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Technical note

The critical problem with high-order anisotropic scattering

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Abstract

A synthetic scattering kernel is used with one-speed transport theory to evaluate the effect of high-order anisotropic scattering on the critical half thickness for a multiplying, unreflected, plane-parallel medium.

1. Introduction

We consider our problem to be defined by the one-speed transport equation (Case and Zweifel, 1967)

$$\mu \frac{\partial}{\partial \tau} \psi(\tau, \mu) + \psi(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) \int_{-1}^{1} P_l(\mu') \psi(\tau, \mu') \mathrm{d}\mu', \tag{1}$$

for $\mu \in [-1,1]$ and $\tau \in (-\tau_0, \tau_0)$, and the boundary conditions

$$\psi(\tau_0, -\mu) = 0, \quad \mu \in (0, 1], \tag{2a}$$

and

$$\psi(-\tau_0, \mu) = 0, \quad \mu \in (0, 1]. \tag{2b}$$

Here τ is the dimensionless spatial variable defined in terms of the total cross-section σ. In addition

$$\varpi = \sigma_s/\sigma, \tag{3}$$

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where σ_s is the scattering cross-section, and the β_l are the coefficients used to define an Lth-order Legendre expansion of the scattering law, i.e.

$$p(\cos\Theta) = \sum_{l=0}^{L} \beta_l P_l(\cos\Theta). \tag{4}$$

For pure scattering we would have $\beta_0 = 1$ in Eq. (4). However, to include isotropic fission, we write

$$\beta_0 = c/\varpi \tag{5}$$

where

$$c = \frac{\nu\sigma_f + \sigma_s}{\sigma}. \tag{6}$$

Here σ_f is the fission cross section and ν is the mean number of neutrons produced per fission. For the critical problem we consider that $\varpi \in (0,1)$ and $c > 1$ are given, along with the β_l, and we seek the critical half thickness τ_0 so that there exists a physically meaningful solution of Eqs. (1) and (2).

In this work, in order to study well the effect of anisotropic scattering on the critical thickness and to make it easy for researchers to evaluate other ways of solving this problem, we use the binomial scattering law (Kaper et al., 1970)

$$p(\cos\Theta) = \frac{L+1}{2^L}(1 + \cos\Theta)^L \tag{7}$$

for which the β coefficients can be computed from (McCormick and Sanchez, 1981)

$$\beta_l = \left(\frac{2l+1}{2l-1}\right)\left(\frac{L+1-l}{L+1+l}\right)\beta_{l-1} \tag{8}$$

with $\beta_0 = 1$. To be very clear, we note that once the β coefficients for the scattering law are computed from Eq. (8), with $\beta_0 = 1$, we then change β_0, as noted by Eq. (5), to take account of isotropic fission.

2. A solution

Here we use a new version (Barichello and Siewert, 1999) of the discrete-ordinates method recently used (Siewert, 2000) to solve Chandrasekhar's basic problem in radiative transfer (Chandrasekhar, 1950). Since much of what we require to solve the critical problem is already available (Siewert, 2000), our presentation here is brief. To begin we consider the symmetry condition

$$\psi(-\tau, -\mu) = \psi(\tau, \mu) \tag{9}$$

and write

$$\mathbf{\Psi}_{\pm}(\tau) = \sum_{j=1}^{N} A_j \big[\mathbf{\Phi}_{\pm}(\nu_j) \mathrm{e}^{-(\tau_0+\tau)/\nu_j} + \mathbf{\Phi}_{\mp}(\nu_j) \mathrm{e}^{-(\tau_0-\tau)/\nu_j} \big]. \tag{10}$$

Here $\mathbf{\Psi}_{\pm}(\tau)$ are vectors the components of which are the angular fluxes evaluated at $\pm\mu_k$ where the μ_k are the N quadrature points used to evaluate integrals defined on the interval [0,1]. In addition, the separation constants ν_j and the elementary vectors $\mathbf{\Phi}_{\pm}(\nu_j)$ are as defined previously (Siewert, 2000). Rewriting Eq. (2a) in terms of the quadrature points, we find

$$\mathbf{\Psi}_{-}(\tau_0) = \mathbf{0}, \tag{11}$$

and so we use Eq. (10) to obtain the condition

$$\sum_{j=1}^{N} A_j \big[\mathbf{\Phi}_{+}(\nu_j) + \mathbf{\Phi}_{-}(\nu_j) \mathrm{e}^{-2\tau_0/\nu_j} \big] = \mathbf{0}. \tag{12}$$

