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An integral equation basic to the BGK model for flow in a cylindrical tube

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Abstract. The classical BGK model for describing the flow of a rarefied gas in a cylindrical tube is used as a starting point for developing an equivalent integral-equation formulation of the considered class of problems. While the problems of Poiseuille flow and thermal creep in a cylindrical tube have already been well solved in terms of a pseudo problem obtained from an integral-equation definition of the problems, the development of the relevant integral equation is given here for a larger class of problems. In particular it is noted that a general inhomogeneous source term in the balance equation and a general inhomogeneous term in the boundary condition are both included in the considered model, and, as a special case, the integral-equation formulation for the case of specular reflection at the surface of the tube is also developed.

Keywords. Discrete ordinates, Rarefied gas dynamics.

1. Introduction

In regard to developing new computational methods for solving the Poiseuille-flow and thermal-creep problems in a cylindrical tube, we note that the papers by Valougeorgis and Thomas [1] and Siewert [2] both made good use of two basic works by Mitsis [3] and Ferziger [4] to reduce the considered problems stated in variables appropriate to cylindrical geometry to related "pseudo problems" defined in terms of plane-geometry variables. Of course, the developed pseudo problems were much more easily solved [1,2] than the original formulation given in terms of variables basic to cylindrical geometry. Since the very useful Mitsis-Ferziger transformations [3,4] were derived from an integral-equation formulation of the BGK model [5], we develop in this work an integral-equation formulation that can be used to describe the flow of a rarefied gas in a cylindrical tube for applications more general than the most basic Poiseuille or thermal-creep problems. To start we follow Williams [6] and consider our problem to be defined by the BGK equation written in the form

$$\xi\Big(\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial\phi}\Big)g(r,\xi,\phi) + g(r,\xi,\phi)$$

$$= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{-\xi'^2} g(r,\xi',\phi')\xi' \, \mathrm{d}\xi' \, \mathrm{d}\phi' + Q(\xi). \quad (1)$$

Here we use $\xi \in [0,\infty)$ and $\phi \in [0,2\pi]$ to denote the magnitude and direction of the component of the particle velocity vector that lies in the plane perpendicular to the axis of the infinitely long tube, and $r \in [0,\mathrm{R}]$ is the radial distance from the centerline of the tube. Continuing to follow Ref. [6], we note that our basic unknown in Eq. (1) is defined in terms of a perturbation of the particle distribution function from a local Maxwellian. To be explicit we note [6] that

$$g(r,\xi,\phi) = \int_{-\infty}^{\infty} \mathrm{e}^{-c_z^2} h(r,\boldsymbol{c}) c_z \, \mathrm{d}c_z \quad (2)$$

where $h(r,\boldsymbol{c})$ is the mentioned perturbation, $\boldsymbol{c}$ is the particle velocity vector and c_z is the component of $\boldsymbol{c}$ that is parallel to the centerline of the tube. The inhomogeneous term $Q(\xi)$ in Eq. (1) is assumed known, and we write our boundary condition as

$$g(\mathrm{R},\xi,\phi) = f(\xi,\phi), \quad \phi \in (\pi/2, 3\pi/2) \quad \text{and} \quad \xi \in [0,\infty), \quad (3)$$

where $f(\xi,\phi)$ is (for the moment) also considered known. If we now impose the symmetry condition

$$f(\xi, 2\pi - \phi) = f(\xi,\phi), \quad \phi \in (\pi/2, \pi], \quad (4)$$

on the boundary data, then we can conclude from Eqs. (1) and (4) that

$$g(r,\xi,2\pi-\phi) = g(r,\xi,\phi), \quad \phi \in [0,\pi], \quad (5)$$

for all r and ξ. And so we let $\mu = \cos\phi$, for $\phi \in [0,\pi]$, and

$$\mathrm{G}(r,\xi,\mu) = g(r,\xi,\arccos\mu), \quad \mu \in [-1,1], \quad (6)$$

in order to rewrite Eqs. (1) and (3) as

$$\xi\Big(\mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\Big)\mathrm{G}(r,\xi,\mu) + \mathrm{G}(r,\xi,\mu)$$

$$= \int_{-1}^{1}\int_0^{\infty} \Psi(\xi',\mu')\mathrm{G}(r,\xi',\mu') \, \mathrm{d}\xi' \, \mathrm{d}\mu' + Q(\xi), \quad (7)$$

for $\mu \in [-1,1]$, $\xi \in [0,\infty)$ and $r \in (0,\mathrm{R})$, and

$$\mathrm{G}(\mathrm{R},\xi,-\mu) = \mathrm{F}(\xi,\mu), \quad \mu \in (0,1] \quad \text{and} \quad \xi \in [0,\infty). \quad (8)$$

Here

$$\Psi(\xi,\mu) = \frac{2\xi\mathrm{e}^{-\xi^2}}{\pi(1-\mu^2)^{1/2}} \quad (9)$$

and

$$\mathrm{F}(\xi,\mu) = f(\xi, \pi - \arccos\mu), \quad \mu \in (0,1]. \quad (10)$$

We consider that Eqs. (7) and (8) are the forms usually encountered in regard to BGK applications defined by cylindrical geometry, and so now we proceed to develop an integral-equation formulation of this problem.

