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An analytical discrete-ordinates solution for dual-mode heat transfer in a cylinder

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Abstract

A modern analytical version of the discrete-ordinates method is used along with Hermite cubic splines and Newton's method to solve a class of coupled nonlinear radiation–conduction heat-transfer problems in a solid cylinder. Computational details of the solution are discussed, and the algorithm is implemented to establish high-quality results for various data sets which include some difficult cases. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The study of combined-mode (radiative and conductive) heat transfer [1,2] is of interest mainly because of engineering applications where, for example, this coupling must be included in the analysis in order to model well the thermal behavior of some materials. We note that Andre and Degiovanni [3,4], Banoczi and Kelley [5] and Klar and Siedow [6] point out that the thermal properties of semi-transparent materials such as glass, polymers, and paper, as well as certain insulating materials, should be discussed in the context of dual-model heat-transfer models. In regard to works [7,8] devoted to this class of radiative-transfer problems, we note that each of the two basic texts [1,2] has a review and a discussion of some methods for developing working solutions to these problems. Of course, because of the considerable mathematical and

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computational complications encountered in these coupled nonlinear heat-transfer problems, it is a challenging task to have an efficient and accurate algorithm for computing all quantities of interest. These aspects of efficiency and accuracy are especially significant when, for example, methods based on iterative schemes (where the direct problem must be solved many times) are to be used to investigate inverse problems in combined-mode heat transfer [9]. In addition to the difficulty of establishing and implementing computational methods for the considered nonlinear problems, the concepts of existence and uniqueness of the solutions to such problems are often ignored issues (also not discussed here) that, in our opinion, warrant additional study along the lines of the work of Kelley [10].

In a paper published in 1991, Siewert and Thomas [11] used a stable version of the spherical-harmonics method [12] to solve some basic problems in combined-mode, radiation and conduction heat transfer [1] for plane–parallel media. Although some good results were reported in Ref. [11], it was also mentioned there that the simple iteration scheme used in that work could fail (very dramatically) for some cases. In 1995, these results were improved by developing [13] an iterative method, based on Newton's method, that proved to be more successful than previous work in solving difficult cases.

In this work we consider a nonlinear dual-mode heat-transfer problem [14] in a solid cylinder, and in order to solve this class of problems, and to include cases with a strong interaction between the two modes of heat transfer, we make use of a modern analytical version [15] of the discrete-ordinates method [16], Hermite cubic splines [17] and Newton's method to establish the desired solution. We therefore consider the equation of transfer written as

$$\left[(1-\mu^2)^{1/2}\left(\cos\phi\,\frac{\partial}{\partial r}-\frac{1}{r}\sin\phi\,\frac{\partial}{\partial\phi}\right)+1\right]I(r,\mu,\phi)=\frac{\varpi}{\pi}\int_0^1\int_0^\pi I(r,\mu',\phi')\,\mathrm{d}\phi'\,\mathrm{d}\mu'+S(r) \tag{1}$$

for $r\in(0,R)$, $\mu\in[0,1]$ and $\phi\in(0,\pi)$. Here the source term is

$$S(r)=(1-\varpi)\Theta^4(r), \tag{2}$$

and we consider the boundary condition, subject to which we must solve Eq. (1), written as

$$I(R,\mu,\phi)=\varepsilon\Theta_0^4+\frac{4\rho}{\pi}\int_0^1\int_0^{\pi/2} I(R,\mu',\phi')(1-\mu'^2)^{1/2}\cos\phi'\,\mathrm{d}\phi'\,\mathrm{d}\mu' \tag{3}$$

for $\mu\in[0,1]$ and $\phi\in[\pi/2,\pi]$. In addition, the normalized [14] temperature distribution $\Theta(r)$ must satisfy the conduction equation and boundary conditions which we write as

$$r\frac{\mathrm{d}^2}{\mathrm{d}r^2}\,\Theta(r)+\frac{\mathrm{d}}{\mathrm{d}r}\,\Theta(r)=\frac{1}{N_\mathrm{c}}\frac{\mathrm{d}}{\mathrm{d}r}\,[rq_\mathrm{r}(r)]-rH \tag{4}$$

and

$$\Theta(R)=\Theta_0\quad\text{and}\quad\frac{\mathrm{d}}{\mathrm{d}r}\Theta(r)\Big|_{r=0}=0. \tag{5}$$

In continuation, we note that since the radiative heat flux

$$q_{\rm r}(r)=\frac{1}{\pi}\int_0^1\int_0^\pi I(r,\mu,\phi)(1-\mu^2)^{1/2}\cos\phi\,{\rm d}\phi\,{\rm d}\mu \tag{6}$$

appears in Eq. (4), it is clear that the two problems (radiation and conduction) are coupled.

In regard to our basic variables, we note that $r\in[0,R]$ is the radial spatial variable (in dimensionless units) and that R is the radius of the cylinder. In addition $\mu=\cos\theta$ and ϕ are the two angular variables that define the direction of photon propagation. To define the physical parameters used here we note that ϖ is the albedo for single scattering, ε is the emissivity of the surface, ρ is the coefficient for diffuse reflection by the surface,

$$N_{\rm c}=\frac{k\beta}{4\sigma n^2 T_{\rm r}^3} \tag{7}$$

is the conduction-to-radiation parameter [1] and

$$H=(k\beta^2 T_{\rm r})^{-1}h. \tag{8}$$

In addition, σ is the Stefan–Boltzmann constant, n is the index of refraction, β is the extinction coefficient, $T_{\rm r}$ is a reference temperature, h is a constant that measures the prescribed heat generation in the medium, k is the thermal conductivity and Θ_0 is the prescribed (normalized) temperature at the surface.

