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Unified solutions to classical flow problems based on the BGK model

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Abstract. A version of the discrete-ordinates method is used to solve in a unified manner some classical flow problems based on the Bhatnagar, Gross and Krook model in the theory of rarefied-gas dynamics. In particular, the thermal-creep problem and the viscous-slip (Kramers') problem are solved for the case of a semi-infinite medium, and the Poiseuille-flow problem, the Couette-flow problem and the thermal-creep problem are all solved for a wide range of the Knudsen number.

Keywords. Discrete ordinates, rarefied gas dynamics.

1. Introduction

Having used in some recent works a new variation of the discrete-ordinates method [1–3] and a method based on an expansion in terms of Hermite polynomials [4] to solve a class of basic problems based on the Bhatnagar, Gross and Krook model [5] in the general area of rarefied-gas dynamics, we would like in this work to revisit and solve in a unified manner a collection of the standard problems often studied [6,7] in regard to flow in plane-parallel media. We note first of all that the literature concerning the basic problems we solve here is very extensive, and so to keep this work to a modest length we do not attempt to review the many works already devoted to this subject. Instead we consider that a recent review article by Sharipov and Seleznev [8] and the books by Cercignani [6] and Williams [7] are available, and so we rely on these books and two basic papers by Loyalka, Petrellis and Storvick [9,10] for the background material we require here. In addition to making use of the older works mentioned, we base the definitions of the considered problems on a recent work [11] by Williams who formulated all of the problems we solve here.

2. The Problems

In this section we present a mathematical formulation of the various problems we intend to solve and we introduce the notation, taken mostly from Refs. [1] and [9–11], we use throughout this work. While our emphasis here is on solving well the considered problems, and not on the derivations of these problems from basic physics, we consider that the recent (parallel) work by Williams [11], that starts with the nonlinear Boltzmann equation and gives unified derivations of all the problems we solve here, is particularly useful for understanding the importance of these classical problems in the kinetic theory of gases.

2.1. Half-Space Thermal-Creep Problem

For the half-space thermal-creep problem, we follow Refs. [9] and [11] and so seek a solution to (what we call) the reduced BGK equation

$$\xi\frac{\partial}{\partial\tau}Y(\tau,\xi)+Y(\tau,\xi)=\int_{-\infty}^{\infty}\Psi(u)Y(\tau,u)\,\mathrm{d}u, \tag{1}$$

for $\tau\in(0,\infty)$ and $\xi\in(-\infty,\infty)$, subject to

$$\lim_{\tau\to\infty}Y(\tau,\xi)=\frac{1}{2}\mathrm{A_T} \tag{2a}$$

and the boundary condition

$$Y(0,\xi)-(1-\alpha)Y(0,-\xi)=\frac{1}{2}\alpha(\xi^2-\frac{1}{2}) \tag{2b}$$

for $\xi\in(0,\infty)$. Here and throughout this work (what we call) the characteristic function is

$$\Psi(u)=\pi^{-1/2}\exp\{-u^2\} \tag{3}$$

and $\alpha\in(0,1]$ is the accommodation coefficient. In regard to the quantities of physical interest that we wish to establish and in order to be consistent with previous work to which we wish to compare our numerical results, we follow the definitions from Ref. [9] and thus will compute the thermal-slip coefficient $\mathrm{A_T}$ and the macroscopic velocity profile

$$q_{\mathrm{T}}(\tau)=2Y_0(\tau) \tag{4}$$

where, in general,

$$Y_0(\tau)=\int_{-\infty}^{\infty}\Psi(u)Y(\tau,u)\,\mathrm{d}u. \tag{5}$$

2.2. Half-Space Viscous-Slip (Kramers') Problem

In regard to the viscous-slip problem for a half space (also known as Kramers' problem) we base our considerations again on Refs. [9] and [11], and so we wish to establish a solution to

$$\xi\frac{\partial}{\partial\tau}Y(\tau,\xi)+Y(\tau,\xi)=\int_{-\infty}^{\infty}\Psi(u)Y(\tau,u)\,\mathrm{d}u, \tag{6}$$

for $\tau\in(0,\infty)$ and $\xi\in(-\infty,\infty)$, subject to

$$\lim_{\tau\to\infty}Y(\tau,\xi)=\mathrm{A_P} \tag{7a}$$

and the boundary condition

$$Y(0,\xi)-(1-\alpha)Y(0,-\xi)=(2-\alpha)\xi \tag{7b}$$

for $\xi\in(0,\infty)$. In regard to quantities of physical interest, we again follow the definitions from Ref. [9], and so we seek to compute the viscous-slip coefficient $\mathrm{A_P}$ and the macroscopic velocity profile

$$q_{\mathrm{P}}(\tau)=\tau+Y_0(\tau)\,. \tag{8}$$

2.3. Poiseuille Flow in a Plane Channel

We note, for example from Refs. [1] and [11], that the Poiseuille-flow problem in a plane channel can be formulated in terms of the reduced BGK equation

$$\xi\frac{\partial}{\partial\tau}Y(\tau,\xi)+Y(\tau,\xi)=\int_{-\infty}^{\infty}\Psi(u)Y(\tau,u)\,\mathrm{d}u, \tag{9}$$