2. Development

In order to reformulate the considered BGK problem defined by Eqs. (7) and (8) in terms of an integral equation, we start our development by making use of the method of characteristics [7], and so we let $\mu = \eta(r)$ and

$$\widehat{\mathrm{G}}(r) = \mathrm{G}[r, \xi, \eta(r)] \tag{11}$$

and note that we can (while thinking of ξ as a parameter) write

$$\frac{\mathrm{d}}{\mathrm{d}r}\widehat{\mathrm{G}}(r) = \frac{\partial}{\partial r}\mathrm{G}(r, \xi, \eta) + \frac{\mathrm{d}\eta}{\mathrm{d}r}\frac{\partial}{\partial \eta}\mathrm{G}(r, \xi, \eta). \tag{12}$$

At this point we let

$$\frac{\mathrm{d}\eta}{\mathrm{d}r} = \frac{1-\eta^2}{r\eta} \tag{13}$$

and rewrite Eq. (12) as

$$\frac{\mathrm{d}}{\mathrm{d}r}\widehat{\mathrm{G}}(r) = \frac{1}{\eta}\Big[\eta\frac{\partial}{\partial r}\mathrm{G}(r, \xi, \eta) + \frac{1-\eta^2}{r}\frac{\partial}{\partial \eta}\mathrm{G}(r, \xi, \eta)\Big] \tag{14}$$

or, after we note Eq. (7),

$$\frac{\mathrm{d}}{\mathrm{d}r}\widehat{\mathrm{G}}(r) + \frac{1}{\xi\eta(r)}\widehat{\mathrm{G}}(r) = H(r) \tag{15}$$

where

$$H(r) = \frac{1}{\xi\eta(r)}\Big[\int_{-1}^{1}\int_{0}^{\infty}\Psi(\xi', \mu')\mathrm{G}(r, \xi', \mu')\,\mathrm{d}\xi'\,\mathrm{d}\mu' + Q(\xi)\Big]. \tag{16}$$

We can solve Eq. (13) to obtain

$$\eta(r) = \pm\nu(r) \tag{17}$$

where

$$\nu(r) = \frac{1}{r}\big(r^2 - a^2\big)^{1/2} \tag{18}$$

and where a is (for the moment) an arbitrary constant. Finding the integrating factor, we write Eq. (15) as

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\widehat{\mathrm{G}}(r)\exp\{\pm r\nu(r)/\xi\}\Big) = H(r)\exp\{\pm r\nu(r)/\xi\}, \tag{19}$$

and so, after noting Eq. (11), we can rewrite Eq. (19) as

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\mathrm{G}[r, \xi, \pm\nu(r)]\exp\{\pm r\nu(r)/\xi\}\Big) = \pm\frac{1}{\xi\nu(r)}\mathrm{S}(r, \xi)\exp\{\pm r\nu(r)/\xi\} \tag{20}$$

where

$$\mathrm{S}(r, \xi) = \int_{-1}^{1}\int_{0}^{\infty}\Psi(\xi', \mu')\mathrm{G}(r, \xi', \mu')\,\mathrm{d}\xi'\,\mathrm{d}\mu' + Q(\xi). \tag{21}$$

At this point we can integrate the plus-sign version of Eq. (20) to obtain

$$G[r,\xi,\nu(r)] = A[r,\xi,\nu(r)] + \frac{1}{\xi}\int_a^r S(x,\xi)\exp\{-[r\nu(r) - x\nu(x)]/\xi\}\frac{dx}{\nu(x)}, \quad (22a)$$

and similarly we can integrate the minus-sign version to find

$$G[r,\xi,-\nu(r)] = G_m[r,\xi,-\nu(r)] + \frac{1}{\xi}\int_r^R S(x,\xi)\exp\{-[x\nu(x) - r\nu(r)]/\xi\}\frac{dx}{\nu(x)} \quad (22b)$$

where we have used Eq. (8) along with the definitions

$$A[r,\xi,\nu(r)] = G(a,\xi,0)\exp\{-r\nu(r)/\xi\} \quad (23a)$$

and

$$G_m[r,\xi,-\nu(r)] = F[\xi,\nu(R)]\exp\{-[R\nu(R) - r\nu(r)]/\xi\}. \quad (23b)$$