2. A reformulation

Noting that the right-hand side of Eq. (3) is defined in terms of the unknown quantity $I(R,\mu,\phi)$, we find it convenient to split, as was done in Ref. [14], our basic problem into two simpler problems that have boundary conditions defined by known quantities, and so we write

$$I(r,\mu,\phi)=\psi(r,\mu,\phi)+\gamma f(r,\mu,\phi), \tag{9}$$

where, first of all, $f(r,\mu,\phi)$ is a solution of the albedo problem defined by

$$\left[(1-\mu^2)^{1/2}\left(\cos\phi\frac{\partial}{\partial r}-\frac{1}{r}\sin\phi\frac{\partial}{\partial\phi}\right)+1\right]f(r,\mu,\phi)=\frac{\varpi}{\pi}\int_0^1\int_0^\pi f(r,\mu',\phi')\,{\rm d}\phi'\,{\rm d}\mu', \tag{10}$$

for $r\in(0,R)$, $\mu\in[0,1]$ and $\phi\in(0,\pi)$, and the boundary condition

$$f(R,\mu,\phi)=1,\qquad \mu\in[0,1]\quad\text{and}\quad\phi\in[\pi/2,\pi]. \tag{11}$$

We then seek $\psi(r,\mu,\phi)$ such that

$$\left[(1-\mu^2)^{1/2}\left(\cos\phi\frac{\partial}{\partial r}-\frac{1}{r}\sin\phi\frac{\partial}{\partial\phi}\right)+1\right]\psi(r,\mu,\phi)=\frac{\varpi}{\pi}\int_0^1\int_0^\pi \psi(r,\mu',\phi')\,{\rm d}\phi'\,{\rm d}\mu'+S(r), \tag{12}$$

for $r \in (0, R)$, $\mu \in [0, 1]$ and $\phi \in (0, \pi)$, and

$$\psi(R, \mu, \phi) = 0, \qquad \mu \in [0, 1] \quad \text{and} \quad \phi \in [\pi/2, \pi]. \tag{13}$$

With these definitions it follows from foregoing equations that the constant γ is defined by

$$\gamma = (1 - \rho A^*)^{-1}[\varepsilon \Theta_0^4 + 4\rho\psi_1(R)], \tag{14}$$

where, in general,

$$\psi_1(r) = \frac{1}{\pi}\int_0^1 \int_0^\pi \psi(r, \mu, \phi)(1 - \mu^2)^{1/2} \cos\phi \, d\phi \, d\mu. \tag{15}$$

In addition, the albedo

$$A^* = \frac{4}{\pi}\int_0^1 \int_0^{\pi/2} f(R, \mu, \phi)(1 - \mu^2)^{1/2} \cos\phi \, d\phi \, d\mu \tag{16}$$

can be expressed as [14]

$$A^* = 1 + 4 f_1(R), \tag{17}$$

where, in general,

$$f_1(r) = \frac{1}{\pi}\int_0^1 \int_0^\pi f(r, \mu, \phi)(1 - \mu^2)^{1/2} \cos\phi \, d\phi \, d\mu. \tag{18}$$

It follows that once we have solved the f and ψ problems we can compute the radiative heat flux from

$$q_r(r) = \psi_1(r) + \gamma f_1(r) \tag{19}$$

which can then be used in Eq. (4), the solution of which we write as

$$\Theta(r) = \Theta_0 + \frac{1}{4}(R^2 - r^2)H - \frac{1}{N_c}\int_r^R q_r(x) \, dx. \tag{20}$$

It is clear that the albedo problem defined by Eqs. (10) and (11) is independent of the temperature distribution $\Theta(r)$ and so can be solved in a direct manner. On the other hand, we see that the ψ problem defined by Eqs. (12) and (13) requires the temperature distribution as shown by Eq. (2). We see also that the temperature distribution, as given by Eq. (20), depends on both the albedo problem and the ψ problem. It is for this reason that we must, in general, solve the temperature problem and the ψ problem, simultaneously.

3. Pseudo problems

Having reformulated our basic problems to be solved, we now make use of some useful transformations, due to Mitsis [18] and generalized in Ref. [19], that allow us to express the solutions we seek in terms of two "pseudo problems." First of all, from Eq. (10) we can conclude that

$$f_1(r) = -\frac{1}{r}(1-\varpi)\int_0^r x f_0(x)\,\mathrm{d}x, \tag{21}$$

where

$$f_0(r) = \frac{1}{\pi}\int_0^1\int_0^\pi f(r,\mu,\phi)\,\mathrm{d}\phi\,\mathrm{d}\mu, \tag{22}$$

and so we consider $f_0(r)$ to be the basic result we require from the albedo problem. It has been shown [19] that $f_0(r)$ can be expressed as

$$f_0(r) = \int_0^1 F(r,\mu)\,\mathrm{d}\mu, \tag{23}$$

where $F(r,\mu)$ is defined by

$$\left[\mu^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) - 1\right]F(r,\mu) + \varpi\int_0^1 F(r,\mu')\,\mathrm{d}\mu' = 0 \tag{24}$$

for $r \in (0,R)$ and $\mu \in [0,1]$, with

$$F(R,\mu) + \Upsilon(\mu)\frac{\partial}{\partial r}F(r,\mu)\bigg|_{r=R} = 1, \quad \mu \in [0,1]. \tag{25}$$