for $\tau\in(-a,a)$ and $\xi\in(-\infty,\infty)$, and the boundary conditions

$$Y(-a,\xi)-(1-\alpha)Y(-a,-\xi)=\alpha\xi^2+a(2-\alpha)\xi \tag{10a}$$

and

$$Y(a,-\xi)-(1-\alpha)Y(a,\xi)=\alpha\xi^2+a(2-\alpha)\xi \tag{10b}$$

for $\xi\in(0,\infty)$. Here $2a$ (the inverse Knudsen number) is the channel width (in nondimensional units). Making use of the definitions from Ref. [10], we will compute the macroscopic velocity profile

$$q_{\mathrm{P}}(\tau)=\frac{1}{2}\big(1-a^2+\tau^2\big)-Y_0(\tau) \tag{11}$$

and the flow rate

$$Q_{\mathrm{P}}=-\frac{1}{2a^2}\int_{-a}^{a}q_{\mathrm{P}}(\tau)\,\mathrm{d}\tau\,. \tag{12}$$

2.4. Couette Flow in a Plane Channel

For the problem of Couette flow, we make use of the extension [11] of the formulations used in Refs. [10] and [12] to the case where the accommodation coefficient α can have any value in the interval $(0,1]$, and so we seek a solution of

$$\xi\frac{\partial}{\partial\tau}Y(\tau,\xi)+Y(\tau,\xi)=\int_{-\infty}^{\infty}\Psi(u)Y(\tau,u)\,\mathrm{d}u, \tag{13}$$

for $\tau\in(-a,a)$ and $\xi\in(-\infty,\infty)$, subject to the boundary conditions [11]

$$Y(-a,\xi)-(1-\alpha)Y(-a,-\xi)=\alpha \tag{14a}$$

and

$$Y(a,-\xi)-(1-\alpha)Y(a,\xi)=-\alpha \tag{14b}$$

for $\xi\in(0,\infty)$. Again $2a$ is the inverse Knudsen number, and for this problem we wish to compute the (constant) normalized stress [11]

$$\mathrm{P}_{xz}=\pi^{1/2}\int_{-\infty}^{\infty}\Psi(u)Y(\tau,u)u\,\mathrm{d}u\,. \tag{15}$$

2.5. Thermal Creep in a Plane Channel

Again we refer to Refs. [10] and [11] and consider the reduced BGK equation

$$\xi\frac{\partial}{\partial\tau}Y(\tau,\xi)+Y(\tau,\xi)=\int_{-\infty}^{\infty}\Psi(u)Y(\tau,u)\,\mathrm{d}u, \tag{16}$$

for $\tau\in(-a,a)$ and $\xi\in(-\infty,\infty)$, and the boundary conditions

$$Y(-a,\xi)-(1-\alpha)Y(-a,-\xi)=\frac{1}{2}\alpha(\xi^2-\frac{1}{2}) \tag{17a}$$

and

$$Y(a,-\xi)-(1-\alpha)Y(a,\xi)=\frac{1}{2}\alpha(\xi^2-\frac{1}{2}) \tag{17b}$$

for $\xi\in(0,\infty)$. As for the problem of Poiseuille flow, we again follow the definitions from Ref. [10] and thus will compute the macroscopic velocity profile

$$q_{\mathrm{T}}(\tau)=Y_0(\tau) \tag{18}$$

and the flow rate

$$Q_{\mathrm{T}}=-\frac{1}{2a^2}\int_{-a}^{a}q_{\mathrm{T}}(\tau)\,\mathrm{d}\tau\,. \tag{19}$$

Having defined the five basic problems we intend to solve in this work, we now formulate the variation of the discrete-ordinates method that we use to solve these problems.

3. A Discrete-Ordinates Solution

The variation of the discrete-ordinates method [13–15] we use in this work was developed in Refs. [1] and [16], and so we can make use of that material now to solve the half-space problems and the problems for finite channels that were formulated in Section 2. We thus approximate the integral term in Eq. (1) by a quadrature formula and write our discrete-ordinates equations as

$$\xi_i \frac{\mathrm{d}}{\mathrm{d}\tau} Y(\tau, \xi_i) + Y(\tau, \xi_i) = \sum_{k=1}^{N} w_k \Psi(\xi_k)[Y(\tau, \xi_k) + Y(\tau, -\xi_k)] \tag{20a}$$

and

$$-\xi_i \frac{\mathrm{d}}{\mathrm{d}\tau} Y(\tau, -\xi_i) + Y(\tau, -\xi_i) = \sum_{k=1}^{N} w_k \Psi(\xi_k)[Y(\tau, \xi_k) + Y(\tau, -\xi_k)] \tag{20b}$$

for $i = 1, 2, \ldots, N$. In writing Eqs. (20) we have taken into account the fact that the characteristic function defined by Eq. (3) is an even function. In addition, we clearly are considering that the N quadrature points $\{\xi_k\}$ and the N weights $\{w_k\}$ are defined for use on the integration interval $[0, \infty)$. We note that it is to this feature of using a "half-range" quadrature scheme that we partially attribute the especially good accuracy we have obtained from the solution reported here.