Going back to make use of Eq. (17) and $\eta(r) = \mu$, we can rewrite Eqs. (22) as

$$G(r,\xi,\mu) = A(r,\xi,\mu) + \frac{1}{\xi}\int_{r(1-\mu^2)^{1/2}}^r S(x,\xi)\exp\{-[r\mu - x\mu_0(x,r,\mu)]/\xi\}\frac{dx}{\mu_0(x,r,\mu)} \quad (24a)$$

and

$$G(r,\xi,-\mu) = G_m(r,\xi,-\mu) + \frac{1}{\xi}\int_r^R S(x,\xi)\exp\{-[x\mu_0(x,r,\mu) - r\mu]/\xi\}\frac{dx}{\mu_0(x,r,\mu)} \quad (24b)$$

for $\mu \in [0,1]$. Here we have used

$$a = r(1-\mu^2)^{1/2}, \quad (25)$$

$$A(r,\xi,\mu) = G[r(1-\mu^2)^{1/2},\xi,0]\exp\{-r\mu/\xi\} \quad (26a)$$

and

$$G_m(r,\xi,-\mu) = F[\xi,\mu_0(R,r,\mu)]\exp\{-[R\mu_0(R,r,\mu) - r\mu]/\xi\} \quad (26b)$$

where

$$\mu_0(x,r,\mu) = \frac{1}{x}\big(x^2 - r^2 + r^2\mu^2\big)^{1/2}. \quad (27)$$

We can set $\mu = 0$ in Eq. (24b) to find

$$G(r,\xi,0) = G_m(r,\xi,0) + \frac{1}{\xi}\int_r^R S(x,\xi)\exp\{-[x\mu_0(x,r,0)]/\xi\}\frac{dx}{\mu_0(x,r,0)}, \quad (28)$$

and now replacing r in Eq. (28) with $r(1-\mu^2)^{1/2}$ and making use of Eq. (26b), we rewrite Eq. (26a) as

$$A(r,\xi,\mu) = G_p(r,\xi,\mu) + \frac{1}{\xi}\int_{r(1-\mu^2)^{1/2}}^R S(x,\xi)\exp\{-[x\mu_0(x,r,\mu) + r\mu]/\xi\}\frac{dx}{\mu_0(x,r,\mu)} \quad (29)$$

where

$$G_p(r,\xi,\mu) = F[\xi,\mu_0(R,r,\mu)]\exp\{-[R\mu_0(R,r,\mu)+r\mu]/\xi\}. \tag{30}$$

Finally, we can use Eq. (29) and rewrite Eqs. (24) as

$$G(r,\xi,\mu) = B(r,\xi,\mu) + \big(\mathcal{I}_p^1 S\big)(r,\xi,\mu) + \big(\mathcal{I}_p^2 S\big)(r,\xi,\mu) \tag{31a}$$

and

$$G(r,\xi,-\mu) = B(r,\xi,-\mu) + \big(\mathcal{I}_m S\big)(r,\xi,-\mu) \tag{31b}$$

for $\mu \in [0,1]$. Here

$$\big(\mathcal{I}_p^1 S\big)(r,\xi,\mu) = \frac{1}{\xi}\int_{r(1-\mu^2)^{1/2}}^{r} S(x,\xi)\exp\{-[r\mu - x\mu_0(x,r,\mu)]/\xi\}\frac{dx}{\mu_0(x,r,\mu)}, \tag{32a}$$

$$\big(\mathcal{I}_p^2 S\big)(r,\xi,\mu) = \frac{1}{\xi}\int_{r(1-\mu^2)^{1/2}}^{R} S(x,\xi)\exp\{-[x\mu_0(x,r,\mu) + r\mu]/\xi\}\frac{dx}{\mu_0(x,r,\mu)} \tag{32b}$$

and

$$\big(\mathcal{I}_m S\big)(r,\xi,-\mu) = \frac{1}{\xi}\int_{r}^{R} S(x,\xi)\exp\{-[x\mu_0(x,r,\mu) - r\mu]/\xi\}\frac{dx}{\mu_0(x,r,\mu)}. \tag{32c}$$

We note that the known boundary terms in Eqs. (31) are given by

$$B(r,\xi,\mu) = F[\xi,\mu_0(R,r,\mu)]\exp\{-[R\mu_0(R,r,\mu)+r\mu]/\xi\}. \tag{33}$$

