tau) + \boldsymbol{Q}(\tau, \mu), \tag{90}$$

where

$$\boldsymbol{I}_{l,\beta}(\tau) = \boldsymbol{P}_l^m(\mu_\beta)\boldsymbol{I}(\tau,\mu_\beta) + \boldsymbol{P}_l^m(-\mu_\beta)\boldsymbol{I}(\tau,-\mu_\beta), \tag{91}$$

to find

$$\boldsymbol{I}(\tau,\mu) = \mu_0 C(\tau:\mu_0,\mu)\boldsymbol{Q}(\mu) + \frac{\varpi}{2}[\boldsymbol{\Xi}(\tau,\mu) + \boldsymbol{\Upsilon}(\tau,\mu)] \tag{92a}$$

and

$$\boldsymbol{I}(\tau,-\mu) = \boldsymbol{R}\mathrm{e}^{-(\tau_0-\tau)/\mu} + \mu_0 T(\tau_0,\tau_0-\tau:\mu_0,\mu)\boldsymbol{Q}(-\mu) + \frac{\varpi}{2}[\boldsymbol{\Xi}(\tau,-\mu) + \boldsymbol{\Upsilon}(\tau,-\mu)] \tag{92b}$$

for $\mu\in[0,1]$ and $\tau\in[0,\tau_0]$. Here

$$\boldsymbol{R} = 2\lambda_0\delta_{0,m}\delta_{1,k}[\mu_0 F_I\,\mathrm{e}^{-\tau_0/\mu_0} + \boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{W}\boldsymbol{M}\boldsymbol{I}_+(\tau_0)]\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \tag{93}$$

the S and C functions are given by Eqs. (80),

$$T(\tau_0,\tau:x,y) = \mathrm{e}^{-(\tau_0-\tau)/x}S(\tau:x,y) \tag{94}$$

and

$$\boldsymbol{Q}(\mu) = \frac{\varpi}{2}\sum_{l=m}^{L}\boldsymbol{P}_l^m(\mu)\boldsymbol{B}_l\boldsymbol{P}_l^m(\mu_0)\boldsymbol{D}_k\boldsymbol{F}. \tag{95}$$

In order to express our results in terms of real quantities, we write

$$\boldsymbol{\Upsilon}(\tau,\pm\mu) = \boldsymbol{\Upsilon}_{\mathrm{r}}(\tau,\pm\mu) + \boldsymbol{\Upsilon}_{\mathrm{c}}(\tau,\pm\mu) \tag{96}$$

where

$$\boldsymbol{\Upsilon}_{\mathrm{r}}(\tau,\mu) = \sum_{l=m}^{L}\boldsymbol{P}_l^m(\mu)\boldsymbol{B}_l\sum_{j=1}^{J_{\mathrm{r}}}[\boldsymbol{A}_l^m(\tau:v_j,\mu) + (-1)^{l-m}\boldsymbol{D}\boldsymbol{B}_l^m(\tau:v_j,\mu)], \tag{97a}$$

$$\boldsymbol{\Upsilon}_{\mathrm{r}}(\tau,-\mu) = \boldsymbol{D}\sum_{l=m}^{L}\boldsymbol{P}_l^m(\mu)\boldsymbol{B}_l\sum_{j=1}^{J_{\mathrm{r}}}[\boldsymbol{C}_l^m(\tau:v_j,\mu) + (-1)^{l-m}\boldsymbol{D}\boldsymbol{D}_l^m(\tau:v_j,\mu)], \tag{97b}$$

$$\boldsymbol{\Upsilon}_{\mathrm{c}}(\tau,\mu) = \sum_{l=m}^{L}\boldsymbol{P}_l^m(\mu)\boldsymbol{B}_l\sum_{j=1}^{J_{\mathrm{c}}}[\boldsymbol{S}_l^m(\tau:v_j,\mu) + (-1)^{l-m}\boldsymbol{D}\boldsymbol{T}_l^m(\tau:v_j,\mu)] \tag{97c}$$

and

$$\boldsymbol{\Upsilon}_{\mathrm{c}}(\tau,-\mu) = \boldsymbol{D}\sum_{l=m}^{L}\boldsymbol{P}_l^m(\mu)\boldsymbol{B}_l\sum_{j=1}^{J_{\mathrm{c}}}[\boldsymbol{U}_l^m(\tau:v_j,\mu) + (-1)^{l-m}\boldsymbol{D}\boldsymbol{V}_l^m(\tau:v_j,\mu)]. \tag{97d}$$

In writing Eqs. (97) we have introduced

$$\boldsymbol{A}_l^m(\tau:\nu_j,\mu)=\nu_j A_j C(\tau:\nu_j,\mu)\boldsymbol{G}_l^m(\nu_j), \tag{98a}$$

$$\boldsymbol{B}_l^m(\tau:\nu_j,\mu)=\nu_j B_j T(\tau_0,\tau:\nu_j,\mu)\boldsymbol{G}_l^m(\nu_j), \tag{98b}$$

$$\boldsymbol{C}_l^m(\tau:\nu_j,\mu)=\nu_j B_j C(\tau_0-\tau:\nu_j,\mu)\boldsymbol{G}_l^m(\nu_j), \tag{98c}$$

$$\boldsymbol{D}_l^m(\tau:\nu_j,\mu)=\nu_j A_j T(\tau_0,\tau_0-\tau:\nu_j,\mu)\boldsymbol{G}_l^m(\nu_j), \tag{98d}$$

$$\boldsymbol{S}_l^m(\tau:\nu_j,\mu)=\mathrm{Re}\{\nu_j C(\tau:\nu_j,\mu)\}\boldsymbol{A}_l^m(1,\nu_j)+\mathrm{Im}\{\nu_j C(\tau:\nu_j,\mu)\}\boldsymbol{A}_l^m(2,\nu_j), \tag{98e}$$

$$\boldsymbol{T}_l^m(\tau:\nu_j,\mu)=\mathrm{Re}\{\nu_j T(\tau_0,\tau:\nu_j,\mu)\}\boldsymbol{B}_l^m(1,\nu_j)+\mathrm{Im}\{\nu_j T(\tau_0,\tau:\nu_j,\mu)\}\boldsymbol{B}_l^m(2,\nu_j), \tag{98f}$$

$$\boldsymbol{U}_l^m(\tau:\nu_j,\mu)=\mathrm{Re}\{\nu_j C(\tau_0-\tau:\nu_j,\mu)\}\boldsymbol{B}_l^m(1,\nu_j)+\mathrm{Im}\{\nu_j C(\tau_0-\tau:\nu_j,\mu)\}\boldsymbol{B}_l^m(2,\nu_j) \tag{98g}$$