Seeking exponential solutions, we substitute

$$Y(\tau, \pm\xi_i) = \phi(\nu, \pm\xi_i)\mathrm{e}^{-\tau/\nu} \tag{21}$$

into Eqs. (20) to find

$$\frac{1}{\nu}\mathbf{M}\mathbf{\Phi}_+ = (\mathbf{I} - \mathbf{W})\mathbf{\Phi}_+ - \mathbf{W}\mathbf{\Phi}_- \tag{22a}$$

and

$$-\frac{1}{\nu}\mathbf{M}\mathbf{\Phi}_- = (\mathbf{I} - \mathbf{W})\mathbf{\Phi}_- - \mathbf{W}\mathbf{\Phi}_+ \tag{22b}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix,

$$\mathbf{\Phi}_\pm = \big[\phi(\nu, \pm\xi_1), \phi(\nu, \pm\xi_2), \ldots, \phi(\nu, \pm\xi_N)\big]^{\mathrm{T}}, \tag{23}$$

the superscript T denotes the transpose operation, the elements of the matrix $\mathbf{W}$ are

$$(\mathbf{W})_{i,j} = w_j \Psi(\xi_j) \tag{24}$$

and

$$\mathbf{M} = \mathrm{diag}\big\{\xi_1, \xi_2, \ldots, \xi_N\big\}. \tag{25}$$

If we now let

$$\boldsymbol{U} = \mathbf{\Phi}_+ + \mathbf{\Phi}_- \tag{26}$$

then we can eliminate between the sum and the difference of Eqs. (22) to find

$$(\mathbf{D} - 2\mathbf{M}^{-1}\mathbf{W}\mathbf{M}^{-1})\mathbf{M}\boldsymbol{U} = \frac{1}{\nu^2}\mathbf{M}\boldsymbol{U} \tag{27}$$

where

$$\mathbf{D} = \mathrm{diag}\{\xi_1^{-2}, \xi_2^{-2}, \ldots, \xi_N^{-2}\}. \tag{28}$$

Multiplying Eq. (27) by a diagonal matrix $\mathbf{T}$, we find

$$(\mathbf{D} - 2\mathbf{V})\boldsymbol{X} = \frac{1}{\nu^2}\boldsymbol{X} \tag{29}$$

where

$$\mathbf{V} = \mathbf{M}^{-1}\mathbf{T}\mathbf{W}\mathbf{T}^{-1}\mathbf{M}^{-1} \tag{30}$$

and

$$\boldsymbol{X} = \mathbf{T}\mathbf{M}\boldsymbol{U}. \tag{31}$$

As discussed in Ref. [16], we can define the elements $t_1, t_2, \ldots, t_N$ of $\mathbf{T}$ so as to make $\mathbf{V}$ symmetric; and therefore, since $\mathbf{V}$ is a symmetric, rank one matrix, we can write our eigenvalue problem in the form

$$(\mathbf{D} - 2\boldsymbol{z}\boldsymbol{z}^{\mathrm{T}})\boldsymbol{X} = \lambda\boldsymbol{X} \tag{32}$$

where $\lambda = 1/\nu^2$ and

$$\boldsymbol{z} = \left[\frac{\sqrt{w_1\Psi(\xi_1)}}{\xi_1}, \frac{\sqrt{w_2\Psi(\xi_2)}}{\xi_2}, \ldots, \frac{\sqrt{w_N\Psi(\xi_N)}}{\xi_N}\right]^{\mathrm{T}}. \tag{33}$$

We note that the eigenvalue problem defined by Eq. (32) is of a form that is encountered when the so-called "divide and conquer" method [17] is used to find the eigenvalues of tridiagonal matrices. In addition, we see from Eq. (28) that, because of the way our basic eigenvalue problem is formulated, we must exclude zero from the set of quadrature points. Of course to exclude zero from the quadrature set is not considered a serious restriction since typical Gauss quadrature schemes do not include the end points of the integration interval.

Considering that we have found the required eigenvalues from Eq. (32), we impose the normalization condition

$$\sum_{k=1}^{N} w_k\Psi(\xi_k)[\phi(\nu,\xi_k) + \phi(\nu,-\xi_k)] = 1 \tag{34}$$

so that we can write our discrete-ordinates solution as

$$Y(\tau, \pm\xi_i) = \sum_{j=1}^{N}\left[\mathrm{A}_j\frac{\nu_j}{\nu_j \mp \xi_i}\mathrm{e}^{-(a+\tau)/\nu_j} + \mathrm{B}_j\frac{\nu_j}{\nu_j \pm \xi_i}\mathrm{e}^{-(a-\tau)/\nu_j}\right] \tag{35}$$

where the arbitrary constants $\{\mathrm{A}_j\}$ and $\{\mathrm{B}_j\}$ are to be determined from the boundary conditions and the separation constants $\{\nu_j\}$ are the reciprocals of the positive square roots of the eigenvalues defined by Eq. (32). It is clear from Eq. (35) that we cannot allow any separation constant to be equal to one of the quadrature points. In addition, the scaling constant a in Eq. (35) is, at this point, also arbitrary (we will use $a = 0$ for half-space applications and $2a$ equal to the full channel width for plane-channel problems).