Here

$$r(\mu) = \mu\frac{K_0(R/\mu)}{K_1(R/\mu)}, \tag{26}$$

where $K_0(z)$ and $K_1(z)$ are modified Bessel functions. Turning now to the ψ problem, we use Eq. (2) and find from Eq. (12) that

$$\psi_1(r) = \frac{1}{r}(1-\varpi)\int_0^r x[\Theta^4(x) - \psi_0(x)]\,\mathrm{d}x, \tag{27}$$

where

$$\psi_0(r) = \frac{1}{\pi}\int_0^1\int_0^\pi \psi(r,\mu,\phi)\,\mathrm{d}\phi\,\mathrm{d}\mu. \tag{28}$$

We note also that $\psi_0(r)$ can be expressed as [19]

$$\psi_0(r) = \int_0^1 \Psi(r,\mu)\,\mathrm{d}\mu, \tag{29}$$

where $\Psi(r,\mu)$ is defined by

$$\left[\mu^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) - 1\right]\Psi(r,\mu) + \varpi\int_0^1 \Psi(r,\mu')\,\mathrm{d}\mu' + (1-\varpi)\Theta^4(r) = 0 \tag{30}$$

for $r \in (0, R)$ and $\mu \in [0, 1]$, with

$$\Psi(R,\mu) + r(\mu)\frac{\partial}{\partial r}\Psi(r,\mu)\bigg|_{r=R} = 0, \quad \mu \in [0,1]. \tag{31}$$

Once we have solved the albedo problem, to establish the albedo A^* and $f_1(r)$, and the coupled ψ and temperature problems, to yield $\psi_1(r)$ and $\Theta(r)$, then we can compute the (normalized) conductive, radiative and total heat fluxes which we express as

$$Q_c(r) = \frac{r}{2}H - \frac{1}{N_c}[\psi_1(r) + \gamma f_1(r)], \tag{32}$$

$$Q_r(r) = \frac{1}{N_c}[\psi_1(r) + \gamma f_1(r)] \tag{33}$$

and

$$Q(r) = \frac{r}{2}H. \tag{34}$$

So we proceed to use the discrete-ordinates method, Hermite cubic splines and Newton's method to develop the solutions we require.

4. The albedo problem

In order to start our discrete-ordinates solution of the albedo problem, expressed in terms of the pseudo problem defined by Eqs. (24) and (25), we approximate the integral term in Eq. (24) by a quadrature formula and write our discrete-ordinates equations as

$$\left[\mu_i^2\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right) - 1\right]F(r,\mu_i) + \varpi\sum_{k=1}^{N} w_k F(r,\mu_k) = 0 \tag{35}$$

for $i = 1, 2, \ldots, N$. In writing Eq. (35) as we have, we clearly are considering that the N quadrature points $\{\mu_k\}$ and the N weights $\{w_k\}$ are defined for use on the integration interval $[0, 1]$. Seeking a Bessel function solution (bounded as $r \to 0$) of Eq. (35), we substitute

$$F(r,\mu_i) = \phi(v,\mu_i)I_0(r/v) \tag{36}$$

into Eq. (35) to find

$$(v^2 - \mu_i^2)\phi(v,\mu_i) = \varpi v^2 \sum_{k=1}^{N} w_k \phi(v,\mu_k) \tag{37}$$

for $i = 1, 2, \ldots, N$. Now if we let $\phi(v, \mu_k)$, $k = 1, 2, \ldots, N$, define the elements of an N vector $\boldsymbol{\Phi}(v)$ we can rewrite Eq. (37) as

$$(\boldsymbol{I} - \lambda \boldsymbol{M}^2)\boldsymbol{\Phi}(v) = \varpi \boldsymbol{W}\boldsymbol{\Phi}(v), \tag{38}$$

where $\lambda = 1/v^2$, $\boldsymbol{I}$ is the $N \times N$ identity matrix, the elements of the $N \times N$ matrix $\boldsymbol{W}$ are given by

$$(\boldsymbol{W})_{i,j} = w_j \tag{39}$$

and

$$\boldsymbol{M} = \mathrm{diag}\{\mu_1, \mu_2, \ldots, \mu_N\}. \tag{40}$$

We note that, not surprisingly, the eigenvalue problem defined by Eq. (38) is essentially the one encountered in Refs. [15,20] in the discrete-ordinates solutions of similar problems in plane geometry, and so we take advantage of those works and rewrite Eq. (38) in the special form [21]

$$(\boldsymbol{D} - \varpi \boldsymbol{z}\boldsymbol{z}^{\mathrm{T}})\boldsymbol{X} = \lambda \boldsymbol{X}, \tag{41}$$

where the superscript T denotes the transpose operation,

$$\boldsymbol{D} = \mathrm{diag}\{\mu_1^{-2}, \mu_2^{-2}, \ldots, \mu_N^{-2}\} \tag{42}$$

and

$$\boldsymbol{z} = [w_1^{1/2}\mu_1^{-1} \quad w_2^{1/2}\mu_2^{-1} \quad \cdots \quad w_N^{1/2}\mu_N^{-1}]^{\mathrm{T}}. \tag{43}$$

Continuing, we note that the eigenvalue problem defined by Eq. (41) is of a form that is encountered when the so-called "divide-and-conquer" method [22] is used to find the eigenvalues of tridiagonal matrices. In addition, we see from Eq. (43) that, because of the way in which our basic eigenvalue problem is formulated, we must exclude zero from the set of quadrature points. Of course to exclude zero from the quadrature set is not considered a serious restriction since typical Gauss quadrature schemes do not include the end points of the integration interval.