For the specific test cases considered in this work, we have found that only one of the separation constants, say ν_1, is imaginary, and so we let $\nu_1 = \mathrm{i}\eta$ and rewrite Eq. (12) as

$$\sin(\tau_0/\eta)\mathbf{\Phi}_I + \cos(\tau_0/\eta)\mathbf{\Phi}_R + \sum_{j=2}^{N} A_j \big[\mathbf{\Phi}_{+}(\nu_j) + \mathbf{\Phi}_{-}(\nu_j) \mathrm{e}^{-2\tau_0/\nu_j} \big] = \mathbf{0}. \tag{13}$$

Since Eq. (12) is homogeneous, we have, in obtaining Eq. (13), introduced the normalization

$$A_1 = (1/2)\mathrm{e}^{\tau_0/\nu_1} \tag{14}$$

and we have let $\mathbf{\Phi}_R$ and $\mathbf{\Phi}_I$ denote respectively the real and imaginary parts of the elementary vector $\mathbf{\Phi}_{+}(\nu_1)$. We can now divide Eq. (13) by $\cos(\tau_0/\eta)$ and rewrite that equation as

$$B_1 \mathbf{\Phi}_I + \sum_{j=2}^{N} B_j \big[\mathbf{\Phi}_{+}(\nu_j) + \mathbf{\Phi}_{-}(\nu_j) \mathrm{e}^{-2\tau_0/\nu_j} \big] = -\mathbf{\Phi}_R \tag{15}$$

Table 1
The critical half thickness for the case $\varpi = 0.9$

Model	$c=1.1$	$c=1.5$	$c=2.0$	$c=2.5$	$c=3.0$	$c=3.5$
$L=0$	2.113310	6.050565(−1)	3.110259(−1)	2.032464(−1)	1.482194(−1)	1.152581(−1)
$L=99$	3.171634	5.817321(−1)	2.691107(−1)	1.710129(−1)	1.238968(−1)	9.645482(−2)
$L=199$	3.151479	5.635481(−1)	2.590671(−1)	1.643650(−1)	1.190616(−1)	9.273101(−2)
$L=299$	3.141080	5.545421(−1)	2.539139(−1)	1.608814(−1)	1.164879(−1)	9.072364(−2)
$L=\infty$	3.110259	5.139124(−1)	2.244387(−1)	1.379532(−1)	9.774342(−2)	7.488080(−2)

where

$$B_1 = \tan(\tau_0/\eta). \tag{16}$$

If we consider that τ_0 is known then Eq. (15) is a system of N linear algebraic equations for the constants B_j, $j=1,2,...,N$. And so we use an iterative approach to find τ_0. We start our calculation with some assumed value of τ_0, we then solve the linear system defined by Eq. (15) and obtain an improved value for τ_0 from Eq. (16). Continuing this process, we found the numerical results given in the accompanying Table 1.

To conclude this work, we note that our table, in addition to providing some good test cases that can be used to evaluate new computational methods in transport theory, shows that the effect of anisotropic scattering on the critical half thickness can be considerable.

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References

Barichello, L.B., Siewert, C.E., 1999. A discrete-ordinates solution for a non-grey model with complete frequency redistribution. Journal of Quantitative Spectroscopy and Radiative Transfer 62, 665.

Case, K.M., Zweifel, P.F., 1967. Linear transport theory. Addison-Wesley, New York.

Chandrasekhar, S., 1950. Radiative transfer. Oxford University Press, London.

Garcia, R.D.M., Siewert, C.E., 1981. Numerical results concerning the effect of anisotropic scattering on the critical-slab problem. Journal of Physics D: Applied Physics 14, L65.

Kaper, H.G., Shultis, J.K., Veninga, J.G., 1970. Numerical evaluation of the slab albedo problem solution in one-speed anisotropic transport theory. Journal of Computational Physics 6, 288.

McCormick, N.J., Sanchez, R., 1981. Inverse problem transport calculations for anisotropic scattering coefficients. Journal of Mathematical Physics 22, 199.

Siewert, C.E., 2000. A concise and accurate solution to Chandrasekhar's basic problem in radiative transfer. Journal of Quantitative Spectroscopy and Radiative Transfer 64, 109.

Siewert, C.E., Williams, M.M.R., 1977. The effect of anisotropic scattering on the critical slab problem in neutron transport theory using a synthetic kernel. Journal of Physics D: Applied Physics 10, 2031.