At this point we find it convenient to modify slightly the discrete-ordinates solution we reported in Ref. [16]. We note that problems based on Eq. (1) are "conservative" since

$$\int_{-\infty}^{\infty} \Psi(\xi)\,\mathrm{d}\xi = 1, \tag{36}$$

and so we expect that one of the eigenvalues defined by Eq. (32) should tend to zero as N tends to infinity. We choose to take this fact into account by explicitly neglecting ν_N, the largest of the computed separation constants $\{\nu_j\}$ and, subsequently, by writing Eq. (35) as

$$Y(\tau, \pm\xi_i) = \mathrm{A}+\mathrm{B}(\tau\mp\xi_i)+\sum_{j=1}^{N-1}\left[\mathrm{A}_j\frac{\nu_j}{\nu_j\mp\xi_i}\mathrm{e}^{-(a+\tau)/\nu_j}+\mathrm{B}_j\frac{\nu_j}{\nu_j\pm\xi_i}\mathrm{e}^{-(a-\tau)/\nu_j}\right]. \tag{37}$$

Of course, the constants A, B, $\{\mathrm{A}_j\}$ and $\{\mathrm{B}_j\}$ that are present in Eq. (37) will, as discussed in Section 4 of this work, be determined by fixing the behavior of $Y(\tau, \xi_i)$ at infinity (for half-space problems) and/or by constraining $Y(\tau, \xi_i)$ to meet discrete-ordinates versions of the relevant boundary conditions. Finally we note that with the discrete-ordinates solution given by Eq. (37) we can use Eq. (34) to obtain, from the definition given by Eq. (5), the discrete-ordinates result [1]

$$Y_0(\tau) = \mathrm{A} + \mathrm{B}\tau + \sum_{j=1}^{N-1}\left[\mathrm{A}_j\mathrm{e}^{-(a+\tau)/\nu_j} + \mathrm{B}_j\mathrm{e}^{-(a-\tau)/\nu_j}\right]. \tag{38}$$

In a similar way we find that

$$Y_1(\tau) = \int_{-\infty}^{\infty} \Psi(u)Y(\tau, u)u\,\mathrm{d}u \tag{39}$$

can be expressed simply in terms of our discrete-ordinates solution, *viz.*

$$Y_1(\tau) = -\frac{1}{2}\mathrm{B}. \tag{40}$$

We consider it important to record some additional comments in regard to Eqs. (37), (38) and (40): as mentioned, in writing Eq. (37) we have not used the largest (infinite) separation constant ν_N and have replaced the two "missing" solutions by the two "exact" terms that appear as the first elements in Eq. (37). Considering subsequently that Eq. (37) is a mixture of exact terms and discrete-ordinates terms, we have, in obtaining Eqs. (38) and (40), integrated the exact terms exactly and the discrete-ordinates terms by making use of our numerical quadrature scheme. Finally, in obtaining Eq. (40), we have replaced the quadrature version of the integral in Eq. (36) with the exact value, *viz.* one.

To conclude this section we should mention that as an alternative to using the analytical expressions for the functions $\phi(\nu_j, \pm\xi_i)$ as we have in Eq. (37), we could, as was done in Ref. [3], make use of the numerical eigenvectors that are available when, for example, DZPACK [18] is used to solve the eigenvalue problem defined by Eq. (32).

4. Solutions to the Problems

Having developed our discrete-ordinates formalism, we are now ready to solve in a unified manner the specific problems defined in Section 2.

4.1. Half-Space Problems

Considering the half-space problems defined by either Eqs. (1) and (2) or by Eqs. (6) and (7), we set the constants B and $\{\mathrm{B}_j\}$ in Eq. (37) all equal to zero and write the desired solution as

$$Y(\tau, \pm\xi_i) = \mathrm{A} + \sum_{j=1}^{N-1} \mathrm{A}_j \frac{\nu_j}{\nu_j \mp \xi_i} \mathrm{e}^{-\tau/\nu_j}\,. \tag{41}$$

Now substituting Eq. (41) into the boundary conditions, either Eq. (2b) or Eq. (7b) evaluated at the quadrature points, we find the system of linear algebraic equations

$$\alpha \mathrm{A} + \sum_{j=1}^{N-1} \mathrm{M}_{i,j} \mathrm{A}_j = \mathrm{F}(\xi_i) \tag{42}$$

for $i = 1, 2, \ldots, N$. Here

$$\mathrm{M}_{i,j} = \nu_j \Big[\frac{\alpha\nu_j + \xi_i(2-\alpha)}{\nu_j^2 - \xi_i^2} \Big] \tag{43}$$

and $\mathrm{F}(\xi)$ is either

$$\mathrm{F}_{\mathrm{T}}(\xi) = \frac{1}{2}\alpha(\xi^2 - \frac{1}{2}) \tag{44a}$$

or

$$\mathrm{F}_{\mathrm{P}}(\xi) = (2-\alpha)\xi \tag{44b}$$

depending on which of the two problems we are considering. Now, of course, all we have to do is to define a quadrature scheme, solve the eigenvalue problem defined by Eq. (32), thus obtaining the separation constants $\{\nu_j\}$, and solve the linear system defined by Eq. (42). In this way all that we seek here is established, *viz.*

$$\mathrm{A}_{\mathrm{T}} = 2\mathrm{A} \tag{45}$$

and

$$q_{\mathrm{T}}(\tau) = 2Y_0(\tau) \tag{46}$$

for the thermal-creep problem and

$$\mathrm{A_P} = \mathrm{A} \tag{47}$$

and

$$q_{\mathrm{P}}(\tau) = \tau + Y_0(\tau) \tag{48}$$

for Kramers' problem. Here

$$Y_0(\tau) = \mathrm{A} + \sum_{j=1}^{N-1} \mathrm{A}_j \mathrm{e}^{-\tau/\nu_j} \,. \tag{49}$$

A discussion of the quadrature scheme we use and some numerical results for these two half-space problems are given in Section 5 of this work.