Considering that we have found the eigenvalues and eigenvectors, λ_j and $\boldsymbol{X}_j$, from Eq. (41), we use

$$v_j = \lambda_j^{-1/2} \tag{44}$$

and

$$\boldsymbol{\Phi}(v_j) = \boldsymbol{S}^{-1}\boldsymbol{X}_j, \tag{45}$$

where

$$\boldsymbol{S} = \mathrm{diag}\{w_1^{1/2}\mu_1, w_2^{1/2}\mu_2, \ldots, w_N^{1/2}\mu_N\}, \tag{46}$$

and write our discrete-ordinates solution as

$$F(r, \mu_i) = \sum_{j=1}^{N} A_j \phi(v_j, \mu_i) \hat{I}_0(r/v_j) \mathrm{e}^{-(R-r)/v_j}. \tag{47}$$

Here the A_j are constants to be determined from the boundary condition

$$F(R, \mu_i) + \Upsilon(\mu_i) \frac{\partial}{\partial r} F(r, \mu_i)\bigg|_{r=R} = 1 \tag{48}$$

for $i=1,2,\dots,N$, with

$$\Upsilon(\mu)=\mu\,\frac{\hat{K}_0(R/\mu)}{\hat{K}_1(R/\mu)}. \tag{49}$$

For computational reasons we use

$$\hat{I}_n(x)=I_n(x)\mathrm{e}^{-x} \quad \text{and} \quad \hat{K}_n(x)=K_n(x)\mathrm{e}^{x}, \tag{50a,b}$$

and to be very clear, we note that $\phi(v_j,\mu_i)$ is the ith element of $\boldsymbol{\Phi}(v_j)$. It is clear that the vectors $\boldsymbol{\Phi}(v_j)$, and thus the elements $\phi(v_j,\mu_i)$, are available from Eq. (45) and the eigenvectors defined by Eq. (41). On the other hand, we can use only the eigenvalues defined by Eq. (41), along with Eq. (44), and then use the analytical expression

$$\phi(v_j,\mu_i)=\frac{\varpi v_j^2}{v_j^2-\mu_i^2}K(v_j), \tag{51a}$$

where

$$K(v_j)=\sum_{i=1}^{N} w_i\phi(v_j,\mu_i) \tag{51b}$$

is, in fact, arbitrary. It follows that Eq. (47) can be used in a discrete-ordinates version of Eq. (23) to find

$$f_0(r)=\sum_{j=1}^{N} A_jK(v_j)\hat{I}_0(r/v_j)\mathrm{e}^{-(R-r)/v_j}, \tag{52}$$

and this result can be used in Eq. (21) to obtain

$$f_1(r)=-(1-\varpi)\sum_{j=1}^{N} v_jA_jK(v_j)\hat{I}_1(r/v_j)\mathrm{e}^{-(R-r)/v_j}. \tag{53}$$

So, all that we require from the albedo problem is established.

5. A particular solution and the spline-driven problems

Looking now at Eq. (30), we note that there is a source term in that equation, and so to develop a discrete-ordinates solution to the problem defined by Eqs. (30) and (31) we must develop a particular solution that can be used with the elementary solutions just employed to solve the albedo problem. So, to be general, we consider a discrete-ordinates version of Eq. (30) written as

$$\left[\mu_i^2\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)-1\right]\Psi(r,\mu_i)+\varpi\sum_{k=0}^{N} w_k\Psi(r,\mu_k)+S(r)=0 \tag{54}$$

for $i=1,2,\dots,N$. At this point we consider that the source term $S(r)$ is known, and so we make use of the method of variation of parameters and express our particular solution as

$$\Psi_{\rm ps}(r,\mu_i)=\sum_{j=1}^{N} C_j\phi(v_j,\mu_i)[V(r,v_j)\hat{I}_0(r/v_j)+U(r,v_j)\hat{K}_0(r/v_j)], \tag{55}$$

where the constants C_j and the functions $U(r,v_j)$ and $V(r,v_j)$ are to be found. We now substitute Eq. (55) into Eq. (54) to find

$$\mu_i^2\sum_{j=1}^{N} C_j\phi(v_j,\mu_i)=1, \quad i=1,2,\dots,N, \tag{56}$$

with

$$U(r,v_j)=\int_0^r xS(x)\hat{I}_0(x/v_j)\,{\rm e}^{-(r-x)/v_j}\,{\rm d}x \tag{57a}$$

and

$$V(r,v_j)=\int_r^R xS(x)\hat{K}_0(x/v_j)\,{\rm e}^{-(x-r)/v_j}\,{\rm d}x. \tag{57b}$$