and

$$\boldsymbol{V}_l^m(\tau:\nu_j,\mu)=\mathrm{Re}\{\nu_j T(\tau_0,\tau_0-\tau:\nu_j,\mu)\}\boldsymbol{A}_l^m(1,\nu_j)+\mathrm{Im}\{\nu_j T(\tau_0,\tau_0-\tau:\nu_j,\mu)\}\boldsymbol{A}_l^m(2,\nu_j) \tag{98h}$$

where

$$\boldsymbol{A}_l^m(1,\nu_j)=A_j^{(1)}\mathrm{Re}\{\boldsymbol{G}_l^m(\nu_j)\}+A_j^{(2)}\mathrm{Im}\{\boldsymbol{G}_l^m(\nu_j)\}, \tag{99a}$$

$$\boldsymbol{A}_l^m(2,\nu_j)=A_j^{(2)}\mathrm{Re}\{\boldsymbol{G}_l^m(\nu_j)\}-A_j^{(1)}\mathrm{Im}\{\boldsymbol{G}_l^m(\nu_j)\}, \tag{99b}$$

$$\boldsymbol{B}_l^m(1,\nu_j)=B_j^{(1)}\mathrm{Re}\{\boldsymbol{G}_l^m(\nu_j)\}+B_j^{(2)}\mathrm{Im}\{\boldsymbol{G}_l^m(\nu_j)\} \tag{99c}$$

and

$$\boldsymbol{B}_l^m(2,\nu_j)=B_j^{(2)}\mathrm{Re}\{\boldsymbol{G}_l^m(\nu_j)\}-B_j^{(1)}\mathrm{Im}\{\boldsymbol{G}_l^m(\nu_j)\}. \tag{99d}$$

We note that $\boldsymbol{G}_l^m(\nu_j)$ is defined by Eq. (34). In order to complete the definitions required to establish Eqs. (92) we now write

$$\Xi(\tau,\pm\mu)=\Xi_r(\tau,\pm\mu)+\Xi_c(\tau,\pm\mu), \tag{100}$$

where

$$\Xi_r(\tau,\mu)=\sum_{l=m}^{L}\boldsymbol{P}_l^m(\mu)\boldsymbol{B}_l\sum_{j=1}^{J_r}\nu_j[X_j(\tau,\mu)+(-1)^{l-m}Y_j(\tau,\mu)\boldsymbol{D}]\boldsymbol{G}_l^m(\nu_j), \tag{101a}$$

$$\Xi_r(\tau,-\mu)=\boldsymbol{D}\sum_{l=m}^{L}\boldsymbol{P}_l^m(\mu)\boldsymbol{B}_l\sum_{j=1}^{J_r}\nu_j[W_j(\tau,\mu)+(-1)^{l-m}Z_j(\tau,\mu)\boldsymbol{D}]\boldsymbol{G}_l^m(\nu_j), \tag{101b}$$

$$\Xi_c(\tau,\mu)=\sum_{l=m}^{L}\boldsymbol{P}_l^m(\mu)\boldsymbol{B}_l\sum_{j=1}^{J_c}[\boldsymbol{X}_l^m(\nu_j,\tau,\mu)+(-1)^{l-m}\boldsymbol{D}\boldsymbol{Y}_l^m(\nu_j,\tau,\mu)] \tag{101c}$$

and

$$\Xi_c(\tau,-\mu)=\boldsymbol{D}\sum_{l=m}^{L}\boldsymbol{P}_l^m(\mu)\boldsymbol{B}_l\sum_{j=1}^{J_c}[\boldsymbol{W}_l^m(\nu_j,\tau,\mu)+(-1)^{l-m}\boldsymbol{D}\boldsymbol{Z}_l^m(\nu_j,\tau,\mu)]. \tag{101d}$$

Here, to define Eqs. (101a) and (101b) we have introduced

$$X_j(\tau,\mu) = \mu_0 a_j \left[\frac{\nu_j C(\tau:\nu_j,\mu) - \mu_0 C(\tau:\mu_0,\mu)}{\nu_j - \mu_0} \right], \tag{102a}$$

$$Y_j(\tau,\mu) = \mu_0 b_j \left[\frac{\mu_0 C(\tau:\mu_0,\mu) - \nu_j e^{-\tau_0/\mu_0} T(\tau_0,\tau:\nu_j,\mu)}{\nu_j + \mu_0} \right], \tag{102b}$$

$$Z_j(\tau,\mu) = \mu_0 a_j \left[\frac{\nu_j T(\tau_0,\tau_0-\tau:\nu_j,\mu) - \mu_0 T(\tau_0,\tau_0-\tau:\mu_0,\mu)}{\nu_j - \mu_0} \right] \tag{102c}$$

and

$$W_j(\tau,\mu) = \mu_0 b_j \left[\frac{\mu_0 T(\tau_0,\tau_0-\tau:\mu_0,\mu) - \nu_j e^{-\tau_0/\mu_0} C(\tau_0-\tau:\nu_j,\mu)}{\nu_j + \mu_0} \right] \tag{102d}$$

with the $\{a_j\}$ and $\{b_j\}$ as defined by Eqs. (79). Finally to complete Eqs. (101c) and (101d) we have used

$$\boldsymbol{X}_l^m(\nu_j,\tau,\mu) = 2[\mathrm{Re}\{\nu_j X_j(\tau,\mu)\}\mathrm{Re}\{\boldsymbol{G}_l^m(\nu_j)\} - \mathrm{Im}\{\nu_j X_j(\tau,\mu)\}\mathrm{Im}\{\boldsymbol{G}_l^m(\nu_j)\}], \tag{103a}$$

$$\boldsymbol{Y}_l^m(\nu_j,\tau,\mu) = 2[\mathrm{Re}\{\nu_j Y_j(\tau,\mu)\}\mathrm{Re}\{\boldsymbol{G}_l^m(\nu_j)\} - \mathrm{Im}\{\nu_j Y_j(\tau,\mu)\}\mathrm{Im}\{\boldsymbol{G}_l^m(\nu_j)\}], \tag{103b}$$

$$\boldsymbol{Z}_l^m(\nu_j,\tau,\mu) = 2[\mathrm{Re}\{\nu_j Z_j(\tau,\mu)\}\mathrm{Re}\{\boldsymbol{G}_l^m(\nu_j)\} - \mathrm{Im}\{\nu_j Z_j(\tau,\mu)\}\mathrm{Im}\{\boldsymbol{G}_l^m(\nu_j)\}] \tag{104a}$$

and

$$\boldsymbol{W}_l^m(\nu_j,\tau,\mu) = 2[\mathrm{Re}\{\nu_j W_j(\tau,\mu)\}\mathrm{Re}\{\boldsymbol{G}_l^m(\nu_j)\} - \mathrm{Im}\{\nu_j W_j(\tau,\mu)\}\mathrm{Im}\{\boldsymbol{G}_l^m(\nu_j)\}]. \tag{104b}$$

To conclude this section we note that while in the beginning of the development of our solution we have used a numerical quadrature scheme to approximate certain integrals, our final results for the four Stokes parameters are continuous in all three variables (τ,μ,ϕ).