4.2. Plane-Channel Problems

Looking now at the problems defined in Section 2 to describe flow in a plane channel, we consider the boundary conditions, subject to which we must solve Eq. (9), written as

$$Y(-a,\xi) - (1-\alpha)Y(-a,-\xi) = \mathrm{F}_1(\xi) \tag{50a}$$

and

$$Y(a,-\xi) - (1-\alpha)Y(a,\xi) = \mathrm{F}_2(\xi) \tag{50b}$$

for $\xi \in (0,\infty)$. To be explicit, we note that

$$\mathrm{F}_1(\xi) = \alpha\xi^2 + a(2-\alpha)\xi \tag{51a}$$

and

$$\mathrm{F}_2(\xi) = \alpha\xi^2 + a(2-\alpha)\xi \tag{51b}$$

for Poiseuille flow,

$$\mathrm{F}_1(\xi) = \alpha \tag{52a}$$

and

$$\mathrm{F}_2(\xi) = -\alpha \tag{52b}$$

for Couette flow and

$$\mathrm{F}_1(\xi) = \frac{1}{2}\alpha(\xi^2 - \frac{1}{2}) \tag{53a}$$

and

$$\mathrm{F}_2(\xi) = \frac{1}{2}\alpha(\xi^2 - \frac{1}{2}) \tag{53b}$$

for thermal creep. So to solve these three problems we substitute Eq. (37) into Eqs. (50) evaluated at the quadrature points to find the system of linear algebraic equations

$$\sum_{j=1}^{N-1}\Big\{\mathrm{M}_{i,j}\mathrm{A}_j+N_{i,j}\mathrm{B}_j\mathrm{e}^{-2a/\nu_j}\Big\}+\alpha\mathrm{A}-\mathrm{B}[\alpha a+\xi_i(2-\alpha)]=\mathrm{F}_1(\xi_i) \tag{54a}$$

and

$$\sum_{j=1}^{N-1}\Big\{\mathrm{M}_{i,j}\mathrm{B}_j+N_{i,j}\mathrm{A}_j\mathrm{e}^{-2a/\nu_j}\Big\}+\alpha\mathrm{A}+\mathrm{B}[\alpha a+\xi_i(2-\alpha)]=\mathrm{F}_2(\xi_i) \tag{54b}$$

for $i=1,2,\dots,N$. Here the matrix elements $\mathrm{M}_{i,j}$ are given by Eq. (43) and

$$N_{i,j}=\nu_j\Big[\frac{\alpha\nu_j-\xi_i(2-\alpha)}{\nu_j^2-\xi_i^2}\Big]. \tag{55}$$

Of course once we have solved Eqs. (54) to find the constants A , B and $\{\mathrm{A}_j,\mathrm{B}_j\}$ we have $Y_0(\tau)$ and $Y_1(\tau)$, as given by Eqs. (38) and (40), established. Again, to be explicit we have

$$q_{\mathrm{P}}(\tau)=\frac{1}{2}\big(1-a^2+\tau^2\big)-Y_0(\tau) \tag{56}$$

and

$$Q_{\mathrm{P}}=\frac{1}{2a^2}\Big[2a\mathrm{A}+\sum_{j=1}^{N-1}\nu_j\big(\mathrm{A}_j+\mathrm{B}_j\big)\big(1-\mathrm{e}^{-2a/\nu_j}\big)\Big]-\frac{1}{2a}\big(1-\frac{2}{3}a^2\big) \tag{57}$$

for the Poiseuille-flow problem,

$$\mathrm{P}_{xz}=-\frac{1}{2}\pi^{1/2}\mathrm{B} \tag{58}$$

for Couette flow and

$$q_{\mathrm{T}}(\tau)=Y_0(\tau) \tag{59}$$

and

$$Q_{\mathrm{T}}=-\frac{1}{2a^2}\Big[2a\mathrm{A}+\sum_{j=1}^{N-1}\nu_j\big(\mathrm{A}_j+\mathrm{B}_j\big)\big(1-\mathrm{e}^{-2a/\nu_j}\big)\Big] \tag{60}$$

for the thermal-creep problem. Here, in general,

$$Y_0(\tau)=\mathrm{A}+\mathrm{B}\,\tau+\sum_{j=1}^{N-1}\Big[\mathrm{A}_j\mathrm{e}^{-(a+\tau)/\nu_j}+\mathrm{B}_j\mathrm{e}^{-(a-\tau)/\nu_j}\Big]. \tag{61}$$

Having developed our discrete-ordinates solution, we are ready to discuss the computational aspects of the solution and to report some numerical results.

