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ON THE NORMAL-MODE EXPANSION TECHNIQUE FOR RADIATIVE TRANSFER IN A SCATTERING, ABSORBING AND EMITTING SLAB WITH SPECULARLY REFLECTING BOUNDARIES

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Abstract—The normal-mode expansion technique has been used to solve the uncoupled radiative heat transfer problem for an absorbing, emitting, isotropically scattering, nonisothermal, gray medium confined between specularly reflecting, gray parallel boundaries held at uniform but different temperatures. Solutions are given for the intensity of radiation, the incident radiation and the net radiation heat flux for a prescribed inhomogeneous source term, represented by a polynomial expansion, in the medium. In addition the solution due to an arbitrary inhomogeneous source term is available from the Green's function developed for the considered problem.

NOMENCLATURE

c,	albedo, the ratio of the scattering to the extinction coefficient;
$E(\tau)$,	incident radiation;
$q(\tau)$,	radiation flux;
$S(\tau)$,	inhomogeneous source term;
T_1, T_2,	temperatures of the boundary surfaces;
$X(z)$,	Case's X-function;
$\varepsilon_1, \varepsilon_2$,	emissivities of the boundary surfaces;
μ,	direction cosine;
τ,	optical variable;
τ_0,	optical thickness between the boundaries.

1. INTRODUCTION

THE RADIATIVE heat-transfer problem in finite plane-parallel geometry for absorbing, emitting gray media has received a great deal of attention. In one of the earlier works, Usiskin and Sparrow [1] presented a numerical solution for such a medium bounded by heated black walls. Heaslet and Warming [2, 3] considered the case of a slab bounded by diffuse reflectors and obtained an analytical solution by utilizing the methods and tabulated functions developed by Chandrasekhar [4] for astrophysical applications. To date, however, the treatment of radiative transfer in scattering, absorbing, emitting media between reflecting, heated parallel surfaces has been restricted to numerical techniques. Love and Grosh [5] examined the case of an isothermal medium with diffusely reflecting boundaries, and Hsia and Love [6] extended the problem to include the effects of nonisothermal temperature distributions.

In the present paper, Case's normal-mode expansion technique [7] is used to obtain solutions to the radiative transfer problem for an absorbing, emitting, isotropically scattering, nonisothermal gray medium bounded by specularly reflecting, diffusely emitting, gray parallel walls each held at uniform but different temperatures. This method, developed by Case [7] for treating one-dimensional neutron transport problems, has been applied only recently in the field of radiative transfer. Siewert and

McCormick [8] obtained a rigorous solution for an absorbing, emitting, anistropically scattering, semi-infinite medium with a linear source function and a free boundary.

One of the earlier applications of the singular eigenfunction expansion technique to problems in finite geometry was made by McCormick and Mendelson [9] who treated the slab albedo problem. Ferziger and Simmons [10] solved the radiative transfer problem for a non-absorbing, non-emitting, perfectly scattering medium (or alternatively for a gray medium in radiative and local thermodynamic equilibrium) bounded by black heated parallel walls. Typical of transport problems with two boundaries, the results of Ferziger and Simmons were not expressed in closed forms; however, they have shown that their analytical approximate solutions were highly accurate. Recently Heaslet and Warming [11] considered non-conservative radiative transfer in semi-infinite and finite media.

The method of normal modes provides an elegant and systematic approach to the solution of one-dimensional, plane-parallel radiative transfer problems. Essentially the method prescribes that the desired solution be written as a linear sum of the eigenfunctions of the homogeneous equation and a particular solution appropriate to the source function of interest. The solution to the problem is thus reduced to that of determining the unknown expansion coefficients appearing in the sum of elementary solutions. These coefficients are determined by constraining the solution to meet the given boundary conditions and by then utilizing the orthogonality properties of these Case eigenfunctions. This procedure is completely analogous to the classical orthogonal expansion treatment of boundary value problems.

2. FORMULATION OF THE PROBLEM

Consider an absorbing, emitting, isotropically scattering, nonisothermal, gray medium bounded by parallel walls of infinite lateral extent and a finite distance apart. It is assumed that the gray opaque walls emit isotropically, reflect specularly and are kept at uniform but different temperatures. Further it is assumed that radiation is the predominant mode of energy transfer, that the system is in steady-state, and that the inhomogeneous source term in the medium, taken to be either a polynomial in the optical variable or that appropriate for the Green's function, is prescribed.

Prediction of the net radiation heat transfer to the walls in such a medium is an important engineering problem. It should be noted that in the absence of participating matter between the walls, the net radiative heat transfer is a function of the temperature and emissivity of the walls, and its determination is a simple matter; however, with the presence of participating matter, the problem requires the solution of the equation of radiative transfer subject to appropriate boundary conditions.