Clearly, Eq. (56) defines a system of linear algebraic equations for the required constants C_j. We can now use some properties of the elementary solutions $\phi(v_j,\mu_i)$ to solve this system (with, of course, the implicitly made assumption that a solution of the system exists). First of all we make use of Eq. (51b) and write Eq. (37) as

$$(1-\mu_i^2/v_j^2)\phi(v_j,\mu_i)=\varpi K(v_j). \tag{58}$$

Following a well-known procedure, we multiply Eq. (58) by $w_i\phi(v_k,\mu_i)$ and sum over i to obtain

$$\sum_{i=1}^{N} w_i\phi(v_j,\mu_i)\phi(v_k,\mu_i)-\frac{1}{v_j^2}\sum_{i=1}^{N} w_i\mu_i^2\phi(v_j,\mu_i)\phi(v_k,\mu_i)=\varpi K(v_j)K(v_k). \tag{59}$$

Interchange j and k in Eq. (59) to find

$$\sum_{i=1}^{N} w_i\phi(v_j,\mu_i)\phi(v_k,\mu_i)-\frac{1}{v_k^2}\sum_{i=1}^{N} w_i\mu_i^2\phi(v_j,\mu_i)\phi(v_k,\mu_i)=\varpi K(v_j)K(v_k). \tag{60}$$

Subtract Eq. (59) from Eq. (60) to deduce that

$$\sum_{i=1}^{N} w_i\mu_i^2\phi(v_j,\mu_i)\phi(v_k,\mu_i)=N(v_j)\delta_{j,k}, \tag{61}$$

where

$$N(v_j)=\sum_{i=1}^{N} w_i\mu_i^2[\phi(v_j,\mu_i)]^2. \tag{62}$$

We can now multiply Eq. (56) by $w_i\phi(v_k,\mu_i)$ and sum over i to find

$$C_k=K(v_k)/N(v_k), \tag{63}$$

and so the required particular solution is established.

As we intend to use Hermite cubic splines [17], as defined in the following section of this paper, to represent the source term in Eq. (30), we now let $S(r)$ be one of the spline functions $\mathscr{F}_k(x)$, for $k=0,1,\dots,K$, and write our solution to

$$\left[\mu_i^2\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)-1\right]\Psi_k(r,\mu_i)+\varpi\sum_{\alpha=1}^{N}w_\alpha\Psi_k(r,\mu_\alpha)+\mathscr{F}_k(r/R)=0 \tag{64}$$

as

$$\Psi_k(r,\mu_i)=\sum_{j=1}^{N}A_{k,j}\phi(v_j,\mu_i)\hat{I}_0(r/v_j)\,\mathrm{e}^{-(R-r)/v_j}+\Psi_{k,\mathrm{ps}}(r,\mu_i), \tag{65}$$

where the particular solution is

$$\Psi_{k,\mathrm{ps}}(r,\mu_i)=\sum_{j=1}^{N}C_j\phi(v_j,\mu_i)[V_k(r,v_j)\hat{I}_0(r/v_j)+U_k(r,v_j)\hat{K}_0(r/v_j)] \tag{66}$$

with

$$U_k(r,v_j)=\int_0^r x\mathscr{F}_k(x/R)\hat{I}_0(x/v_j)\,\mathrm{e}^{-(r-x)/v_j}\,\mathrm{d}x \tag{67a}$$

and

$$V_k(r,v_j)=\int_r^R x\mathscr{F}_k(x/R)\hat{K}_0(x/v_j)\,\mathrm{e}^{-(x-r)/v_j}\,\mathrm{d}x. \tag{67b}$$

To find the constants $A_{k,j}$, we substitute Eq. (65) into a discrete-ordinates version of Eq. (31) to find

$$\sum_{j=1}^{N}A_{k,j}\phi(v_j,\mu_i)[\hat{I}_0(R/v_j)+(1/v_j)\Upsilon(\mu_i)\hat{I}_1(R/v_j)]=R_k(\mu_i) \tag{68}$$

for $i=1,2,\dots,N$. Here,

$$R_k(\mu_i)=-\Psi_{k,\mathrm{ps}}(R,\mu_i)-\Upsilon(\mu_i)\left.\frac{\partial}{\partial r}\Psi_{k,\mathrm{ps}}(r,\mu_i)\right|_{r=R} \tag{69}$$

or

$$R_k(\mu_i)=-\sum_{j=1}^{N}C_j\phi(v_j,\mu_i)U_k(R,v_j)[\hat{K}_0(R/v_j)-(1/v_j)\Upsilon(\mu_i)\hat{K}_1(R/v_j)]. \tag{70}$$

Of course once the required constants are found we can evaluate a discrete-ordinates version of

$$\psi_{k,0}(r)=\int_0^1\Psi_k(r,\mu)\,\mathrm{d}\mu \tag{71}$$

to find, after imposing the (arbitrary) normalization

$$K(v_j) = 1, \tag{72}$$

$$\psi_{k,0}(r) = \sum_{j=1}^{N} \{A_{k,j}\hat{I}_0(r/v_j)\mathrm{e}^{-(R-r)/v_j} + C_j[V_k(r,v_j)\hat{I}_0(r/v_j) + U_k(r,v_j)\hat{K}_0(r/v_j)]\}. \tag{73}$$

We also define

$$\psi_{k,1}(r) = \frac{1}{r}\int_0^r x[\mathscr{F}_k(x/R) - (1-\varpi)\psi_{k,0}(x)]\,\mathrm{d}x \tag{74}$$

and find, after using Eq. (73) and noting in particular Eqs. (56), (58) and (72), that

$$\psi_{k,1}(r) = -(1-\varpi)\sum_{j=1}^{N} v_j[A_{k,j}\hat{I}_1(r/v_j)\,\mathrm{e}^{-(R-r)/v_j} + C_j S_k(r,v_j)], \tag{75}$$

where

$$S_k(r,v_j) = V_k(r,v_j)\hat{I}_1(r/v_j) - U_k(r,v_j)\hat{K}_1(r/v_j). \tag{76}$$