7. Computational details and numerical results

Of course the first thing we must do in order to evaluate our discrete-ordinates solution numerically is to define a quadrature scheme. In that regard, we consider it important to note that our discrete-ordinates solution is essentially independent of the quadrature scheme to be used. The only two restrictions we have imposed are that the N quadrature points $\{\mu_k\}$ and the N weights $\{w_k\}$ must be defined for use on the integration interval $[0,1]$ and, because of the way our basic eigenvalue problem is formulated, that we must exclude zero from the set of quadrature points. While our experience with the discrete-ordinates method for the general polarization model considered here is essentially zero, we have made some observations from the scalar theory that have guided us here. First of all, in a recent work [9] concerning the equivalence between the spherical-harmonics method and the classical discrete-ordinates method that uses a quadrature scheme defined for use on the integration interval $[-1, 1]$, we confirmed that the weights and nodes defined by the zeros of the associated Legendre functions $P^m_{m+2N}(\mu)$ were a natural choice for a "full-range" quadrature scheme. We therefore can suggest that a "half-range" quadrature scheme defined in terms of the "weight function" $(1-\mu^2)^m$ on the integration interval [0,1] seems the natural choice [10,11] for applications based on the scalar model of the equation of transfer. As reported by Chalhoub and Garcia in Refs. [10,11], this quadrature scheme has been used to good effect in radiative-transfer calculations, and so the extension of this scheme to the case of polarization is something that deserves consideration. On the other hand, we have seen [12] a case where the inclusion in the boundary data of a "step function" was well solved by subdividing the integration interval $[0, 1]$ so as to have a "break point" that coincided with the rise in the step-function boundary data. And so we consider there to be some merit in using a simple integration scheme that can easily be mapped onto the integration interval [0,1] or various subintervals of that basic interval. While we intend to investigate (in future work) the effectiveness of other integration schemes, in this work we follow a simple approach: we start with the usual Gauss–Legendre scheme (of order N) defined by the zeros of the Legendre polynomial $P_N(\mu)$ for use on the integration interval $[-1, 1]$, and then we map (linearly) this scheme into a scheme defined for use on the interval [0,1].

Having defined our quadrature scheme, our next computational job is to compute the separation constants $\{\nu_j\}$ and both the direct and the adjoint eigenvectors. Considering Eq. (39a) to define the basic eigenvalue problem to be solved, we clearly can express the direct vectors $\mathbf{\Phi}_{\pm}(\nu_j)$ in terms of the eigenvectors of the matrix $\boldsymbol{FE}$. Not surprisingly, we can express the adjoint vectors $\mathbf{\Psi}_{\pm}(\nu_j)$ in terms of the left eigenvectors of $\boldsymbol{FE}$. And so we have used the subroutine DGEEV from the LAPACK collection [13] to compute the eigenvalues and both the left and right eigenvectors of $\boldsymbol{FE}$. With the separation constants $\{\nu_j\}$ and the direct and adjoint vectors available, we have used the subroutines DGETRF and DGETRS, also from the LAPACK package, to find the required A and B constants from the linear system defined by Eqs. (84). Therefore, we consider our solution established.

In order to test our FORTRAN implementation of the discrete-ordinates solution developed in this work, we first looked at the numerical results for the two test problems we have already solved, initially with a method based on generalized spherical harmonics [14] and then with the F_N method [6]. The first of these two test problems [6,14] is defined by a set of "Greek constants" with $L = 13$ and the second with $L = 60$. In Ref. [14] the numerical results for the four Stokes parameters were reported for the $L = 13$ problem and the $L = 60$ problem with five figures of accuracy, and in Ref. [6] results for the same problems were reported with six figures of accuracy. In evaluating our implementation of the discrete-ordinates solution developed here, we found with $N = 30, 40, \ldots, 90$ that we could confirm with five figures of agreement all the results given in Refs. [6,14], and, as a matter of fact, all of our results for the Stokes parameters I, Q and U and, with only a few exceptions, all of our results for the Stokes parameter V agreed with six figures of accuracy with the results of Ref. [6]. Continuing with an evaluation of our solution, we considered next the three basic problems that Wauben and Hovenier [15] used to test their solution, based on the "adding/doubling" method [16], for the general class of polarization problems considered here. Wauben and Hovenier's results were reported with five figures of accuracy, and so, again with $N = 30, 40, \ldots, 90$, we found essentially perfect agreement with previously reported, highly accurate solutions.

In the process of evaluating the numerical results obtained from our FORTRAN implementation of the discrete-ordinates solution developed here, we saw again a situation that deserves comment. For polarization, in contrast to the scalar theory where only the intensity is sought, three of the Stokes parameters, Q, U and V, can be positive, negative and, in fact, zero. And so there clearly will be certain values of the independent variables for which a numerical computation (carried out on a machine with a finite word length) such as the one reported here (and elsewhere [6,14,15]) can yield values for which none of the significant figures are correct. While we may think of these special cases as truly exceptional, we must at the same time not take all suggestions of achieved accuracy to be definitive statements.

Naturally we would now like to report some numerical results, but in order not to be excessive in our tabulations we have chosen to focus our attention of the second of the three test problems introduced by Wauben and Hovenier [15]. Quoting from Ref. [15], we note that this problem is for scattering in an atmosphere of randomly oriented oblate spheroids with aspect ratio 1.999987, size parameter 3 and index of refraction $1.53 - 0.006i$. We note that the Greek constants for this $L = 11$ problem have been accurately computed and reported by Kuik, de Haan and Hovenier [17]. However, to be complete we choose to list in Table 1 these defining Greek constants in the notation used in this work. Continuing, we follow Ref. [15] and consider the case of $\tau_0 = 1$, $\varpi = 0.973527$ and $\mu_0 = 0.6$. While from Eqs. (13a) and (15) it is clear that the solution we have developed is valid even for a beam of incident light that is already polarized we consider, in regard to our tabulations of results, that the incident beam is defined by having $F_I = 1$ as the only nonzero component of $\boldsymbol{F}$.

Hence, we list in Tables 2–9 our results for the four Stokes parameters. We note that in listing our results with six digits, we are of the opinion that essentially all of these results are correct to within one unit in the last place. While we have no definitive proof that all six digits of our results are correct we have done two things to establish the confidence we do have. As mentioned, our results agree with the five-figure results of Wauben and Hovenier [15] and we have seen what we consider to be a reasonable degree of stability in the results as we varied the number of quadrature points N from 30 to 90, say.