The equation of radiative transfer for one-dimensional, plane-parallel, emitting, absorbing, isotropically scattering, gray media can be written in the form [4]

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} + I(\tau, \mu) = (1 - c)\, I_b(\tau) + \frac{c}{2} \int_{-1}^{1} I(\tau, \mu')\, \mathrm{d}\mu', \qquad (1)$$

where the τ is the optical variable and μ is the cosine of the angle between the directed intensity and the positive τ axis. The constant c is the single-scattering albedo, which is the ratio of the scattering coefficient to the extinction coefficient, and $I_b(\tau)$ is the integrated Planck's function.

The bounding surfaces are positioned at $\tau = 0$ and $\tau = \tau_0$, are kept at uniform temperatures T_1 and T_2, and have emissivities ε_1 and ε_2 respectively. The boundary conditions for equation (1) are then given in the forms

$$I(0, \mu) = \varepsilon_1 \left(\frac{\bar{\sigma} T_1^4}{\pi}\right) + (1 - \varepsilon_1)\, I(0, -\mu), \qquad \mu \varepsilon (0, 1), \qquad (2a)$$

and

$$I(\tau_0, -\mu) = \varepsilon_2 \left(\frac{\bar{\sigma} T_2^4}{\pi}\right) + (1 - \varepsilon_2) I(\tau_0, \mu), \qquad \mu \varepsilon(0, 1). \tag{2b}$$

Here $\bar{\sigma}$ is the Stefan–Boltzmann constant.

Once the intensity of radiation is determined, the incident radiation $E(\tau)$ and the net radiation flux $q(\tau)$ are easily obtained, *i.e.*

$$E(\tau) = 2\pi \int_{-1}^{1} I(\tau, \mu)\, d\mu \tag{3a}$$

and

$$q(\tau) = 2\pi \int_{-1}^{1} I(\tau, \mu)\, \mu\, d\mu. \tag{3b}$$

3. GENERAL ANALYSIS

In this section we use the method of normal modes to construct the solution of the radiative transfer problem described previously. For convenience in the analysis, equations (1) and (2) are written more compactly as

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} + I(\tau, \mu) = S(\tau) + \frac{c}{2} \int_{-1}^{1} I(\tau, \mu')\, d\mu', \tag{4}$$

$$I(0, \mu) = a_1 + b_1 I(0, -\mu), \qquad \mu\, \varepsilon(0, 1), \tag{5a}$$

and

$$I(\tau_0, -\mu) = a_2 + b_2 I(\tau_0, \mu), \qquad \mu\, \varepsilon(0, 1). \tag{5b}$$

Here we have defined

$$S(\tau) \triangleq (1 - c)\, I_b(\tau), \tag{6a}$$

$$a_i \triangleq \varepsilon_i \left(\frac{\bar{\sigma} T_i^4}{\pi}\right), \qquad i = 1 \text{ or } 2, \tag{6b}$$

and

$$b_i \triangleq (1 - \varepsilon_i), \qquad i = 1 \text{ or } 2. \tag{6c}$$

In equation (4), $S(\tau)$ represents the inhomogeneous source term the form of which, at this point, is left arbitrary. In the following section we specialize our results for several explicit sources of interest and give the corresponding particular solutions to equation (4).

We proceed then to write the desired solution as a linear sum of the normal modes introduced by Case [7] and a particular solution; the former solutions all satisfy the homogeneous version of equation (4), whereas the particular solution $I_p(\tau, \mu)$ cannot be given until the function $S(\tau)$ is specified. Thus

$$I(\tau, \mu) = A(\eta_0)\, \phi(\eta_0, \mu)\, e^{-\tau/\eta_0} + A(-\eta_0)\, \phi(-\eta_0, \mu)\, e^{\tau/\eta_0} + \int_{-1}^{1} A(\eta)\, \phi(\eta, \mu)\, e^{-\tau/\eta}\, d\eta + I_p(\tau, \mu), \tag{7}$$

where

$$\phi(\pm \eta_0, \mu) = \frac{c\eta_0}{2} \frac{1}{\eta_0 \mp \mu}, \tag{8a}$$

$$\phi(\eta, \mu) = \frac{c\eta}{2} \frac{P}{\eta - \mu} + \lambda(\eta)\, \delta(\eta - \mu), \tag{8b}$$

$$\lambda(\eta) = 1 - c\eta \tanh^{-1}(\eta), \tag{9a}$$

and the discrete eigenvalues $\pm\eta_0$ are the two zeros of the dispersion function

$$\Lambda(z) = 1 + c\frac{z}{2} \int_{-1}^{1} \frac{d\mu}{\mu - z} = 1 - cz \tanh^{-1}\left(\frac{1}{z}\right). \tag{9b}$$

Here P is a mnemonic symbol used to denote the Cauchy principal-value function, and $\delta(x)$ denotes the Dirac delta function; the continuum eigenfunctions $\phi(\eta, \mu)$, $\eta\, \varepsilon(-1, 1)$, are generalized functions.