At this point we introduce new integration variables in Eqs. (32), *viz.*

$$s = [r\mu - x\mu_0(x,r,\mu)]/\xi \tag{34a}$$

in Eq. (32a),

$$s = [r\mu + x\mu_0(x,r,\mu)]/\xi \tag{34b}$$

in Eq. (32b) and

$$s = [x\mu_0(x,r,\mu) - r\mu]/\xi \tag{34c}$$

in Eq. (32c) to find that we can write Eqs. (31) as

$$G(r,\xi,\mu) = B(r,\xi,\mu) + \int_0^{s_0(r,\xi,\mu)} S[(r^2+s^2\xi^2-2rs\xi\mu)^{1/2},\xi]e^{-s}\,ds \tag{35a}$$

and

$$G(r,\xi,-\mu) = B(r,\xi,-\mu) + \int_0^{s_0(r,\xi,-\mu)} S[(r^2+s^2\xi^2+2rs\xi\mu)^{1/2},\xi]e^{-s}\,ds \tag{35b}$$

for $\mu \in [0,1]$. Here

$$s_0(r,\xi,\mu) = [(R^2 - r^2 + r^2\mu^2)^{1/2} + r\mu]/\xi. \tag{36}$$

As we now wish to derive an integral equation for

$$G(r) = \int_{-1}^{1}\int_0^{\infty} \Psi(\xi,\mu)G(r,\xi,\mu)\,d\xi\,d\mu \tag{37}$$

we multiply Eqs. (31) by $\Psi(\xi,\mu)$ and integrate over all ξ and μ. In the process of combining and simplifying the resulting equations, we first find three integrals to consider:

$$U(r,\xi) = \frac{1}{\xi}\int_0^1 \int_{r(1-\mu^2)^{1/2}}^{r} U(r,\xi,x,\mu)\,\mathrm{d}x\,\mathrm{d}\mu, \tag{38a}$$

$$\mathrm{V}(r,\xi) = \frac{1}{\xi}\int_0^1 \int_{r(1-\mu^2)^{1/2}}^{\mathrm{R}} \mathrm{V}(r,\xi,x,\mu)\,\mathrm{d}x\,\mathrm{d}\mu \tag{38b}$$

and

$$\mathrm{W}(r,\xi) = \frac{1}{\xi}\int_0^1 \int_{r}^{\mathrm{R}} \mathrm{W}(r,\xi,x,\mu)\,\mathrm{d}x\,\mathrm{d}\mu \tag{38c}$$

where

$$U(r,\xi,x,\mu) = \mathrm{S}(x,\xi)\frac{\exp\{-[r\mu - x\mu_0(x,r,\mu)]/\xi\}}{\mu_0(x,r,\mu)(1-\mu^2)^{1/2}}, \tag{39a}$$

$$\mathrm{V}(r,\xi,x,\mu) = \mathrm{S}(x,\xi)\frac{\exp\{-[r\mu + x\mu_0(x,r,\mu)]/\xi\}}{\mu_0(x,r,\mu)(1-\mu^2)^{1/2}} \tag{39b}$$

and

$$\mathrm{W}(r,\xi,x,\mu) = \mathrm{S}(x,\xi)\frac{\exp\{-[x\mu_0(x,r,\mu) - r\mu]/\xi\}}{\mu_0(x,r,\mu)(1-\mu^2)^{1/2}}. \tag{39c}$$

Interchanging the order of integrations in Eqs. (38), we find

$$\xi U(r,\xi) = \int_0^r \int_{(1-x^2/r^2)^{1/2}}^{1} U(r,\xi,x,\mu)\,\mathrm{d}\mu\,\mathrm{d}x, \tag{40a}$$

$$\xi\mathrm{V}(r,\xi) = \int_0^r \int_{(1-x^2/r^2)^{1/2}}^{1} \mathrm{V}(r,\xi,x,\mu)\,\mathrm{d}\mu\,\mathrm{d}x + \int_r^{\mathrm{R}} \int_0^1 \mathrm{V}(r,\xi,x,\mu)\,\mathrm{d}\mu\,\mathrm{d}x \tag{40b}$$

and

$$\xi\mathrm{W}(r,\xi) = \int_r^{\mathrm{R}} \int_0^1 \mathrm{W}(r,\xi,x,\mu)\,\mathrm{d}\mu\,\mathrm{d}x. \tag{40c}$$

Defining $X(r,\xi)$ to be $\mathrm{W}(r,\xi)$ plus the second term in Eq. (40b), we write

$$\xi X(r,\xi) = \int_r^{\mathrm{R}} \int_{-1}^1 \mathrm{W}(r,\xi,x,\mu)\,\mathrm{d}\mu\,\mathrm{d}x. \tag{41}$$

We can now change the μ variable in Eq. (41) with the transformation

$$x\mu_0(x,r,\mu) - r\mu = p(x,r,\alpha), \tag{42}$$

where [4]

$$p(x,r,\alpha) = (x^2 + r^2 - 2xr\alpha)^{1/2}, \tag{43}$$

to obtain

$$\xi X(r,\xi) = \int_r^{\mathrm{R}} x\mathrm{S}(x,\xi)\int_{-1}^1 \frac{\exp\{-p(x,r,\alpha)/\xi\}}{p(x,r,\alpha)(1-\alpha^2)^{1/2}}\,\mathrm{d}\alpha\,\mathrm{d}x. \tag{44}$$