Table 1
The Greek constants

l	α_l	β_l	γ_l	δ_l	ϵ_l	ζ_l
0	0.0	1.0	0.0	0.915207	0.0	0.0
1	0.0	2.104031	0.0	2.095727	0.0	0.0
2	3.726079	2.095158	−0.116688	2.008624	0.065456	3.615946
3	2.202868	1.414939	−0.209370	1.436545	0.221658	2.240516
4	1.190694	0.703593	−0.227137	0.706244	0.097752	1.139473
5	0.391203	0.235001	−0.144524	0.238475	0.052458	0.365605
6	0.105556	0.064039	−0.052640	0.056448	0.009239	0.082779
7	0.020484	0.012837	−0.012400	0.009703	0.001411	0.013649
8	0.003097	0.002010	−0.002093	0.001267	0.000133	0.001721
9	0.000366	0.000246	−0.000267	0.000130	0.000011	0.000172
10	0.000035	0.000024	−0.000027	0.000011	0.000001	0.000014
11	0.000003	0.000002	−0.000002	0.000001	0.000000	0.000001

Table 2
The Stokes parameter $I_*(\eta\tau_0, \mu, \phi)$ for $\tau_0 = 1.0$, $\varpi = 0.973527$, $\mu_0 = 0.6$ and $\phi - \phi_0 = 0$

μ	$\eta = 0$	$\eta = 0.125$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 0.875$	$\eta = 1$
−1.0	5.06873(−2)	4.26589(−2)	3.45653(−2)	1.97273(−2)	7.87443(−3)	3.36768(−3)	
−0.9	9.31376(−2)	7.94053(−2)	6.49881(−2)	3.77006(−2)	1.51276(−2)	6.39107(−3)	
−0.8	1.34207(−1)	1.16434(−1)	9.67628(−2)	5.79246(−2)	2.40896(−2)	1.03704(−2)	
−0.7	1.85879(−1)	1.64182(−1)	1.38667(−1)	8.59562(−2)	3.73760(−2)	1.65616(−2)	
−0.6	2.52631(−1)	2.27083(−1)	1.94938(−1)	1.25328(−1)	5.72304(−2)	2.62245(−2)	
−0.5	3.39136(−1)	3.10078(−1)	2.70587(−1)	1.80764(−1)	8.70580(−2)	4.13932(−2)	
−0.4	4.50034(−1)	4.18565(−1)	3.71549(−1)	2.58872(−1)	1.32485(−1)	6.57126(−2)	
−0.3	5.88275(−1)	5.57063(−1)	5.03716(−1)	3.68551(−1)	2.03527(−1)	1.06556(−1)	
−0.2	7.51295(−1)	7.25362(−1)	6.68926(−1)	5.18889(−1)	3.19032(−1)	1.81520(−1)	
−0.1	9.26796(−1)	9.14221(−1)	8.57350(−1)	7.03838(−1)	5.07724(−1)	3.42394(−1)	
−0.0	1.06308	1.11180	1.05977	8.93477(−1)	7.09516(−1)	6.13083(−1)	
0.0		1.11180	1.05977	8.93477(−1)	7.09516(−1)	6.13083(−1)	4.87453(−1)
0.1		9.19330(−1)	1.15086	1.08622	8.92798(−1)	7.90299(−1)	6.79443(−1)
0.2		6.82308(−1)	1.01841	1.16823	1.04785	9.55108(−1)	8.50181(−1)
0.3		5.57642(−1)	8.96182(−1)	1.15474	1.12495	1.06016	9.75609(−1)
0.4		4.76481(−1)	7.96034(−1)	1.10009	1.13783	1.10186	1.04220
0.5		4.12231(−1)	7.05067(−1)	1.02038	1.10231	1.09016	1.05364
0.6		3.52991(−1)	6.13447(−1)	9.17822(−1)	1.02520	1.03104	1.01398
0.7		2.92560(−1)	5.14665(−1)	7.90758(−1)	9.08171(−1)	9.26470(−1)	9.24776(−1)
0.8		2.27271(−1)	4.04229(−1)	6.36433(−1)	7.50251(−1)	7.75782(−1)	7.85298(−1)
0.9		1.53983(−1)	2.77491(−1)	4.49583(−1)	5.46313(−1)	5.73783(−1)	5.90163(−1)
1.0		4.54880(−2)	8.60062(−2)	1.53099(−1)	2.03658(−1)	2.23429(−1)	2.39759(−1)

Table 3
The Stokes parameter $Q(\eta\tau_0, \mu, \phi)$ for $\tau_0 = 1.0$, $\varpi = 0.973527$, $\mu_0 = 0.6$ and $\phi - \phi_0 = 0$

μ	$\eta = 0$	$\eta = 0.125$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 0.875$	$\eta = 1$
−1.0	−2.62388(−3)	−2.11088(−3)	−1.63442(−3)	−8.51291(−4)	−3.07300(−4)	−1.23574(−4)	
−0.9	−6.21108(−3)	−5.22959(−3)	−4.19045(−3)	−2.30412(−3)	−8.42624(−4)	−3.19935(−4)	
−0.8	−9.17777(−3)	−7.96328(−3)	−6.54890(−3)	−3.82480(−3)	−1.54351(−3)	−6.41954(−4)	
−0.7	−1.35127(−2)	−1.18673(−2)	−9.90044(−3)	−6.03325(−3)	−2.63786(−3)	−1.18363(−3)	
−0.6	−1.97780(−2)	−1.73519(−2)	−1.45497(−2)	−9.10013(−3)	−4.21295(−3)	−1.99352(−3)	
−0.5	−2.82242(−2)	−2.45834(−2)	−2.06154(−2)	−1.31086(−2)	−6.33962(−3)	−3.12183(−3)	
−0.4	−3.87698(−2)	−3.34643(−2)	−2.80114(−2)	−1.80457(−2)	−9.08595(−3)	−4.63829(−3)	
−0.3	−5.09421(−2)	−4.35707(−2)	−3.63804(−2)	−2.37574(−2)	−1.25383(−2)	−6.67691(−3)	
−0.2	−6.38561(−2)	−5.40974(−2)	−4.49935(−2)	−2.98046(−2)	−1.68197(−2)	−9.57142(−3)	
−0.1	−7.64043(−2)	−6.40333(−2)	−5.27985(−2)	−3.50601(−2)	−2.17787(−2)	−1.42900(−2)	
−0.0	−8.68152(−2)	−7.27597(−2)	−5.93089(−2)	−3.84013(−2)	−2.42138(−2)	−1.90606(−2)	
0.0		−7.27597(−2)	−5.93089(−2)	−3.84013(−2)	−2.42138(−2)	−1.90606(−2)	−1.53052(−2)
0.1		−5.15009(−2)	−5.61917(−2)	−3.98378(−2)	−2.44346(−2)	−1.86760(−2)	−1.41112(−2)
0.2		−2.82499(−2)	−3.73359(−2)	−3.28191(−2)	−2.15191(−2)	−1.63539(−2)	−1.19181(−2)
0.3		−1.52587(−2)	−2.16678(−2)	−2.10324(−2)	−1.42534(−2)	−1.06188(−2)	−7.23430(−3)
0.4		−7.39748(−3)	−1.07231(−2)	−1.03064(−2)	−6.11829(−3)	−3.74253(−3)	−1.42386(−3)
0.5		−2.96196(−3)	−4.18485(−3)	−3.21080(−3)	−2.90669(−4)	1.32304(−3)	2.92812(−3)
0.6		−1.31489(−3)	−1.82488(−3)	−9.20302(−4)	1.22004(−3)	2.41834(−3)	3.64474(−3)
0.7		−2.04401(−3)	−3.29648(−3)	−3.72131(−3)	−2.48701(−3)	−1.57050(−3)	−5.32187(−4)
0.8		−4.55546(−3)	−7.75632(−3)	−1.08255(−2)	−1.09647(−2)	−1.03845(−2)	−9.51833(−3)
0.9		−7.65293(−3)	−1.32406(−2)	−1.95935(−2)	−2.15730(−2)	−2.15180(−2)	−2.09886(−2)
1.0		−6.29411(−3)	−1.11544(−2)	−1.75869(−2)	−2.08984(−2)	−2.17389(−2)	−2.21768(−2)