The functions $A(\eta_0)$, $A(-\eta_0)$ and $A(\eta)$, $\eta\, \varepsilon(-1, 1)$, are arbitrary expansion coefficients which must be determined such that the solution given by equation (7) meets the boundary conditions of the problem. Before pursuing this, however, we note that the incident radiation and the flux may be obtained trivially from equations (3) and (7):

$$E(\tau) = 2\pi\Big[A(\eta_0) e^{-\tau/\eta_0} + A(-\eta_0)\, e^{\tau/\eta_0} + \int_{-1}^{1} A(\eta)\, e^{-\tau/\eta}\, d\eta + \int_{-1}^{1} I_p(\tau, \mu)\, d\mu\Big], \tag{10a}$$

and

$$q(\tau) = 2\pi(1-c)\Big[A(\eta_0)\,\eta_0\,e^{-\tau/\eta_0} - A(-\eta_0)\,\eta_0\,e^{\tau/\eta_0} + \int_{-1}^{1} A(\eta)\,\eta\,e^{-\tau/\eta}\,d\eta + (1-c)^{-1}\int_{-1}^{1} I_p(\tau,\mu)\,\mu\,d\mu\Big]. \tag{10b}$$

Here we have used the fact

$$\int_{-1}^{1} \mu^{\alpha}\,\phi(\xi,\mu)\,d\mu = [\xi(1-c)]^{\alpha}, \qquad \alpha = 0 \text{ or } 1, \qquad \xi = \pm\eta_0 \text{ or } \varepsilon(-1,1). \tag{11}$$

If we now define two functions (considered known),

$$f_1(\mu) \triangleq a_1 + b_1\,I_p(0,-\mu) - I_p(0,\mu), \quad \mu\varepsilon(0,1), \tag{12a}$$

and

$$f_2(\mu) \triangleq a_2 + b_2\,I_p(\tau_0,\mu) - I_p(\tau_0,-\mu), \quad \mu\varepsilon(0,1), \tag{12b}$$

and introduce equation (7) into equations (5), the boundary conditions can be cast in the forms

$$f_1(\mu) + [b_1A(\eta_0) - A(-\eta_0)]\,\phi(-\eta_0,\mu) + \int_0^1 [b_1A(\eta) - A(-\eta)]\,\phi(-\eta,\mu)\,d\eta = [A(\eta_0) - b_1A(-\eta_0)]\,\phi(\eta_0,\mu) + \int_0^1 [A(\eta) - b_1A(-\eta)]\,\phi(\eta,\mu)\,d\eta, \quad \mu\varepsilon(0,1), \tag{13a}$$

and

$$f_2(\mu) + [b_2A(-\eta_0)\,e^{\tau_0/\eta_0} - A(\eta_0)\,e^{-\tau_0/\eta_0}]\,\phi(-\eta_0,\mu) + \int_0^1 [b_2A(-\eta)\,e^{\tau_0/\eta} - A(\eta)\,e^{-\tau_0/\eta}]\,\phi(-\eta,\mu)\,d\eta = [A(-\eta_0)\,e^{\tau_0/\eta_0} - b_2A(\eta_0)\,e^{-\tau_0/\eta_0}]\,\phi(\eta_0,\mu) + \int_0^1 [A(-\eta)\,e^{\tau_0/\eta} - b_2A(\eta)\,e^{-\tau_0/\eta}]\,\phi(\eta,\mu)\,d\eta, \quad \mu\varepsilon(0,1). \tag{13b}$$

In order to determine the four unknown expansion coefficients, $A(\eta_0)$, $A(-\eta_0)$, $A(\eta)$ and $A(-\eta)$, we must solve equations (13). The half-range completeness theorem, proved by Case [7] and reviewed briefly in the Appendix of this paper, states that $\phi(\eta_0,\mu)$ and $\phi(\eta,\mu)$, $\eta\varepsilon(0,1)$, form a complete basis set for the expansion of arbitrary functions defined for $\mu\varepsilon(0,1)$. The right-hand sides of equations (13) are two such expansions. These two equations are coupled singular integral equations; they may, however, be reduced to coupled, non-singular, Fredholm-type, integral equations by following the methods of Muskhelishvili [12]. We prefer to use, as an alternative method, the half-range orthogonality theorem proved by Kuščer, McCormick and Summerfield [13]. This theorem and the results for the scalar products of interest here are recorded in the Appendix for reference.