6. The spline functions

The Hermite cubic spline functions we use in this work are taken from Schultz [17] and were also used in the context of dual-mode heat transfer in Ref. [11]. To be specific and to define the notation we use, we list these splines here. First of all, we consider there to be $M+1$ knots ζ_α defined on the interval [0,1] by

$$\zeta_\alpha = (\alpha/M)^m, \quad \alpha = 0,1,\dots,M. \tag{77}$$

In this work, we use either the linear distribution ($m=1$) or the quadratic distribution ($m=2$). So to approximate a function, say $Y(r)$ defined on the interval $[0,R]$, in terms of the spline functions we write

$$Y(r) = \sum_{\alpha=0}^{K} a_\alpha \mathscr{F}_\alpha(r/R), \tag{78}$$

where the a_α are constants and where $K = 2M+1$. We note that there are two spline functions $\mathscr{F}_\alpha(x)$ associated with each knot and that the spline functions are defined differently for even or odd values of α. So we write

$$\mathscr{F}_{2\beta}(x) = \Phi_\beta(x) \quad \text{and} \quad \mathscr{F}_{2\beta+1}(x) = \Psi_\beta(x) \tag{79a,b}$$

for $\beta = 0,1,\dots,M$. Making use of the definitions

$$p_\alpha(x) = \frac{x - \zeta_{\alpha-1}}{\zeta_\alpha - \zeta_{\alpha-1}} \tag{80a}$$

and

$$g_\alpha(x) = \frac{\zeta_{\alpha+1} - x}{\zeta_{\alpha+1} - \zeta_\alpha} \tag{80b}$$

and considering that the spline functions are zero unless otherwise defined, we can write the Φ functions as

$$\Phi_0(x) = g_0^2(x)[3 - 2g_0(x)], \quad x \in [\zeta_0, \zeta_1], \tag{81a}$$

$$\Phi_\alpha(x) = \begin{cases} p_\alpha^2(x)[3 - 2p_\alpha(x)], & x \in [\zeta_{\alpha-1}, \zeta_\alpha], \\ g_\alpha^2(x)[3 - 2g_\alpha(x)], & x \in [\zeta_\alpha, \zeta_{\alpha+1}], \end{cases} \tag{81b}$$

for $\alpha = 1, 2, \ldots, M - 1$, and

$$\Phi_M(x) = p_M^2(x)[3 - 2p_M(x)], \quad x \in [\zeta_{M-1}, \zeta_M]. \tag{81c}$$

In a similar way we can write Ψ functions as

$$\Psi_0(x) = x g_0^2(x), \quad x \in [\zeta_0, \zeta_1], \tag{82a}$$

$$\Psi_\alpha(x) = \begin{cases} (x - \zeta_\alpha) p_\alpha^2(x), & x \in [\zeta_{\alpha-1}, \zeta_\alpha], \\ (x - \zeta_\alpha) g_\alpha^2(x), & x \in [\zeta_\alpha, \zeta_{\alpha+1}], \end{cases} \tag{82b}$$

for $\alpha = 1, 2, \ldots, M - 1$, and

$$\Psi_M(x) = (x - \zeta_M) p_M^2(x), \quad x \in [\zeta_{M-1}, \zeta_M]. \tag{82c}$$

To conclude this discussion, we note [17] that it is a property of the Hermite cubic splines that the coefficients in Eq. (78) can be computed from

$$a_{2\alpha} = Y(r)|_{r=\zeta_\alpha R} \tag{83a}$$

and

$$a_{2\alpha+1} = R \frac{\mathrm{d}}{\mathrm{d}r} Y(r)\bigg|_{r=\zeta_\alpha R} \tag{83b}$$

for $\alpha = 0, 1, \ldots, M$.

7. The coupled problems

Summarizing our development to this point, we note that the temperature distribution is given by

$$\Theta(r) = \Theta_0 + \frac{1}{4}(R^2 - r^2)H - \frac{1}{N_\mathrm{c}} \int_r^R q_\mathrm{r}(x)\,\mathrm{d}x, \tag{84}$$

where

$$q_\mathrm{r}(r) = \psi_1(r) + \gamma f_1(r) \tag{85}$$

and where $f_1(r)$ is given by Eq. (53),

$$\gamma = (1 - \rho A^*)^{-1}[\varepsilon \Theta_0^4 + 4\rho \psi_1(R)] \tag{86}$$

and

$$A^* = 1 + 4f_1(R). \tag{87}$$

Recalling that

$$\psi_1(r) = \frac{1}{r}(1-\varpi)\int_0^r x[\Theta^4(x) - \psi_0(x)]\,\mathrm{d}x, \tag{88}$$

where

$$\psi_0(r) = \int_0^1 \Psi(r,\mu)\,\mathrm{d}\mu, \tag{89}$$

we must simultaneously consider Eq. (84) and

$$\left[\mu^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) - 1\right]\Psi(r,\mu) + \varpi\int_0^1 \Psi(r,\mu')\,\mathrm{d}\mu' + (1-\varpi)\Theta^4(r) = 0 \tag{90}$$

for $r \in (0,R)$ and $\mu \in [0,1]$, with

$$\Psi(R,\mu) + \Upsilon(\mu)\,\frac{\partial}{\partial r}\Psi(r,\mu)\bigg|_{r=R} = 0, \quad \mu \in [0,1]. \tag{91}$$