Table 4
The Stokes parameter $I(\eta\tau_0, \mu, \phi)$ for $\tau_0 = 1.0$, $\varpi = 0.973527$, $\mu_0 = 0.6$ and $\phi - \phi_0 = \pi/2$

μ	$\eta = 0$	$\eta = 0.125$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 0.875$	$\eta = 1$
−1.0	5.06873(−2)	4.26589(−2)	3.45653(−2)	1.97273(−2)	7.87443(−3)	3.36768(−3)	
−0.9	6.19358(−2)	5.29317(−2)	4.34998(−2)	2.54886(−2)	1.03835(−2)	4.45134(−3)	
−0.8	7.50297(−2)	6.52843(−2)	5.45545(−2)	3.30394(−2)	1.39113(−2)	6.04941(−3)	
−0.7	8.99675(−2)	7.98953(−2)	6.80382(−2)	4.28129(−2)	1.88025(−2)	8.36589(−3)	
−0.6	1.06628(−1)	9.69057(−2)	8.42961(−2)	5.53915(−2)	2.55668(−2)	1.17151(−2)	
−0.5	1.24626(−1)	1.16321(−1)	1.03663(−1)	7.15698(−2)	3.50052(−2)	1.66194(−2)	
−0.4	1.42981(−1)	1.37756(−1)	1.26291(−1)	9.24119(−2)	4.84616(−2)	2.40304(−2)	
−0.3	1.59427(−1)	1.59789(−1)	1.51602(−1)	1.19147(−1)	6.83470(−2)	3.59013(−2)	
−0.2	1.69216(−1)	1.78600(−1)	1.76763(−1)	1.52103(−1)	9.91433(−2)	5.69394(−2)	
−0.1	1.64533(−1)	1.86973(−1)	1.94416(−1)	1.85519(−1)	1.46773(−1)	1.00973(−1)	
−0.0	1.29793(−1)	1.77920(−1)	1.97733(−1)	2.05728(−1)	1.88061(−1)	1.70182(−1)	
0.0		1.77920(−1)	1.97733(−1)	2.05728(−1)	1.88061(−1)	1.70182(−1)	1.35883(−1)
0.1		1.16227(−1)	1.72145(−1)	2.11318(−1)	2.09519(−1)	1.98866(−1)	1.79368(−1)
0.2		7.48873(−2)	1.31715(−1)	1.96362(−1)	2.16153(−1)	2.14139(−1)	2.03778(−1)
0.3		5.55501(−2)	1.05304(−1)	1.76048(−1)	2.10875(−1)	2.16833(−1)	2.14906(−1)
0.4		4.53023(−2)	8.90714(−2)	1.59092(−1)	2.01891(−1)	2.13409(−1)	2.17953(−1)
0.5		3.97668(−2)	7.93758(−2)	1.46961(−1)	1.93582(−1)	2.08614(−1)	2.17679(−1)
0.6		3.71787(−2)	7.42707(−2)	1.39530(−1)	1.87810(−1)	2.04965(−1)	2.17049(−1)
0.7		3.67356(−2)	7.27487(−2)	1.36510(−1)	1.85440(−1)	2.03829(−1)	2.17862(−1)
0.8		3.80593(−2)	7.43069(−2)	1.37739(−1)	1.86992(−1)	2.06079(−1)	2.21317(−1)
0.9		4.09882(−2)	7.87342(−2)	1.43218(−1)	1.92910(−1)	2.12398(−1)	2.28341(−1)
1.0		4.54880(−2)	8.60062(−2)	1.53099(−1)	2.03658(−1)	2.23429(−1)	2.39759(−1)

Table 5
The Stokes parameter $Q(\eta\tau_0, \mu, \phi)$ for $\tau_0 = 1.0$, $\varpi = 0.973527$, $\mu_0 = 0.6$ and $\phi - \phi_0 = \pi/2$

μ	$\eta = 0$	$\eta = 0.125$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 0.875$	$\eta = 1$
−1.0	2.62388(−3)	2.11088(−3)	1.63442(−3)	8.51291(−4)	3.07300(−4)	1.23574(−4)	
−0.9	2.78623(−3)	2.21987(−3)	1.69697(−3)	8.46122(−4)	2.80937(−4)	1.06325(−4)	
−0.8	3.11982(−3)	2.48963(−3)	1.89812(−3)	9.19572(−4)	2.73592(−4)	8.91437(−5)	
−0.7	3.63237(−3)	2.94652(−3)	2.27523(−3)	1.11479(−3)	3.14678(−4)	8.79028(−5)	
−0.6	4.31252(−3)	3.60399(−3)	2.85874(−3)	1.47720(−3)	4.39137(−4)	1.22453(−4)	
−0.5	5.12123(−3)	4.45680(−3)	3.66908(−3)	2.05686(−3)	6.92072(−4)	2.19892(−4)	
−0.4	5.97337(−3)	5.46694(−3)	4.70772(−3)	2.91122(−3)	1.13897(−3)	4.22613(−4)	
−0.3	6.69791(−3)	6.52916(−3)	5.92994(−3)	4.10251(−3)	1.88848(−3)	8.09959(−4)	
−0.2	6.96260(−3)	7.39290(−3)	7.16647(−3)	5.65675(−3)	3.14128(−3)	1.56643(−3)	
−0.1	6.20014(−3)	7.56684(−3)	7.97278(−3)	7.33260(−3)	5.21027(−3)	3.24346(−3)	
−0.0	3.29274(−3)	6.44585(−3)	7.76800(−3)	8.34815(−3)	7.33298(−3)	6.30252(−3)	
0.0		6.44585(−3)	7.76800(−3)	8.34815(−3)	7.33298(−3)	6.30252(−3)	4.24678(−3)
0.1		3.44085(−3)	5.97680(−3)	8.31441(−3)	8.42086(−3)	7.90324(−3)	6.88282(−3)
0.2		1.95236(−3)	4.13453(−3)	7.29816(−3)	8.56175(−3)	8.58558(−3)	8.17683(−3)
0.3		1.39770(−3)	3.17155(−3)	6.31946(−3)	8.19829(−3)	8.62820(−3)	8.69933(−3)
0.4		1.27191(−3)	2.83084(−3)	5.79826(−3)	7.89813(−3)	8.55063(−3)	8.92357(−3)
0.5		1.42944(−3)	2.97177(−3)	5.83844(−3)	7.97980(−3)	8.72477(−3)	9.25053(−3)
0.6		1.83087(−3)	3.55401(−3)	6.50354(−3)	8.64499(−3)	9.40786(−3)	9.98214(−3)
0.7		2.48000(−3)	4.59362(−3)	7.87638(−3)	1.00680(−2)	1.08174(−2)	1.13758(−2)
0.8		3.40531(−3)	6.14451(−3)	1.00722(−2)	1.24342(−2)	1.31729(−2)	1.36843(−2)
0.9		4.65455(−3)	8.29234(−3)	1.32437(−2)	1.59585(−2)	1.67186(−2)	1.71810(−2)
1.0		6.29411(−3)	1.11544(−2)	1.75869(−2)	2.08984(−2)	2.17389(−2)	2.21768(−2)