In order to isolate the discrete coefficient $[A(\eta_0) - b_1A(-\eta_0)]$ on the right-hand side of equation (13a), we multiply that equation by $W(\mu)\,\phi(\eta_0,\mu)$ and integrate over μ from zero to one. Similarly the discrete coefficient $[A(-\eta_0)\,e^{\tau_0/\eta_0} - b_2A(\eta_0)\,e^{-\tau_0/\eta_0}]$ is isolated by applying this same operation to equation (13b). We utilize the half-range orthogonality theorem and the various normalization integrals given in the Appendix to obtain

$$-\left(\frac{c\eta_0}{2}\right)^2 X(\eta_0)[A(\eta_0) - b_1A(-\eta_0)] = F_1(\eta_0) + \left(\frac{c\eta_0}{2}\right)^2 X(-\eta_0)[b_1A(\eta_0) - A(-\eta_0)] + \frac{c^2}{4}\eta_0\int_0^1 \eta X(-\eta)[b_1A(\eta) - A(-\eta)]\,d\eta \tag{14a}$$

and

$$\left(\frac{c\eta_0}{2}\right)^2 X(\eta_0)[b_2A(\eta_0)\,e^{-\tau_0/\eta_0} - A(-\eta_0)\,e^{\tau_0/\eta_0}] = F_2(\eta_0) - \left(\frac{c\eta_0}{2}\right)^2 X(-\eta_0)[A(\eta_0)\,e^{-\tau_0/\eta_0} - b_2A(-\eta_0)^{\tau_0/\eta_0}] - \frac{c^2}{4}\eta_0\int_0^1 \eta X(-\eta)[A(\eta)\,e^{-\tau_0/\eta} - b_2A(-\eta)\,e^{\tau_0/\eta}]\,d\eta. \tag{14b}$$

Here we have defined

$$F_\alpha(\xi) \triangleq \int_0^1 W(\mu)\,\phi(\xi,\mu)\,f_\alpha(\mu)\,d\mu, \qquad \alpha = 1 \text{ and } 2. \tag{15}$$

Further, the weight function $W(\mu)$ is given by

$$W(\mu) = \frac{c\mu}{2(1-c)(\eta_0+\mu)X(-\mu)}, \qquad \mu\varepsilon(0,1), \tag{16}$$

and $X(z)$ is Case's X-function [7], *viz.*

$$X(z) = \frac{1}{1-z}\exp\left\{\frac{1}{\pi}\int_0^1 \tan^{-1}\left[\frac{c\pi\mu}{2(1-c\mu\tanh^{-1}\mu)}\right] \times \frac{d\mu}{\mu-z}\right\}. \tag{17}$$

In addition to being well tabulated for the variables of interest [14], the X-function is related to Chandrasekhar's H-function [4] by

$$H(z) \equiv [\sqrt{(1-c)}(\eta_0+z)X(-z)]^{-1}, \tag{18}$$

and thus $W(\mu)$ may be written as

$$W(\mu) = \frac{c}{2}(1-c)^{-\frac{1}{2}}\,\mu H(\mu). \tag{19}$$

Similarly, we now project equations (13) on to the continuum eigenfunctions by multiplying those equations by $W(\mu)\,\phi(\eta',\mu)$, $\eta'\varepsilon(0,1)$, and integrating over the half-range, $\mu\varepsilon(0,1)$. Again we utilize the orthogonality theorem to obtain (after interchanging η and η')

$$W(\eta)\Lambda^+(\eta)\Lambda^-(\eta)[A(\eta)-b_1A(-\eta)] = F_1(\eta) + c\eta_0\eta X(-\eta_0)\,\phi(-\eta_0,\eta)[b_1A(\eta_0)-A(-\eta_0)] + \frac{c\eta}{2}\int_0^1(\eta_0+\eta')X(-\eta')\,\phi(-\eta',\eta) \times [b_1A(\eta')-A(-\eta')]\,d\eta', \qquad \eta\varepsilon(0,1), \tag{20a}$$

and

$$W(\eta)\Lambda^+(\eta)\Lambda^-(\eta)[A(-\eta)e^{\tau_0/\eta}-b_2A(\eta)e^{-\tau_0/\eta}] = F_2(\eta) + c\eta_0\eta X(-\eta_0)\,\phi(-\eta_0,\eta) \times [b_2A(-\eta_0)e^{\tau_0/\eta_0}-A(\eta_0)e^{-\tau_0/\eta_0}] - \frac{c\eta}{2}\int_0^1(\eta_0+\eta')X(-\eta')\,\phi(-\eta',\eta)[A(\eta)e^{-\tau_0/\eta} - b_2A(-\eta)e^{\tau_0/\eta}]\,d\eta', \qquad \eta\varepsilon(0,1). \tag{20b}$$

Here

$$\Lambda^+(\eta)\Lambda^-(\eta) = \lambda^2(\eta) + \left(\frac{c\eta\pi}{2}\right)^2, \tag{21a}$$

or alternatively

$$\Lambda^+(\eta)\Lambda^-(\eta) = 1/g(c,\eta), \tag{21b}$$

where $g(c,\eta)$ is a function well tabulated by Case, de Hoffmann and Plazcek [15].