Next we change μ to $-\mu$ in the first term in Eq. (40b), and we then change the μ variable in the resulting expressions with the transformation given by Eq. (42). Finally we change the μ variable in Eq. (40a) with the transformation

$$r\mu - x\mu_0(x, r, \mu) = p(x, r, \alpha) \tag{45}$$

and add the result to the transformed version of the first term from Eq. (40b) to obtain

$$\xi Y(r, \xi) = \int_0^r x\mathrm{S}(x, \xi) \int_{-1}^1 \frac{\exp\{-p(x, r, \alpha)/\xi\}}{p(x, r, \alpha)(1-\alpha^2)^{1/2}} \,\mathrm{d}\alpha \,\mathrm{d}x. \tag{46}$$

Continuing with our development of an integral equation for $\mathrm{G}(r)$, we can now use Eqs. (44) and (46) to rewrite Eq. (37) as

$$\mathrm{G}(r) = \mathrm{B}(r) + \frac{2}{\pi}\int_0^\infty \mathrm{e}^{-\xi^2} \int_0^{\mathrm{R}} x\mathrm{S}(x, \xi) \int_{-1}^1 \frac{\exp\{-p(x, r, \mu)/\xi\}}{p(x, r, \mu)(1-\mu^2)^{1/2}} \,\mathrm{d}\mu \,\mathrm{d}x \,\mathrm{d}\xi. \tag{47}$$

Here the “boundary term” is

$$\mathrm{B}(r) = \int_{-1}^1 \int_0^\infty \Psi(\xi, \mu)\mathrm{F}[\xi, \mu_0(\mathrm{R}, r, \mu)] \exp\{-s_0(r, \xi, \mu)\} \,\mathrm{d}\xi \,\mathrm{d}\mu \tag{48}$$

where we have made use of Eqs. (27), (33) and (36).

At this point we wish to make use of some Bessel-function identities [8,9] to find a more convenient form for Eq. (47). First of all, in terms of the modified Bessel functions of the second kind, we note that

$$\mathrm{K}_{1/2}(x) = \left(\frac{\pi}{2x}\right)^{1/2} \mathrm{e}^{-x} \tag{49}$$

and that [9]

$$\mathrm{K}_{1/2}(x) = \left(\frac{2x}{\pi}\right)^{1/2} \int_1^\infty y\mathrm{K}_0(yx) \frac{\mathrm{d}y}{(y^2-1)^{1/2}}, \tag{50}$$

and so using Eqs. (49) and (50), we can rewrite Eq. (47) as

$$\mathrm{G}(r) = \mathrm{B}(r) + \frac{4}{\pi^2}\int_0^\infty \frac{\mathrm{e}^{-\xi^2}}{\xi} \int_0^{\mathrm{R}} x\mathrm{S}(x, \xi) \int_{-1}^1 \int_1^\infty \frac{y\mathrm{K}_0[yp(x, r, \mu)/\xi]}{(y^2-1)^{1/2}(1-\mu^2)^{1/2}} \,\mathrm{d}y \,\mathrm{d}\mu \,\mathrm{d}x \,\mathrm{d}\xi. \tag{51}$$

Introducing a new variable $\tau = \xi/y$, we can rewrite Eq. (51) as

$$\mathrm{G}(r) = \mathrm{B}(r) + \frac{4}{\pi^2}\int_0^\infty \xi\mathrm{e}^{-\xi^2} \int_0^{\mathrm{R}} x\mathrm{S}(x, \xi) \int_0^\xi \frac{1}{\tau^2(\xi^2-\tau^2)^{1/2}} \int_{-1}^1 \frac{\mathrm{K}_0[p(x, r, \mu)/\tau]}{(1-\mu^2)^{1/2}} \,\mathrm{d}\mu \,\mathrm{d}\tau \,\mathrm{d}x \,\mathrm{d}\xi. \tag{52}$$

Quoting from Ref. [8], we wish to make use of an addition theorem that can be written as

$$\mathrm{K}_0[(x^2+r^2-2xr\cos\alpha)^{1/2}]=\sum_{n=-\infty}^{\infty}\mathrm{F}_n(x,r)\mathrm{e}^{in\alpha} \tag{53}$$

where

$$\mathrm{F}_n(x,r)=\begin{cases}\mathrm{I}_n(x)\mathrm{K}_n(r), & x<r,\\ \mathrm{K}_n(x)\mathrm{I}_n(r), & x>r,\end{cases} \tag{54}$$

and $\mathrm{I}_n(x)$ and $\mathrm{K}_n(x)$ are modified Bessel functions of the first and second kind. We can integrate Eq. (53) to find

$$\int_0^{2\pi}\mathrm{K}_0[(x^2+r^2-2xr\cos\alpha)^{1/2}]\,\mathrm{d}\alpha=2\pi\mathrm{F}_0(x,r) \tag{55a}$$

or

$$\int_{-1}^{1}\frac{\mathrm{K}_0[p(x,r,\mu)]}{(1-\mu^2)^{1/2}}\,\mathrm{d}\mu=\pi\mathrm{F}_0(x,r). \tag{55b}$$