Table 1. The slip coefficients $\mathrm{A_T}$ and $\mathrm{A_P}$

α	$\mathrm{A_T}$	$\mathrm{A_P}$
0.01	5.028545(−1)	1.766386(2)
0.10	5.283566(−1)	1.710313(1)
0.20	5.563021(−1)	8.224902
0.30	5.838476(−1)	5.255112
0.40	6.110039(−1)	3.762619
0.50	6.377813(−1)	2.861190
0.60	6.641898(−1)	2.255410
0.70	6.902391(−1)	1.818667
0.80	7.159384(−1)	1.487654
0.90	7.412966(−1)	1.227198
1.00	7.663225(−1)	1.016191

5. Numerical Results

The first thing we must do is to define the quadrature scheme to be used in our discrete-ordinates solution. In this work we have used one of the (nonlinear) transformations

$$u(\xi) = \exp\{-\xi\} \tag{62a}$$

or

$$u(\xi) = \frac{1}{1+\xi} \tag{62b}$$

to map $\xi \in [0, \infty)$ into $u \in [0, 1]$, and we then used a Gauss-Legendre scheme mapped (linearly) onto the interval $[0, 1]$. Of course other quadrature schemes could be used. In fact we note that recent works by Garcia [19] and Gander and Karp [20] have reported special quadrature schemes for use in the general area of particle-transport theory. Such an approach clearly could be used here. In fact the choice of a quadrature scheme based on the integration interval $[0, \infty)$ with a weight function as defined by Eq. (3) seems a natural choice for this work. However, we have found the use of a mapping defined by either of Eqs. (62) followed by the use of the Gauss-Legendre integration formulas to be so effective that we have not developed any special-purpose quadrature schemes.

Having defined our quadrature scheme and in developing a FORTRAN implementation of our solution, we found the required separation constants $\{\nu_j\}$ by using the special numerical package DZPACK [18] that was developed to take advantage of the special structure of Eq. (32) to solve our eigenvalue problem. The required separation constants were then available as the reciprocals of the square

Table 2. The macroscopic velocity profile $q_{\mathrm{T}}(\tau)$ for a half space

τ	$\alpha = 0.20$	$\alpha = 0.40$	$\alpha = 0.60$	$\alpha = 0.80$	$\alpha = 1.00$
0.0	4.377443(−1)	3.786517(−1)	3.225358(−1)	2.692240(−1)	2.185558(−1)
0.2	4.760866(−1)	4.529236(−1)	4.304719(−1)	4.086951(−1)	3.875591(−1)
0.4	4.938872(−1)	4.877829(−1)	4.816888(−1)	4.756064(−1)	4.695371(−1)
0.6	5.058782(−1)	5.113450(−1)	5.164243(−1)	5.211386(−1)	5.255086(−1)
0.8	5.146970(−1)	5.287087(−1)	5.420738(−1)	5.548278(−1)	5.670040(−1)
1.0	5.214841(−1)	5.420912(−1)	5.618698(−1)	5.808647(−1)	5.991177(−1)
1.4	5.312026(−1)	5.612809(−1)	5.902961(−1)	6.183047(−1)	6.453593(−1)
1.8	5.377202(−1)	5.741684(−1)	6.094128(−1)	6.435172(−1)	6.765404(−1)
2.0	5.401959(−1)	5.790673(−1)	6.166854(−1)	6.531161(−1)	6.884206(−1)
2.5	5.448495(−1)	5.882823(−1)	6.303740(−1)	6.711948(−1)	7.108100(−1)
3.0	5.480059(−1)	5.945373(−1)	6.396728(−1)	6.834851(−1)	7.260418(−1)
5.0	5.537261(−1)	6.058853(−1)	6.565604(−1)	7.058284(−1)	7.537610(−1)
7.0	5.554011(−1)	6.092125(−1)	6.615180(−1)	7.123957(−1)	7.619180(−1)
10.0	5.560896(−1)	6.105812(−1)	6.635590(−1)	7.151015(−1)	7.652814(−1)
15.0	5.562784(−1)	6.109567(−1)	6.641194(−1)	7.158449(−1)	7.662061(−1)
20.0	5.562989(−1)	6.109976(−1)	6.641805(−1)	7.159260(−1)	7.663071(−1)

roots of these eigenvalues. We then used the subroutines DGECO and DGESL from the LINPACK package [21] to solve the linear system defined by Eqs. (42) or (54), and so the solutions to the various problems were considered established.

Finally, but importantly, we note that since the function $\Psi(u)$ defined by Eq. (3) can be zero, from a computational point-of-view, we can have some, say a total of N_0, of the separation constants $\{\nu_j\}$ equal to some of the quadrature points $\{\xi_i\}$. Of course this is not allowed in Eq. (35), and so, since the quadrature points where $\Psi(\xi_k)$ is effectively zero make no contribution to the right-hand side of Eqs. (20), we have seen that we can simply omit these quadrature points from our calculation. Of course, in omitting these N_0 quadrature points we have effectively changed N to $N - N_0$ in some aspects of our final solution.