Equations (14) and (20) are the four basic equations from which the unknown expansion coefficients $A(\eta_0)$, $A(-\eta_0)$, $A(\eta)$ and $A(-\eta)$, $\eta\varepsilon(0,1)$, must be determined. These equations can be written considerably more concisely in matrix notation:

$$\mathbf{M}\mathbf{A}(\eta_0) = \mathbf{G}(\eta_0) + \int_0^1 \mathbf{B}(\eta')\,\mathbf{A}(\eta')\,K_0(\eta')\,d\eta' \tag{22a}$$

and

$$\mathbf{M}(\eta)\,\mathbf{A}(\eta) = \mathbf{G}(\eta) + \mathbf{B}(\eta_0)\,\mathbf{A}(\eta_0)\,K_1(\eta) + \int_0^1 \mathbf{B}(\eta')\,\mathbf{A}(\eta')\,K(\eta'\to\eta)\,d\eta', \qquad \eta\varepsilon(0,1), \tag{22b}$$

where

$$\mathbf{A}(\xi) = \begin{vmatrix} A(\xi) \\ A(-\xi) \end{vmatrix}, \qquad \xi = \eta_0 \text{ or } \varepsilon(0,1). \tag{23}$$

In order to obtain equations (22), we have simply rearranged equations (14) and (20), invoked several of the previously given definitions, and introduced

$$\mathbf{M} \triangleq \begin{vmatrix} b_1 e^{-2z_0/\eta_0} - 1 & b_1 - e^{-2z_0/\eta_0} \\ (b_2 - e^{-2z_0/\eta_0})e^{-\tau_0/\eta_0} & (b_2 e^{-2z_0/\eta_0} - 1)e^{\tau_0/\eta_0} \end{vmatrix}, \tag{24a}$$

$$\mathbf{M}(\eta) \triangleq \begin{vmatrix} 1 & -b_1 \\ -b_2\, e^{-\tau_0/\eta} & e^{\tau_0/\eta} \end{vmatrix}, \tag{24b}$$

and

$$\mathbf{B}(\xi) \triangleq \begin{vmatrix} b_1 & -1 \\ -e^{-\tau_0/\xi} & b_2 e^{\tau_0/\xi} \end{vmatrix}, \quad \xi = \eta_0 \text{ or } \varepsilon(0,1). \tag{25}$$

In addition $\mathbf{G}(\xi)$ is a vector whose two components are the following:

$$g_\alpha(\eta_0) \triangleq \left(\frac{2}{c\eta_0}\right)^2 \frac{1}{X(\eta_0)} F_\alpha(\eta_0), \qquad \alpha = 1 \text{ and } 2; \tag{26a}$$

$$g_\alpha(\eta) \triangleq \left(\frac{2}{c\eta}\right)(1-c)(\eta_0+\eta)\, g(c,\eta) \times X(-\eta)\, F_\alpha(\eta), \qquad \alpha = 1 \text{ and } 2. \tag{26b}$$

Finally

$$K_0(\eta) = \frac{\eta X(-\eta)}{\eta_0 X(\eta_0)}, \tag{27}$$

$$K_1(\eta) = c(1-c)\,\eta_0^2\, g(c,\eta)\, X(-\eta)\, X(-\eta_0), \tag{28}$$

$$K(\eta' \to \eta) = \frac{c(1-c)\,\eta'}{2} \times \frac{(\eta_0+\eta)(\eta_0+\eta')}{\eta+\eta'}\, g(c,\eta)\, X(-\eta)\, X(-\eta') \tag{29}$$

and

$$e^{2z_0/\eta_0} = -X(\eta_0)/X(-\eta_0), \tag{30}$$

where z_0 is the Milne problem extrapolation distance,

$$z_0 = \frac{c\eta_0}{2}\int_0^1 \left[1 + \frac{c\mu^2}{1-\mu^2}\right] g(c,\mu) \tanh^{-1}\left(\frac{\mu}{\eta_0}\right) d\mu; \tag{31}$$

this quantity also has been tabulated for various values of c. (See Case, de Hoffmann, and Placzek [15 p. 136]).

Up to this point our analysis has been mathematically rigorous; clearly, however, approximations must now be made, for it is highly unlikely that analytical solutions to equations (22) exist. It follows that the degree of precision with which we can complete the desired solution will be measured by how accurately we determine $\mathbf{A}(\eta_0)$ and $\mathbf{A}(\eta)$ from equations (22). Although these equations are formidable analytically, they certainly pose no problem for existing computing facilities. Thus if highly accurate "bench mark" solutions are sought, an iterative numerical procedure could be used to solve these equations to any desired degree of accuracy. Bond and Siewert [16] have solved numerically a set of equations similar to equations (22) for a lattice problem in neutron transport theory. Their work illustrates the merits of iterative solutions.