Using Eq. (55b), we can rewrite Eq. (52) as

$$\mathrm{G}(r)=\mathrm{B}(r)+\frac{4}{\pi}\int_0^\infty\xi\mathrm{e}^{-\xi^2}\int_0^{\mathrm{R}}x\mathrm{S}(x,\xi)\int_0^\xi\frac{\mathrm{F}_0(x/\tau,r/\tau)}{\tau^2(\xi^2-\tau^2)^{1/2}}\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}\xi. \tag{56}$$

We now note from Eq. (21) that

$$\mathrm{S}(x,\xi)=\mathrm{G}(x)+Q(\xi), \tag{57}$$

and so we interchange some orders of integration and write Eq. (56) as

$$\mathrm{G}(r)=\mathrm{B}(r)+\mathrm{C}(r)+\frac{4}{\pi}\int_0^{\mathrm{R}}x\mathrm{G}(x)\int_0^\infty\frac{\mathrm{F}_0(x/\tau,r/\tau)}{\tau^2}\int_\tau^\infty\frac{\xi\mathrm{e}^{-\xi^2}}{(\xi^2-\tau^2)^{1/2}}\,\mathrm{d}\xi\,\mathrm{d}\tau\,\mathrm{d}x \tag{58}$$

where the known term that results from the inhomogeneous term in Eq. (1) is

$$\mathrm{C}(r)=\frac{4}{\pi}\int_0^{\mathrm{R}}x\int_0^\infty\frac{\mathrm{F}_0(x/\tau,r/\tau)}{\tau^2}\int_\tau^\infty\frac{\xi Q(\xi)\mathrm{e}^{-\xi^2}}{(\xi^2-\tau^2)^{1/2}}\,\mathrm{d}\xi\,\mathrm{d}\tau\,\mathrm{d}x. \tag{59}$$

Now, we can evaluate the integral over ξ in Eq. (58) to obtain the desired integral equation for $\mathrm{G}(r)$, *viz.*

$$\mathrm{G}(r)=\mathrm{B}(r)+\mathrm{C}(r)+\int_0^{\mathrm{R}}x\mathrm{G}(x)\mathrm{K}(x\to r)\,\mathrm{d}x \tag{60}$$

where the kernel of the integral equation is

$$\mathrm{K}(x\to r)=\frac{2}{\pi^{1/2}}\int_0^\infty\mathrm{e}^{-\tau^2}\mathrm{F}_0(x/\tau,r/\tau)\frac{\mathrm{d}\tau}{\tau^2} \tag{61}$$

and the two inhomogeneous terms $\mathrm{B}(r)$ and $\mathrm{C}(r)$ are given by Eqs. (48) and (59).

3. Poiseuille and thermal-creep flow

Having obtained, for the general case, the desired integral equation for $\mathrm{G}(r)$, we now wish to look at two special cases. We consider the inhomogeneous source term in Eq. (1) written as [6]

$$Q(\xi) = \frac{1}{2}\big[k_1 - k_2(\xi^2 - 1)\big] \tag{62}$$

where k_1 and k_2 are constants. Substituting Eq. (62) into Eq. (59) and using the identity

$$x[\mathrm{K}_0(x)\mathrm{I}_1(x) + \mathrm{I}_0(x)\mathrm{K}_1(x)] = 1, \tag{63}$$

we find

$$\mathrm{C}(r) = \frac{k_1}{2} - \frac{\mathrm{R}}{\pi^{1/2}}\int_0^\infty \mathrm{e}^{-\tau^2}\mathrm{K}_1(\mathrm{R}/\tau)\mathrm{I}_0(r/\tau)\big[k_1 - k_2(\tau^2 - 1/2)\big]\frac{\mathrm{d}\tau}{\tau}. \tag{64}$$