In order to illustrate one of the merits of our developed discrete-ordinates solutions to the considered problems, we list some typical results in Tables 1–8. We note that these numerical results are valid for a wide range of the Knudsen number and are given with what we believe to be seven figures of accuracy for all but the Couette-flow problem where we believe we have eight figures of accuracy for the computation of the normalized stress (Table 8). Of course, we have no definitive proof of the accuracy of our results, but we have done various things to establish the confidence we have. First of all, we have found basic agreement, except for the typographical errors [22] in Tables I, II and V of Ref. [10], with

Table 3. The macroscopic velocity profile $q_{\mathrm{P}}(\tau)$ for a half space

τ	$\alpha = 0.20$	$\alpha = 0.40$	$\alpha = 0.60$	$\alpha = 0.80$	$\alpha = 1.00$
0.0	7.622844	3.238196	1.805584	1.109556	7.071068(−1)
0.2	8.063202	3.646240	2.182908	1.457664	1.027415
0.4	8.357992	3.928844	2.453793	1.717270	1.276161
0.6	8.617267	4.180609	2.698309	1.954782	1.506903
0.8	8.858622	4.416772	2.929448	2.181057	1.728463
1.0	9.089163	4.643505	3.152488	2.400515	1.944445
1.4	9.530897	5.080070	3.584033	2.827181	2.366367
1.8	9.957489	5.503391	4.004171	3.244222	2.780389
2.0	1.016727(1)	5.711974	4.211588	3.450500	2.985559
2.5	1.068513(1)	6.227655	4.725143	3.961982	3.495016
3.0	1.119681(1)	6.737914	5.234016	4.469502	4.001214
5.0	1.321680(1)	8.755485	7.249226	6.482403	6.011854
7.0	1.522221(1)	1.076024(1)	9.253350	8.485905	8.014746
10.0	1.822430(1)	1.376209(1)	1.225495(1)	1.148726(1)	1.101587(1)
15.0	2.322484(1)	1.876256(1)	1.725536(1)	1.648761(1)	1.601616(1)
20.0	2.822489(1)	2.376261(1)	2.225540(1)	2.148765(1)	2.101619(1)

the results reported in Refs. [9] and [10] for the various problems we have solved here, and we have confirmed precisely the Poiseuille-flow results from Ref. [23]. In addition, we have increased the value of N used in our computations until we found stability in the final results. We have also used numerical linear-algebra packages other than those mentioned and both nonlinear maps given by Eqs. (62) to obtain the same results as given in our tables. Finally, we have established our numerical results with two independent FORTRAN implementations of our solutions.

We note that we have typically used $N = 50$ to generate the results listed in our tables, and to have an idea of the computational time required for a typical case, we note that our FORTRAN implementation (no special effort was made to make the code especially efficient) of our discrete-ordinates solution (with $N = 50$) runs in less than 0.2 seconds on a 166 MHz Pentium-based PC.

6. Final Comments

In regard to additional work in the general area of rarefied gas dynamics [24–26], we note that our variation of the discrete-ordinates method has been used to solve a heat-transfer problem in a plane channel where the coupled effects

Table 4. The macroscopic velocity profile $q_{\mathrm{T}}(\tau)$ for a channel of width $2a = 2.0$

τ	$\alpha = 0.50$	$\alpha = 0.80$	$\alpha = 0.88$	$\alpha = 0.96$	$\alpha = 1.00$
0.0	2.439084(–1)	2.420532(–1)	2.417017(–1)	2.413991(–1)	2.412645(–1)
0.1	2.434617(–1)	2.413836(–1)	2.409771(–1)	2.406209(–1)	2.404602(–1)
0.2	2.421049(–1)	2.393495(–1)	2.387756(–1)	2.382569(–1)	2.380167(–1)
0.3	2.397858(–1)	2.358715(–1)	2.350111(–1)	2.342144(–1)	2.338381(–1)
0.4	2.364086(–1)	2.308045(–1)	2.295260(–1)	2.283235(–1)	2.277487(–1)
0.5	2.318176(–1)	2.239126(–1)	2.220646(–1)	2.203091(–1)	2.194636(–1)
0.6	2.257644(–1)	2.148202(–1)	2.122195(–1)	2.097328(–1)	2.085296(–1)
0.7	2.178377(–1)	2.029065(–1)	1.993179(–1)	1.958715(–1)	1.941984(–1)
0.8	2.072854(–1)	1.870416(–1)	1.821364(–1)	1.774110(–1)	1.751119(–1)
0.9	1.924101(–1)	1.646908(–1)	1.579355(–1)	1.514142(–1)	1.482365(–1)
1.0	1.643019(–1)	1.227565(–1)	1.126193(–1)	1.028313(–1)	9.806183(–2)