Table 6
The Stokes parameter $U(\eta\tau_0, \mu, \phi)$ for $\tau_0 = 1.0$, $\varpi = 0.973527$, $\mu_0 = 0.6$ and $\phi - \phi_0 = \pi/2$

μ	$\eta = 0$	$\eta = 0.125$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 0.875$	$\eta = 1$
−1.0	0.0	0.0	0.0	0.0	0.0	0.0	
−0.9	−2.70682(−3)	−2.32098(−3)	−1.90171(−3)	−1.09314(−3)	−4.19776(−4)	−1.66426(−4)	
−0.8	−4.21321(−3)	−3.71468(−3)	−3.12021(−3)	−1.88805(−3)	−7.74963(−4)	−3.22767(−4)	
−0.7	−5.58993(−3)	−5.08235(−3)	−4.38449(−3)	−2.79790(−3)	−1.22661(−3)	−5.35536(−4)	
−0.6	−6.87877(−3)	−6.47620(−3)	−5.75341(−3)	−3.88473(−3)	−1.82101(−3)	−8.32433(−4)	
−0.5	−8.04140(−3)	−7.88798(−3)	−7.24489(−3)	−5.20505(−3)	−2.61968(−3)	−1.25494(−3)	
−0.4	−8.97454(−3)	−9.26130(−3)	−8.84683(−3)	−6.82571(−3)	−3.72289(−3)	−1.87750(−3)	
−0.3	−9.46757(−3)	−1.04490(−2)	−1.04781(−2)	−8.81412(−3)	−5.30923(−3)	−2.85284(−3)	
−0.2	−9.12219(−3)	−1.11151(−2)	−1.18724(−2)	−1.11459(−2)	−7.70550(−3)	−4.54879(−3)	
−0.1	−7.33915(−3)	−1.06850(−2)	−1.24261(−2)	−1.32930(−2)	−1.12967(−2)	−8.03633(−3)	
−0.0	−3.05581(−3)	−8.66077(−3)	−1.16471(−2)	−1.41899(−2)	−1.40600(−2)	−1.32138(−2)	
0.0		−8.66077(−3)	−1.16471(−2)	−1.41899(−2)	−1.40600(−2)	−1.32138(−2)	−1.11765(−2)
0.1		−4.98561(−3)	−9.20740(−3)	−1.38774(−2)	−1.51134(−2)	−1.48596(−2)	−1.39511(−2)
0.2		−3.50673(−3)	−7.20511(−3)	−1.26561(−2)	−1.51698(−2)	−1.55119(−2)	−1.52561(−2)
0.3		−3.09382(−3)	−6.35554(−3)	−1.16892(−2)	−1.47890(−2)	−1.55381(−2)	−1.57622(−2)
0.4		−3.10519(−3)	−6.16966(−3)	−1.12480(−2)	−1.44912(−2)	−1.54348(−2)	−1.59356(−2)
0.5		−3.31725(−3)	−6.35008(−3)	−1.12245(−2)	−1.44072(−2)	−1.54050(−2)	−1.60283(−2)
0.6		−3.60514(−3)	−6.69341(−3)	−1.14251(−2)	−1.44548(−2)	−1.54238(−2)	−1.60673(−2)
0.7		−3.85298(−3)	−6.99735(−3)	−1.15834(−2)	−1.43989(−2)	−1.52870(−2)	−1.58844(−2)
0.8		−3.89731(−3)	−6.97343(−3)	−1.12723(−2)	−1.37808(−2)	−1.45445(−2)	−1.50490(−2)
0.9		−3.40711(−3)	−6.03854(−3)	−9.59532(−3)	−1.15695(−2)	−1.21429(−2)	−1.25068(−2)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 7
The Stokes parameter $V(\eta\tau_0, \mu, \phi)$ for $\tau_0 = 1.0$, $\varpi = 0.973527$, $\mu_0 = 0.6$ and $\phi - \phi_0 = \pi/2$