If we now substitute

$$\mathrm{G}(r) = \frac{1}{\pi^{1/2}}\mathrm{Z}(r) - \frac{1}{4}\big(2k_1 + k_2\big) \tag{65}$$

into Eq. (60) and make use of Eq. (64), we find

$$\mathrm{Z}(r) = \int_0^{\mathrm{R}} x\mathrm{Z}(x)\mathrm{K}(x \to r)\,\mathrm{d}x + \mathrm{S}(r) \tag{66}$$

where the known term is

$$\mathrm{S}(r) = \pi^{1/2}\mathrm{B}(r) + \frac{\pi^{1/2}}{2}k_1 + k_2\mathrm{R}\int_0^\infty \tau\mathrm{e}^{-\tau^2}\mathrm{K}_1(\mathrm{R}/\tau)\mathrm{I}_0(r/\tau)\,\mathrm{d}\tau. \tag{67}$$

To obtain the Poiseuille-flow problem, we set $\mathrm{B}(r) = 0$, $k_1 = 1$ and $k_2 = 0$ to find from Eqs. (65), (66) and (67)

$$\mathrm{G_P}(r) = \frac{1}{\pi^{1/2}}\mathrm{Z_P}(r) - \frac{1}{2}, \tag{68}$$

$$\mathrm{Z_P}(r) = \int_0^{\mathrm{R}} x\mathrm{Z_P}(x)\mathrm{K}(x \to r)\,\mathrm{d}x + \mathrm{S_P}(r) \tag{69}$$

and

$$\mathrm{S_P}(r) = \frac{1}{2}\pi^{1/2}. \tag{70}$$

In a similar way, to obtain the thermal-creep problem, we set $\mathrm{B}(r) = 0$, $k_1 = 0$ and $k_2 = 1$ to find from Eqs. (65), (66) and (67)

$$\mathrm{G_T}(r) = \frac{1}{\pi^{1/2}}\mathrm{Z_T}(r) - \frac{1}{4}, \tag{71}$$

$$\mathrm{Z_T}(r) = \int_0^{\mathrm{R}} x\mathrm{Z_T}(x)\mathrm{K}(x \to r)\,\mathrm{d}x + \mathrm{S_T}(r) \tag{72}$$

and

$$\mathrm{S_T}(r) = \mathrm{R}\int_0^\infty \tau \mathrm{e}^{-\tau^2} \mathrm{K}_1(\mathrm{R}/\tau)\mathrm{I}_0(r/\tau)\,\mathrm{d}\tau. \tag{73}$$

We note that Eqs. (69) and (72) are exactly the equations that were solved in Refs. [1] and [2] by making use of the Mitsis-Ferziger transformations to find pseudo problems that are more easily solved than the original formulation.

4. Reflecting surface

The special cases discussed here in Section 3 are defined for the situation where the inhomogeneous term in the boundary condition is zero; however if we allow reflection at the surface of the tube we could have, *e.g.*

$$\mathrm{F}(\xi,\mu) = (1-\alpha)\mathrm{G}(\mathrm{R},\xi,\mu), \quad \mu \in [0,1], \tag{74}$$

and then Eq. (60) would have to be modified since $\mathrm{B}(r)$, as given by Eq. (48), could not be considered a known term. And so continuing this development, we find, from Eqs. (31a) and (33), that

$$\mathrm{F}(\xi,\mu) = \frac{(1-\alpha)\big[\big(\mathcal{I}_\mathrm{p}^1\mathrm{S}\big)(\mathrm{R},\xi,\mu) + \big(\mathcal{I}_\mathrm{p}^2\mathrm{S}\big)(\mathrm{R},\xi,\mu)\big]}{1-(1-\alpha)\exp\{-2\mathrm{R}\mu/\xi\}}. \tag{75}$$

If we let

$$\Delta(x,\xi,\mu) = \frac{1}{\xi}\mathrm{S}(x,\xi)\frac{\exp\{-\mathrm{R}\mu/\xi\}}{\mu_0(x,\mathrm{R},\mu)}\big[\exp\{x\mu_0(x,\mathrm{R},\mu)/\xi\} + \exp\{-x\mu_0(x,\mathrm{R},\mu)/\xi\}\big] \tag{76}$$

we can rewrite Eq. (75) as

$$\mathrm{F}(\xi,\mu) = \frac{1-\alpha}{1-(1-\alpha)\exp\{-2\mathrm{R}\mu/\xi\}}\int_{\mathrm{R}(1-\mu^2)^{1/2}}^{\mathrm{R}} \Delta(x,\xi,\mu)\,\mathrm{d}x. \tag{77}$$

And so we can write Eq. (48) as

$$\mathrm{B}(r) = (1-\alpha)\int_0^\infty\int_{-1}^1 \int_{r(1-\mu^2)^{1/2}}^{\mathrm{R}} \Psi(\xi,\mu)\Gamma(x,\xi,r,\mu)\exp\{-2\mathrm{R}\mu_0(\mathrm{R},r,\mu)/\xi\}\,\mathrm{d}x\,\mathrm{d}\mu\,\mathrm{d}\xi \tag{78}$$

where

$$\Gamma(x,\xi,r,\mu) = \mathrm{S}(x,\xi)\frac{\exp\{-[r\mu + x\mu_0(x,r,\mu)]/\xi\} + \exp\{-[r\mu - x\mu_0(x,r,\mu)]/\xi\}}{\xi\mu_0(x,r,\mu)[1-(1-\alpha)\exp\{-2\mathrm{R}\mu_0(\mathrm{R},r,\mu)/\xi\}]}. \tag{79}$$