Fortunately analytical approximations can be obtained from equations (22) which should yield solutions of sufficient accuracy. Ferziger and Simmons [10] obtained two different approximate solutions to a related problem; they showed that the lowest-order solution was better than classical diffusion theory, whereas the second-order solution was highly accurate.

In the present analysis, the lowest-order solution is obtained by neglecting the continuum coefficients entirely; the discrete solutions are thus readily available from equation (22a):

$$\mathbf{A}_1(\eta) \equiv 0; \qquad \mathbf{A}_1(\eta_0) = \mathbf{M}^{-1}\mathbf{G}(\eta_0). \tag{32}$$

The second-order continuum solution is found by neglecting the contribution from the kernel $K(\eta' \to \eta)$ in equation (22b) and by using in that equation the lowest-order $\mathbf{A}(\eta_0)$. Finally $\mathbf{A}_2(\eta)$

is substituted into equation (22a) to yield $\mathbf{A}_2(\eta_0)$. Thus

$$\mathbf{A}_2(\eta) = \mathbf{M}^{-1}(\eta)\,[\mathbf{G}(\eta) + \mathbf{B}(\eta_0)\,\mathbf{A}_1(\eta_0)\,K_1(\eta)]$$

and (33a)

$$\mathbf{A}_2(\eta_0) = \mathbf{M}^{-1}[\mathbf{G}(\eta_0) + \int_0^1 \mathbf{B}(\eta')\,\mathbf{A}_2(\eta') \times K_0(\eta')\,\mathrm{d}\eta']. \quad (33b)$$

Having completed the general analysis for this problem, we proceed to determine explicitly the particular solutions and the vectors $\mathbf{G}(\eta_0)$ and $\mathbf{G}(\eta)$ for several inhomogeneous source terms.

4. PARTICULAR SOLUTIONS FOR VARIOUS SOURCE FUNCTIONS

Basic to the analysis in the previous section is the need to determine a particular solution to equation (4) for sources of interest. In addition we should like to construct the vectors $\mathbf{G}(\eta_0)$ and $\mathbf{G}(\eta)$ required in the solutions for the expansion coefficients $\mathbf{A}(\eta_0)$ and $\mathbf{A}(\eta)$. We examine two basic types of sources: the first being a polynomial in the optical variable τ, and the second defining the Green's function problem.

Focusing our attention on the first type of source, we consider

$$S^{(i)}(\tau) = \tau^i, \qquad i = 0, 1, 2 \ldots. \quad (34)$$

We investigate only the first three values of i in the above; however, the procedure used here is clearly valid for arbitrary i. In addition, particular solutions corresponding to linear sums of the above sources are obtained by superposition. If we assume a solution of the form $I_p(\tau, \mu) = A_{\alpha\beta}\tau^\alpha\mu^\beta$, the following are found immediately:

$$I_p^{(0)}(\tau, \mu) = \frac{1}{1-c}, \quad (35a)$$

$$I_p^{(1)}(\tau, \mu) = \frac{1}{1-c}[\tau - \mu] \quad (35b)$$

and

$$I_p^{(2)}(\tau, \mu) = \frac{1}{1-c}\left[\tau^2 - 2\mu\tau + 2\mu^2 + \frac{2c}{3(1-c)}\right].$$

Particular solutions for an arbitrary polynomial inhomogeneous source term have been determined by Lundquist and Horak [17]. We now construct the vectors $\mathbf{G}^{(1)}(\eta_0)$ and $\mathbf{G}^{(1)}(\eta)$ corresponding to $i = 1$. The results for other cases are given without proof. We note from equations (12) that

$$f_1^{(1)}(\mu) = a_1 + \frac{1}{1-c}(b_1 + 1)\,\mu \quad (36a)$$

and

$$f_2^{(1)}(\mu) = a_2 + \frac{1}{1-c}[\tau_0(b_2 - 1) - \mu(b_2 + 1)]. \quad (36b)$$

In order to construct the $\mathbf{G}$-vectors of interest, we need the following integrals:

$$\int_0^1 W(\mu)\,\phi(\xi, \mu)\,\mu^\alpha \mathrm{d}\mu \triangleq \frac{c\xi}{2}\,\Theta(\alpha, \xi),$$

$$\xi = \eta_0 \text{ or } \epsilon(0, 1), \qquad \alpha = 0, 1 \text{ and } 2. \quad (37a)$$

Here

$$\Theta(\alpha, \xi) = \begin{Bmatrix} \gamma^{(0)} & \alpha = 0 \\ \gamma^{(1)} + \xi - \eta_0 & \alpha = 1 \\ \gamma^{(2)} + (\xi - \eta_0)(\gamma^{(1)} + \xi) & \alpha = 2 \end{Bmatrix}, \quad (37b)$$

where

$$\gamma^{(\alpha)} \triangleq \int_0^1 \gamma(\mu)\,\mu^\alpha\,\mathrm{d}\mu. \quad (37c)$$