So now we introduce a spline representation of the source term and write

$$(1-\varpi)\Theta^4(r) = \sum_{k=0}^{K} a_k \mathscr{F}_k(r/R), \tag{92}$$

where the constants a_k are to be determined. Since we have expressed the source term in Eq. (90) in terms of splines, we can now write

$$\psi_1(r) = \sum_{k=0}^{K} a_k \psi_{k,1}(r), \tag{93}$$

where $\psi_{k,1}(r)$ is given by Eq. (75). So to complete our solution we simply have to determine the required constants $\{a_k\}$. To find defining equations for these constants, we use Eqs. (92) and (93) in Eq. (84) to obtain

$$\sum_{k=0}^{K} a_k \mathscr{F}_k(r/R) = (1-\varpi)\left[\Gamma(r) + \sum_{k=0}^{K} a_k \Gamma_k(r)\right]^4, \tag{94}$$

where

$$\Gamma(r) = \Theta_0 + \frac{1}{4}(R^2 - r^2)H + \frac{1}{N_\mathrm{c}}L(r)(1-\rho A^*)^{-1}\varepsilon\Theta_0^4 \tag{95}$$

and

$$\Gamma_k(r) = \frac{1}{N_\mathrm{c}}[4\rho(1-\rho A^*)^{-1}L(r)\psi_{k,1}(R) + L_k(r)]. \tag{96}$$

Here

$$L(r) = (1-\varpi)\sum_{j=1}^{N} v_j^2 A_j[\hat{I}_0(R/v_j) - \hat{I}_0(r/v_j)\mathrm{e}^{-(R-r)/v_j}] \tag{97}$$

and

$$L_k(r) = (1-\varpi)\sum_{j=1}^{N} v_j^2\{A_{k,j}[\hat{I}_0(R/v_j) - \hat{I}_0(r/v_j)\mathrm{e}^{-(R-r)/v_j}] + C_j T_k(r, v_j)\} \tag{98}$$

with

$$T_k(r, v_j) = U_k(R, v_j)\hat{K}_0(R/v_j) - U_k(r, v_j)\hat{K}_0(r/v_j) - V_k(r, v_j)\hat{I}_0(r/v_j). \tag{99}$$

In order to generate from Eq. (94) a finite set of discrete equations from which we can determine the coefficients $\{a_k\}$ required in the approximation given by Eq. (92), we follow procedures typically used when working with splines: we evaluate Eq. (94) and the derivative (with respect to r) of that equation at $r_\alpha = \zeta_\alpha R$. In this way, we obtain (since $K = 2M + 1$) the system of $K + 1$ nonlinear algebraic equations

$$\sum_{k=0}^{K} a_k \mathscr{F}_k(\zeta_\alpha) = (1-\varpi)\left[\Gamma(r_\alpha) + \sum_{k=0}^{K} a_k \Gamma_k(r_\alpha)\right]^4 \tag{100a}$$

and

$$\sum_{k=0}^{K} a_k \mathscr{F}'_k(\zeta_\alpha) = 4R(1-\varpi)\left[\Gamma(r_\alpha) + \sum_{k=0}^{K} a_k \Gamma_k(r_\alpha)\right]^3 \left[\Gamma'(r_\alpha) + \sum_{k=0}^{K} a_k \Gamma'_k(r_\alpha)\right] \tag{100b}$$

for $\alpha = 0, 1, 2, \ldots, M$. We note from the basic definitions given by Eqs. (81) and (82) that

$$\mathscr{F}_{2k}(\zeta_\alpha) = \delta_{\alpha,k} \tag{101a}$$

and

$$\mathscr{F}_{2k+1}(\zeta_\alpha) = 0 \tag{101b}$$

and that

$$\mathscr{F}'_{2k}(\zeta_\alpha) = 0 \tag{102a}$$

and

$$\mathscr{F}'_{2k+1}(\zeta_\alpha) = \delta_{\alpha,k} \tag{102b}$$

for $k, \alpha = 0, 1, 2, \ldots, M$. So we can rewrite Eqs. (100) as

$$a_{2\alpha} = (1-\varpi)\left[\Gamma(r_\alpha) + \sum_{k=0}^{K} a_k \Gamma_k(r_\alpha)\right]^4 \tag{103a}$$

and

$$a_{2\alpha+1} = 4R(1-\varpi)\left[\Gamma(r_\alpha) + \sum_{k=0}^{K} a_k \Gamma_k(r_\alpha)\right]^3 \left[\Gamma'(r_\alpha) + \sum_{k=0}^{K} a_k \Gamma'_k(r_\alpha)\right] \tag{103b}$$

for $\alpha = 0, 1, 2, \ldots, M$. Of course we can differentiate Eqs. (95) and (96) to find the derivatives we require in Eq. (103b). So we can write

$$\Gamma'(r) = -\frac{1}{2} rH + \frac{1}{N_c} L'(r)(1 - \rho A^*)^{-1} \varepsilon \Theta_0^4 \tag{104}$$

and

$$\Gamma_k'(r) = \frac{1}{N_c}[4\rho(1 - \rho A^*)^{-1} L'(r)\psi_{k,1}(R) + L_k'(r)], \tag{105}$$

where

$$L'(r) = -(1 - \varpi) \sum_{j=1}^{N} v_j A_j \hat{I}_1(r/v_j) \mathrm{e}^{-(R-r)/v_j} \tag{106}$$

and

$$L_k'(r) = (1 - \varpi) \sum_{j=1}^{N} v_j \{-A_{k,j} \hat{I}_1(r/v_j) \mathrm{e}^{-(R-r)/v_j} + C_j v_j T_k'(r, v_j)\} \tag{107}$$

with

$$v_j T_k'(r, v_j) = U_k(r, v_j) \hat{K}_1(r/v_j) - V_k(r, v_j) \hat{I}_1(r/v_j). \tag{108}$$