We can now interchange the orders of integrations in Eq. (78) and use the transformations given by, say, Eqs. (43) and (45) to find ultimately that we can express the boundary term for Eq. (60) as

$$\mathrm{B}(r) = \frac{2}{\pi}\int_0^{\mathrm{R}}\int_0^{\infty}\int_{-1}^{1} x\mathrm{S}(x,\xi)\frac{\exp\{-\xi^2 - p(x,r,\mu)/\xi\}}{p(x,r,\mu)(1-\mu^2)^{1/2}}\mathrm{T}(x,r,\mu,\xi)\,\mathrm{d}\mu\,\mathrm{d}\xi\,\mathrm{d}x \quad (80)$$

where $p(x,r,\mu)$ is defined by Eq. (43),

$$\mathrm{T}(x,r,\mu,\xi) = \frac{2(1-\alpha)\exp\{-2\mathrm{R}\mu_0[\mathrm{R},r,\beta(x,r,\mu)]/\xi\}}{1-(1-\alpha)\exp\{-2\mathrm{R}\mu_0[\mathrm{R},r,\beta(x,r,\mu)]/\xi\}} \quad (81)$$

and

$$\beta(x,r,\mu) = (r-\mu x)/p(x,r,\mu). \quad (82)$$

At this point we are able to summarize the integral-equation formulation of the considered problem for the case of general $Q(\xi)$ coupled with the reflecting surface condition as given by Eq. (74). And so we have found that we can, for this case, write Eq. (60) as

$$\mathrm{G}(r) = \mathrm{C}(r) + \mathrm{C}_{\mathrm{s}}(r) + \int_0^{\mathrm{R}} x\mathrm{G}(x)[\mathrm{K}(x\to r) + \mathrm{K}_{\mathrm{s}}(x\to r)]\,\mathrm{d}x \quad (83)$$

where we have made use of Eqs. (80) and (57), where $\mathrm{K}(x \to r)$ is given by Eq. (61) and where

$$\mathrm{K}_{\mathrm{s}}(x\to r) = \frac{2}{\pi}\int_0^{\infty}\int_{-1}^{1}\exp\{-\xi^2 - p(x,r,\mu)/\xi\}\frac{\mathrm{T}(x,r,\mu,\xi)}{p(x,r,\mu)(1-\mu^2)^{1/2}}\,\mathrm{d}\mu\,\mathrm{d}\xi. \quad (84)$$

In addition, the first of the two inhomogeneous terms in Eq. (83) is given by Eq. (59), and

$$\mathrm{C}_{\mathrm{s}}(r) = \frac{2}{\pi}\int_0^{\mathrm{R}}\int_0^{\infty}\int_{-1}^{1} xQ(\xi)\frac{\exp\{-\xi^2 - p(x,r,\mu)/\xi\}}{p(x,r,\mu)(1-\mu^2)^{1/2}}\mathrm{T}(x,r,\mu,\xi)\,\mathrm{d}\mu\,\mathrm{d}\xi\,\mathrm{d}x. \quad (85)$$

To conclude this section we note that we have added a subscript to two factors in Eq. (83) to remind us that these terms define the effect of specular reflection.

5. Concluding comments

To summarize the results developed in this work, we note, first of all, that our most basic result is given by Eq. (60), supplemented with Eqs. (48) and (59), for general, but specified, inhomogeneous source terms $Q(\xi)$ and $\mathrm{F}(\xi,\mu)$ as used in Eqs. (7) and (10). On the other hand, Eq. (83), with reference to Eqs. (59), (61), (84) and (85), is our most general result for the case of specular reflection. In addition, we have found and summarized, in Eqs. (69) and (72), the classical results for the cases of Poiseuille and thermal-creep flow in a cylindrical tube.

It is clear that in developing the integral-equation formulation for the various cases considered here, we have provided the background material necessary to start a solution based on the relevant integral equation. However, we must note that while the integral-equation formulation has been used [1,2] to solve the classical Poiseuille and thermal-creep problems, additional work is required to solve the more general class of problems discussed here.

Finally we should note that while the derivations given here are somewhat extensive, we believe the reported details to be useful for understanding better the importance of the integral-equation formulation of the BGK problems considered. We also believe that the results established in this work can be used, in principle, by researchers in other areas [10] of particle transport theory where problems in cylindrical geometry are of interest.

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