μ	$\eta = 0$	$\eta = 0.125$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 0.875$	$\eta = 1$
−1.0	0.0	0.0	0.0	0.0	0.0	0.0	
−0.9	−8.99395(−5)	−9.45319(−5)	−8.87138(−5)	−6.20642(−5)	−2.77337(−5)	−1.15977(−5)	
−0.8	−8.56577(−5)	−9.97471(−5)	−1.00258(−4)	−7.81599(−5)	−3.93004(−5)	−1.80316(−5)	
−0.7	−5.14919(−5)	−7.67690(−5)	−8.77224(−5)	−8.04354(−5)	−4.64895(−5)	−2.33521(−5)	
−0.6	3.82341(−6)	−3.26279(−5)	−5.63656(−5)	−7.10939(−5)	−4.96274(−5)	−2.75105(−5)	
−0.5	7.49254(−5)	2.90766(−5)	−8.32986(−6)	−5.03104(−5)	−4.82908(−5)	−3.02415(−5)	
−0.4	1.56395(−4)	1.05212(−4)	5.49784(−5)	−1.73843(−5)	−4.15905(−5)	−3.11575(−5)	
−0.3	2.40494(−4)	1.91489(−4)	1.31807(−4)	2.92045(−5)	−2.78175(−5)	−2.96131(−5)	
−0.2	3.13066(−4)	2.79500(−4)	2.17907(−4)	9.17183(−5)	−3.12603(−6)	−2.40852(−5)	
−0.1	3.44228(−4)	3.50035(−4)	3.00093(−4)	1.69548(−4)	4.27337(−5)	−8.65908(−6)	
−0.0	2.33318(−4)	3.63690(−4)	3.48805(−4)	2.40812(−4)	1.18320(−4)	5.31552(−5)	
0.0		3.63690(−4)	3.48805(−4)	2.40812(−4)	1.18320(−4)	5.31552(−5)	−4.97495(−5)
0.1		2.11540(−4)	2.95754(−4)	2.70101(−4)	1.73798(−4)	1.20241(−4)	5.52843(−5)
0.2		1.08690(−4)	1.87000(−4)	2.25666(−4)	1.80882(−4)	1.46077(−4)	1.02806(−4)
0.3		5.61112(−5)	1.07077(−4)	1.52068(−4)	1.42044(−4)	1.26207(−4)	1.04415(−4)
0.4		2.46521(−5)	5.04435(−5)	7.93694(−5)	8.25373(−5)	7.89324(−5)	7.32669(−5)
0.5		4.58957(−6)	1.00384(−5)	1.63738(−5)	1.90666(−5)	2.11706(−5)	2.52917(−5)
0.6		−8.02145(−6)	−1.79104(−5)	−3.38318(−5)	−3.87511(−5)	−3.56385(−5)	−2.71035(−5)
0.7		−1.50244(−5)	−3.52052(−5)	−6.92387(−5)	−8.42061(−5)	−8.29678(−5)	−7.40412(−5)
0.8		−1.72809(−5)	−4.23271(−5)	−8.71848(−5)	−1.10720(−4)	−1.12567(−4)	−1.05920(−4)
0.9		−1.48020(−5)	−3.79271(−5)	−8.17611(−5)	−1.07733(−4)	−1.12021(−4)	−1.08944(−4)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 8
The Stokes parameter $I(\eta\tau_0, \mu, \phi)$ for $\tau_0 = 1.0$, $\varpi = 0.973527$, $\mu_0 = 0.6$ and $\phi - \phi_0 = \pi$

μ	$\eta = 0$	$\eta = 0.125$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 0.875$	$\eta = 1$
−1.0	5.06873(−2)	4.26589(−2)	3.45653(−2)	1.97273(−2)	7.87443(−3)	3.36768(−3)	
−0.9	4.49363(−2)	3.83950(−2)	3.16314(−2)	1.87386(−2)	7.81148(−3)	3.42290(−3)	
−0.8	4.95588(−2)	4.29605(−2)	3.59227(−2)	2.19649(−2)	9.46818(−3)	4.21487(−3)	
−0.7	5.54912(−2)	4.89254(−2)	4.16034(−2)	2.63509(−2)	1.18019(−2)	5.35782(−3)	
−0.6	6.19199(−2)	5.57088(−2)	4.83056(−2)	3.18639(−2)	1.49295(−2)	6.94693(−3)	
−0.5	6.84106(−2)	6.30655(−2)	5.59609(−2)	3.87231(−2)	1.91563(−2)	9.19467(−3)	
−0.4	7.44304(−2)	7.06903(−2)	6.44950(−2)	4.72940(−2)	2.50375(−2)	1.25100(−2)	
−0.3	7.89824(−2)	7.78699(−2)	7.35195(−2)	5.79874(−2)	3.35858(−2)	1.77429(−2)	
−0.2	8.01523(−2)	8.29108(−2)	8.16526(−2)	7.07287(−2)	4.66688(−2)	2.69450(−2)	
−0.1	7.51771(−2)	8.29356(−2)	8.56728(−2)	8.26216(−2)	6.65726(−2)	4.61143(−2)	
−0.0	5.93784(−2)	7.61084(−2)	8.33481(−2)	8.76235(−2)	8.22105(−2)	7.53201(−2)	
0.0		7.61084(−2)	8.33481(−2)	8.76235(−2)	8.22105(−2)	7.53201(−2)	6.04997(−2)
0.1		4.81348(−2)	7.00090(−2)	8.63151(−2)	8.80624(−2)	8.49382(−2)	7.76334(−2)
0.2		2.95259(−2)	5.13545(−2)	7.72739(−2)	8.77079(−2)	8.84673(−2)	8.55910(−2)
0.3		2.07108(−2)	3.91681(−2)	6.67897(−2)	8.29734(−2)	8.70780(−2)	8.79923(−2)
0.4		1.58301(−2)	3.14343(−2)	5.81591(−2)	7.72710(−2)	8.36674(−2)	8.74252(−2)
0.5		1.28841(−2)	2.64106(−2)	5.17402(−2)	7.22956(−2)	8.01998(−2)	8.60001(−2)
0.6		1.10828(−2)	2.32179(−2)	4.74187(−2)	6.88413(−2)	7.78133(−2)	8.51326(−2)
0.7		1.01614(−2)	2.15833(−2)	4.53652(−2)	6.77033(−2)	7.75917(−2)	8.61683(−2)
0.8		1.03325(−2)	2.19948(−2)	4.67328(−2)	7.07013(−2)	8.16497(−2)	9.14855(−2)
0.9		1.31130(−2)	2.72720(−2)	5.64093(−2)	8.41720(−2)	9.68474(−2)	1.08352(−1)
1.0		4.54880(−2)	8.60062(−2)	1.53099(−1)	2.03658(−1)	2.23429(−1)	2.39759(−1)

Table 9
The Stokes parameter $Q(\eta\tau_0, \mu, \phi)$ for $\tau_0 = 1.0$, $\varpi = 0.973527$, $\mu_0 = 0.6$ and $\phi - \phi_0 = \pi$