In addition, $\gamma^{(0)} \equiv 1$, $\gamma^{(1)} = I(c)$ which is tabulated by Case, de Hoffmann and Placzek [15], and $\gamma^{(2)}$ is easily obtained since the X-function is also well known numerically [14]. Noting equation (15), we write

$$F_1^{(1)}(\xi) = \frac{c\xi}{2}\left\{a_1 + \frac{1}{1-c}(b_1 + 1)\Theta(1, \xi)\right\} \quad (38a)$$

and

$$F_2^{(1)}(\xi) = \frac{c\xi}{2}\left\{a_2 + \frac{1}{1-c}[\tau_0(b_2 - 1) - (b_2 + 1)\,\Theta(1, \xi)]\right\}. \quad (38b)$$

The **G**-vectors for the three considered cases may now be written by inspection:

$$\mathbf{G}^{(0)}(\xi) = \pi(\xi)\begin{vmatrix} a_1 + \dfrac{1}{1-c}(b_1 - 1) \\ a_1 + \dfrac{1}{1-c}(b_2 - 1)\end{vmatrix} \tag{39a}$$

$$\mathbf{G}^{(1)}(\xi) = \pi(\xi)\begin{vmatrix} a_1 + \dfrac{1}{1-c}(b_1 + 1)\,\Theta(1,\xi) \\ a_2 + \dfrac{1}{1-c}[\tau_0(b_2 - 1) - (b_2 + 1)\,\Theta(1,\xi)]\end{vmatrix} \tag{39b}$$

and

$$\mathbf{G}^{(2)}(\xi) = \pi(\xi)\begin{vmatrix} a_1 + \dfrac{b_1 - 1}{1-c}\left[2\Theta(2,\xi) + \dfrac{2c}{3(1-c)}\right] \\ a_2 + \dfrac{1}{1-c}\left\{(b_2 - 1)\left[2\Theta(2,\xi) + \dfrac{2c}{3(1-c)} + \tau_0^2\right] - (b_2 + 1)\,2\tau_0\,\Theta(1,\xi)\right\}\end{vmatrix}, \tag{39c}$$

where

$$\pi(\xi) \triangleq \begin{cases} \dfrac{2}{c\eta_0 X(\eta_0)} & \xi = \eta_0 \\ (1-c)(\eta_0 + \eta)\,g(c,\eta)\,X(-\eta) & \xi = \eta \end{cases} \tag{40}$$

As illustrated above, the analysis given in Section 3 is sufficiently general to be able to incorporate any inhomogeneous source term represented by a polynomial in the optical variable. The solution technique developed here requires only that we be able to find a particular solution to equation (4) for a given source. This particular solution can always be obtained; if, in fact, we find the particular solution associated with the source

$$S(\tau,\mu) = \delta(\tau - \tau_1)\,\delta(\mu - \mu_1), \qquad \tau_1 \epsilon (0,\tau_0), \quad \mu_1 \epsilon (-1,1), \tag{41}$$

then we would have available the particular solution for any source distribution.

A particular solution associated with equation (41) is clearly the well known *infinite-medium* Green's function [14]:

$$I_p(\tau_1,\mu_1 \to \tau,\mu) = \pm\left[\frac{\phi(\pm\eta_0,\mu_1)\,\phi(\pm\eta_0,\mu)\,e^{\mp(\tau-\tau_1)/\eta_0}}{N(\pm\eta_0)} + \int_0^1 d\eta\,\frac{\phi(\pm\eta,\mu_1)\,\phi(\pm\eta,\mu)\,e^{\mp(\tau-\tau_1)/\eta}}{N(\pm\eta)}\right], \qquad \tau \gtrless \tau_1, \tag{42}$$

where

$$N(\pm\eta) = \pm\,\eta\Lambda^+(\eta)\,\Lambda^-(\eta) \tag{43a}$$

and

$$N(\pm\eta_0) = \pm\frac{c}{2}\eta_0^3\left[\frac{c}{\eta_0^2 - 1} - \frac{1}{\eta_0^2}\right] \tag{43b}$$

In order now to complete the solution for the Green's function for the considered problem [equations (4), (5) and (41)], equation (42) is simply entered into equation (12) to give the appropriate **G**-vectors. Since this procedure yields the Green's function and thus, in essence, the complete solution for *any* source function, we must expect the corresponding **G**-vectors to be reasonably complex. For this reason they are not given here explicitly; however, to calculate these vectors is straightforward and requires no information additional to that presented here.

In conclusion we note that the present analysis may be extended quite easily to include the effects of linearly anisotropic scattering. This extension can be realized by utilizing the methods developed by McCormick and Kuščer [18].

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APPENDIX

Completeness and Orthogonality Theorems

Here we simply state the half-range completeness theorem presented initially by Case [7] and the half-range orthogonality theorem proved by Kuščer, McCormick and Summerfield [13]. In addition, a summary of the necessary normalization integrals [13] used to develop the results in the main text is given.