Table 5. The thermal-creep flow rate Q_{T}

$2a$	$\alpha = 0.50$	$\alpha = 0.80$	$\alpha = 0.88$	$\alpha = 0.96$	$\alpha = 1.00$
0.05	–1.653689	–1.080865	–9.775525(–1)	–8.867589(–1)	–8.452893(–1)
0.10	–1.266442	–8.659805(–1)	–7.914282(–1)	–7.253123(–1)	–6.949272(–1)
0.30	–7.580824(–1)	–5.712072(–1)	–5.338214(–1)	–4.999736(–1)	–4.841992(–1)
0.50	–5.705723(–1)	–4.551664(–1)	–4.310383(–1)	–4.089057(–1)	–3.984993(–1)
0.70	–4.649666(–1)	–3.864581(–1)	–3.695099(–1)	–3.538098(–1)	–3.463781(–1)
0.90	–3.953837(–1)	–3.392428(–1)	–3.268193(–1)	–3.152202(–1)	–3.097001(–1)
1.00	–3.685435(–1)	–3.205049(–1)	–3.097630(–1)	–2.997001(–1)	–2.948999(–1)
2.00	–2.245046(–1)	–2.129203(–1)	–2.101613(–1)	–2.075212(–1)	–2.062429(–1)
5.00	–1.075322(–1)	–1.116570(–1)	–1.127116(–1)	–1.137481(–1)	–1.142597(–1)
7.00	–8.036291(–2)	–8.533480(–2)	–8.661542(–2)	–8.787780(–2)	–8.850228(–2)
9.00	–6.421908(–2)	–6.907720(–2)	–7.033298(–2)	–7.157270(–2)	–7.218664(–2)

of temperature and density must be resolved simultaneously [2], and new works [27–30] devoted to flow in cylindrical tubes, to the temperature-jump problem and to binary gas mixtures also have been completed. And so in this basic work, we believe we have shown our unified discrete-ordinates solutions to be very effective (especially accurate and easy to implement) for what we consider to be a set of classical problems based on the BGK model. It seems, therefore, that we are jus-

Table 6. The macroscopic velocity profile $q_{\mathrm{P}}(\tau)$ for a channel of width $2a = 2.0$

τ	$\alpha = 0.50$	$\alpha = 0.80$	$\alpha = 0.88$	$\alpha = 0.96$	$\alpha = 1.00$
0.0	–3.652222	–2.319616	–2.117410	–1.948801	–1.874577
0.1	–3.644836	–2.312148	–2.109921	–1.941293	–1.867059
0.2	–3.622577	–2.289638	–2.087351	–1.918663	–1.844401
0.3	–3.585117	–2.251759	–2.049368	–1.880582	–1.806271
0.4	–3.531852	–2.197901	–1.995366	–1.826440	–1.752062
0.5	–3.461789	–2.127072	–1.924350	–1.755244	–1.680778
0.6	–3.373321	–2.037666	–1.834718	–1.665394	–1.590822
0.7	–3.263728	–1.926991	–1.723784	–1.554211	–1.479519
0.8	–3.127917	–1.790039	–1.586566	–1.416741	–1.341927
0.9	–2.954020	–1.615281	–1.411628	–1.241643	–1.166756
1.0	–2.676407	–1.340372	–1.137527	–9.683813(–1)	–8.939247(–1)

Table 7. The Poiseuille flow rate Q_{P}

$2a$	$\alpha = 0.50$	$\alpha = 0.80$	$\alpha = 0.88$	$\alpha = 0.96$	$\alpha = 1.00$
0.05	5.223296	3.089711	2.738340	2.437354	2.302256
0.10	4.556406	2.707741	2.406046	2.148241	2.032714
0.30	3.778472	2.244771	2.001067	1.794509	1.702474
0.50	3.544371	2.102266	1.876620	1.686342	1.601874
0.70	3.437669	2.038767	1.822011	1.639850	1.559186
0.90	3.383887	2.009241	1.797636	1.620223	1.541800
1.00	3.368218	2.001867	1.792059	1.616312	1.538678
2.00	3.376574	2.041385	1.838563	1.669366	1.594857
5.00	3.774402	2.438234	2.235059	2.065478	1.990767
7.00	4.088108	2.746112	2.541436	2.370375	2.294932
9.00	4.410190	3.063464	2.857565	2.685295	2.609254

tified in believing that the methods reported here can now be extended to solve even more challenging problems based on improved physical models derived from the Boltzmann equation.

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Table 8. The normalized stress P_{xz} for Couette flow with $\alpha = 1$

$2a$	P_{xz}
1.00(−7)	9.9999991(−1)
1.00(−6)	9.9999911(−1)
1.00(−5)	9.9999114(−1)
1.00(−4)	9.9991142(−1)
1.00(−3)	9.9911754(−1)
5.00(−3)	9.9564273(−1)
1.00(−2)	9.9139801(−1)
2.00(−2)	9.8317550(−1)
1.00(−1)	9.2579682(−1)
1.00	6.0072919(−1)
1.25	5.5110153(−1)
1.50	5.0958224(−1)
1.75	4.7421167(−1)
2.00	4.4364669(−1)
2.50	3.9334018(−1)
3.00	3.5353372(−1)
4.00	2.9431859(−1)
5.00	2.5224614(−1)
7.00	1.9627566(−1)
1.00(1)	1.4731246(−1)
2.00(1)	8.0447692(−2)
1.00(2)	1.7371483(−2)
1.00(3)	1.7688589(−3)
1.00(4)	1.7720937(−4)
1.00(5)	1.7724178(−5)
1.00(6)	1.7724502(−6)
1.00(7)	1.7724535(−7)

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