μ	$\eta = 0$	$\eta = 0.125$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 0.875$	$\eta = 1$
−1.0	−2.62388(−3)	−2.11088(−3)	−1.63442(−3)	−8.51291(−4)	−3.07300(−4)	−1.23574(−4)	
−0.9	−2.63222(−4)	−1.87243(−4)	−1.27675(−4)	−6.34884(−5)	−3.60705(−5)	−2.07293(−5)	
−0.8	4.96872(−4)	4.47223(−4)	3.76448(−4)	1.99866(−4)	4.91073(−5)	9.03753(−6)	
−0.7	1.06032(−3)	9.27390(−4)	7.62051(−4)	3.97628(−4)	1.01516(−4)	1.99151(−5)	
−0.6	1.54267(−3)	1.35038(−3)	1.10854(−3)	5.76842(−4)	1.40667(−4)	2.02162(−5)	
−0.5	1.96215(−3)	1.73618(−3)	1.43610(−3)	7.54266(−4)	1.75017(−4)	1.33873(−5)	
−0.4	2.29534(−3)	2.07154(−3)	1.73934(−3)	9.33582(−4)	2.06991(−4)	−1.03515(−6)	
−0.3	2.48585(−3)	2.31639(−3)	1.99227(−3)	1.10897(−3)	2.33236(−4)	−2.95783(−5)	
−0.2	2.43740(−3)	2.39646(−3)	2.14181(−3)	1.26552(−3)	2.44688(−4)	−9.22060(−5)	
−0.1	1.99821(−3)	2.18280(−3)	2.08167(−3)	1.37060(−3)	2.46442(−4)	−2.42524(−4)	
−0.0	8.81090(−4)	1.50522(−3)	1.63451(−3)	1.26636(−3)	3.39180(−4)	−3.01309(−4)	
0.0		1.50522(−3)	1.63451(−3)	1.26636(−3)	3.39180(−4)	−3.01309(−4)	−1.21124(−3)
0.1		3.85693(−4)	7.12616(−4)	7.52905(−4)	1.60992(−4)	−3.09457(−4)	−9.11160(−4)
0.2		−1.32626(−4)	−6.17051(−5)	−1.79012(−5)	−3.71866(−4)	−6.96814(−4)	−1.11838(−3)
0.3		−3.29597(−4)	−4.73271(−4)	−6.65904(−4)	−1.01901(−3)	−1.28299(−3)	−1.60287(−3)
0.4		−3.68727(−4)	−6.34178(−4)	−1.07855(−3)	−1.57478(−3)	−1.86067(−3)	−2.16466(−3)
0.5		−3.05520(−4)	−6.14083(−4)	−1.26237(−3)	−1.96412(−3)	−2.32656(−3)	−2.68057(−3)
0.6		−1.82142(−4)	−4.80470(−4)	−1.28178(−3)	−2.21069(−3)	−2.68107(−3)	−3.12859(−3)
0.7		−7.25545(−5)	−3.59285(−4)	−1.30730(−3)	−2.47551(−3)	−3.06905(−3)	−3.63287(−3)
0.8		−1.52845(−4)	−5.52927(−4)	−1.77788(−3)	−3.22729(−3)	−3.94833(−3)	−4.62849(−3)
0.9		−9.22746(−4)	−1.92041(−3)	−3.95770(−3)	−5.85205(−3)	−6.69706(−3)	−7.45424(−3)
1.0		−6.29411(−3)	−1.11544(−2)	−1.75869(−2)	−2.08984(−2)	−2.17389(−2)	−2.21768(−2)

8. The conservative case

We now would like to discuss briefly the conservative case, which we have implicitly excluded from our analysis in the preceding sections of this work. While it is possible that the term conservative case can be interpreted in various ways, we use the term here to mean those values of the albedo for single scattering ϖ and the Greek constants α, β, γ, δ, ε and ζ, for a given L, which allow the separation constant ν in Eq. (27) to be unbounded. Continuing, we see from Eq. (27) that allowing ν to be unbounded implies that Eqs. (24) can be satisfied by a solution of the form

$$\boldsymbol{I}(\tau, \pm\mu_i) = \boldsymbol{\Phi}(\pm\mu_i). \tag{105}$$

So now if we substitute Eq. (105) into Eqs. (24) we find

$$\boldsymbol{\Phi}(\mu_i) = \frac{\varpi}{2}\sum_{l=m}^{L} \boldsymbol{P}_l^m(\mu_i)\boldsymbol{B}_l\boldsymbol{G}_l^m \tag{106a}$$

and

$$\boldsymbol{\Phi}(-\mu_i) = \frac{\varpi}{2}\sum_{l=m}^{L} \boldsymbol{P}_l^m(-\mu_i)\boldsymbol{B}_l\boldsymbol{G}_l^m \tag{106b}$$

for $i = 1, 2, \ldots, N$. Here

$$\boldsymbol{G}_l^m = \sum_{n=1}^{N} w_n[\boldsymbol{P}_l^m(\mu_n)\boldsymbol{\Phi}(\mu_n) + \boldsymbol{P}_l^m(-\mu_n)\boldsymbol{\Phi}(-\mu_n)]. \tag{107}$$

We now multiply Eq. (106a) by $w_i\boldsymbol{P}_\alpha^m(\mu_i)$ and Eq. (106b) by $w_i\boldsymbol{P}_\alpha^m(-\mu_i)$, sum the resulting equations over the index i and then add the results to obtain

$$[(2\alpha+1)\boldsymbol{I} - \varpi\boldsymbol{B}_\alpha]\boldsymbol{G}_\alpha^m = \boldsymbol{0} \tag{108}$$

for $\alpha = m, m+1, \ldots, L$. At this point we can rewrite Eq. (108), for $\alpha = m, m+1, \ldots, L$, as

$$\mathrm{diag}\{\boldsymbol{A}_m, \boldsymbol{A}_{m+1}, \ldots, \boldsymbol{A}_L\}\begin{bmatrix} \boldsymbol{G}_m^m \\ \boldsymbol{G}_{m+1}^m \\ \vdots \\ \boldsymbol{G}_L^m \end{bmatrix} = \boldsymbol{0}, \tag{109}$$

where

$$\boldsymbol{A}_l = (2l+1)\boldsymbol{I} - \varpi\boldsymbol{B}_l. \tag{110}$$

Clearly any combination of parameters that makes the coefficient matrix in Eq. (109) singular will yield what we have defined to be a conservative case. However, more can be said. In a fundamental work concerning the expansion of the scattering matrix in terms of generalized spherical functions, van der Mee and Hovenier [18] have shown that the matrices $\boldsymbol{A}_l$, for $l > 0$, cannot be singular for any physically meaningful case. Based on this result of van der Mee and Hovenier [18] we see that we can have a conservative case only if $m = 0$ and only if

$$(1 - \varpi\beta_0)(1 - \varpi\delta_0) = 0. \tag{111}$$

Now, since $\beta_0 = 1$, we see that $\varpi = 1$ yields the conservative case. We note also that in the (unlikely) event [18] that $\delta_0 = 1$ both factors in Eq. (111) would be zero. Finally since the $m = 0$ problem splits into two two-vector problems (an I–Q problem and a U–V problem) we clearly must, if $\varpi = 1$, modify our developed discrete-ordinates solution of the I–Q problem and, if $\delta_0 = 1$, also the U–V problem.

From an analytical and computational point of view, the problem with the conservative case is that the largest separation constant becomes infinite, and so the exponential form introduced by Eq. (27), does not always generate the two independent forms of the solution that are needed. Rather than report the (minor) modifications to our solution that are required for the conservative case, we note that these modifications have been developed and used [4] for the scalar version of this problem. Similar results can be expected for the polarization model considered here.

9. Concluding comment

In regard to the literature on the subject of seriously developed computational methods that have been shown capable of establishing, for the polarization problem considered here, the complete Stokes vector for arbitrary values of the independent variables we know only of Refs. [6,14,15]. In fact, it seems that the solution given here and the ones developed in Refs. [6,14] are the only ones that are continuous in all variables and that have been shown to work well for quite general cases.

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