THEOREM I: *The eigenfunctions* $\phi(\eta_0, \mu)$ *and* $\phi(\eta, \mu)$ $\eta\varepsilon(0, 1)$ *are complete on the half-range in the sense that an arbitrary function* $\Psi(\mu)$ *defined for* $\mu\varepsilon(0, 1)$ *can be expanded in the form*

$$\Psi(\mu) = A(\eta_0)\,\phi(\eta_0, \mu) + \int_0^1 A(\eta)\,\phi(\eta, \mu)\,d\eta, \qquad \mu\varepsilon(0, 1). \tag{A-1}$$

THEOREM II: *The eigenfunctions* $\phi(\eta_0, \mu)$ *and* $\phi(\eta, \mu)$ $\eta\varepsilon(0, 1)$ *are complete on the half-range in with respect to the weight function* $W(\mu)$, *i.e.*

$$\int_0^1 \phi(\xi, \mu)\,\phi(\xi', \mu)\,W(\mu)\,d\mu = 0, \qquad \xi \neq \xi';\, \xi, \xi' = \eta_0 \text{ or } \varepsilon(0, 1). \tag{A-2}$$

Normalization Integrals

In all of the following formulae, $\eta, \eta'\varepsilon(0, 1)$.

$$\int_0^1 \phi(\eta, \mu)\,\phi(\eta', \mu)\,W(\mu)\,d\mu = W(\eta)\,\Lambda^+(\eta)\,\Lambda^-(\eta)\,\delta(\eta - \eta') \tag{A-3}$$

$$\int_0^1 \phi(\eta_0, \mu)\,\phi(\eta, \mu)\,W(\mu)\,d\mu = 0 \tag{A-4}$$

$$\int_0^1 \phi(-\eta_0, \mu)\,\phi(\eta, \mu)\,W(\mu)\,d\mu = c\eta\eta_0 X(-\eta_0)\,\phi(-\eta_0, \eta) \tag{A-5}$$

$$\int_0^1 \phi(\pm\eta_0, \mu)\,\phi(\eta_0, \mu)\,W(\mu)\,d\mu = \mp\left(\frac{c\eta_0}{2}\right)^2 X(\pm\eta_0) \tag{A-6}$$

$$\int_0^1 \phi(-\eta,\mu)\,\phi(\eta_0,\mu)\,W(\mu)\,\mathrm{d}\mu = \tfrac{1}{4}c^2\eta\eta_0 X(-\eta) \qquad \text{(A-7)}$$

$$\int_0^1 \phi(-\eta,\mu)\,\phi(\eta',\mu)\,W(\mu)\,\mathrm{d}\mu = \tfrac{1}{2}c\eta'\phi(-\eta,\eta')(\eta_0+\eta)\,X(-\eta). \qquad \text{(A-8)}$$

Résumé—La technique de développement en mode normal a été employée pour résourdre le problème de transport de chaleur par rayonnement non couplé pour un milieu gris absorbant, émetteur, diffusant isotropiquement et non isotherme confiné entre des limites parallèles grises et réfléchissant de façon spéculaire portées à des températures uniformes mais différentes. On donne des solutions pour l'intensité du rayonnement, le rayonnement incident et le flux de chaleur de rayonnement net pour un terme de source non homogène imposé dans le milieu et représenté par un développement polynomial. En outre, la solution due à un terme arbitraire de source non homogène est disponible à partir de la fonction de Green exposée pour le problème considéré.

Zusammenfassung—Mit Hilfe der "Normal Mode Expansion Technique" wurde die entkoppelte Wärmeübertragung durch Strahlung in einem absorbierenden, emittierenden isotrop streuenden, nichtisothermen, grauen Medium berechnet. Die grauen, parallelen Begrenzungen des Mediums sollen spiegelnd reflektieren und konstante aber unterschiedliche Temperaturen haben. Lösungen werden für die Strahlungsintensität, für die auf die Begrenzungen einfallende Strahlung und für den Nettostrahlungsstrom, für den Fall angegeben, dass ein inhomogener Quellen-Ausdruck in Form eines Polynoms, vorgeschrieben ist.

Zusätzlich ist eine Lösung, die auf einen willkürlichen inhomogenen Quellen-Ausdruck zurückgeht, mit Hilfe der Green-Funktion verfügbar, die für den betrachteten Fall entwickelt wird.

Аннотация—Приводится решение несвязанной задачи лучистого переноса тепла для поглощающей, излучающей изотропно рассеивающей, неизотермической серой среды, заключенной между зеркально-отражающими серыми параллельными границами с постоянными, но различными температурами. Представлены решения для интенсивности излучения падающего излучения и результирующего теплового потока излучения в среде для заданного неоднородного источника, представленного полиноминальным разложением. Кроме того получено решение, опирающееся на функцию Грина, для данной задачи.