Now to complete our solution we must solve the nonlinear system of algebraic equations, defined by Eqs. (103), to find the required constants $\{a_k\}$. To have an approach that can be effective in some simple cases we can use a direct iterative procedure to solve Eqs. (103). We thus start our iteration process with the initial values

$$a_{2\alpha} = (1 - \varpi)\Theta_0^4 \tag{109a}$$

and

$$a_{2\alpha+1} = 0 \tag{109b}$$

for $\alpha = 0, 1, 2, \ldots, M$, and these results can now be used on the right-hand sides of Eqs. (103) to get the next iterates. This process clearly can be continued. However, we have found cases where this simple iteration procedure does not converge, and so as a second iterative procedure we can use Newton's method. We let $\boldsymbol{a}$ be the vector with $K + 1$ components $\{a_k\}$. We also introduce a vector $\boldsymbol{B}(\boldsymbol{a})$ that allows us to write Eqs. (103) as

$$\boldsymbol{B}(\boldsymbol{a}) = \boldsymbol{0}. \tag{110}$$

It follows that we can write our (Newton) iterative solution of Eq. (110) as

$$\boldsymbol{a}_{j+1} = \boldsymbol{a}_j - \boldsymbol{J}^{-1}(\boldsymbol{a}_j)\boldsymbol{B}(\boldsymbol{a}_j), \quad j = 0, 1, 2, \ldots \;. \tag{111}$$

where the Jacobian matrix is

$$\boldsymbol{J}(\boldsymbol{a}) = \left[\frac{\partial}{\partial a_0}\boldsymbol{B}(\boldsymbol{a}) \quad \frac{\partial}{\partial a_1}\boldsymbol{B}(\boldsymbol{a}) \quad \cdots \quad \frac{\partial}{\partial a_K}\boldsymbol{B}(\boldsymbol{a}) \right]. \tag{112}$$

8. Computational aspects and numerical results

As the important aspects of the numerical implementation of our discrete-ordinates solution have already been discussed [15,20], we can be brief here. To start, we define our quadrature scheme $\{w_k, \mu_k\}$ by linearly mapping the Gauss–Legendre scheme onto the interval $[0,1]$. We then use the driver program RG from the EISPACK collection [23] to find the eigenvalues and eigenvectors defined by Eq. (41). So, after noting Eqs. (44) and (45), we have the required separation constants and the associated elementary solutions. To find the constants $\{A_j\}$ and $\{A_{k,j}\}$ we use the subroutine DGECO and DGESL from the LINPACK collection [24] to solve the linear systems defined by Eqs. (48) and (68). Following these procedures, we have all that we require to evaluate the quantities $\Gamma(r_\alpha)$, $\Gamma_k(r_\alpha)$, $\Gamma'(r_\alpha)$ and $\Gamma'_k(r_\alpha)$, and so all we have to do is to solve Eqs. (103) to obtain the constants $\{a_k\}$ which we do, as mentioned in the previous section of the paper, by iterating on Eqs. (103) with either a simple recursive method or with Newton's method.

Of course in implementing our solution we must evaluate the U and V functions as given by Eqs. (67). We change some variables and rewrite these functions in the forms

$$U_k(r, v_j) = R^2 E_k(r/R, R/v_j) \tag{113a}$$

and

$$V_k(r, v_j) = R^2 G_k(r/R, R/v_j), \tag{113b}$$

where

$$E_k(x, y) = \int_0^x \tau \mathscr{F}_k(\tau) \hat{I}_0(\tau y) \mathrm{e}^{-(x-\tau)y}\, \mathrm{d}\tau, \quad x \in [0,1], \quad y \geqslant 0, \tag{114a}$$

and

$$G_k(x, y) = \int_x^1 \tau \mathscr{F}_k(\tau) \hat{K}_0(\tau y) \mathrm{e}^{-(\tau-x)y}\, \mathrm{d}\tau, \quad x \in [0,1], \quad y > 0. \tag{114b}$$

Considering that $[\alpha_k, \beta_k]$ is the support of the spline function $\mathscr{F}_k(x)$, i.e

$$\mathscr{F}_k(x) = 0, \quad x \notin [\alpha_k, \beta_k], \tag{115}$$

we can write

$$E_k(x, y) = 0, \quad x \leqslant \alpha_k, \tag{116a}$$

and

$$E_k(x, y) = \int_{\alpha_k}^{\min\{x, \beta_k\}} \tau \mathscr{F}_k(\tau) \hat{I}_0(\tau y) \mathrm{e}^{-(x-\tau)y}\, \mathrm{d}\tau, \quad x > \alpha_k. \tag{116b}$$

In a similar way, we can write

$$G_k(x, y) = 0, \quad x \geqslant \beta_k, \tag{117a}$$

and

$$G_k(x, y) = \int_{\max\{x, \alpha_k\}}^{\beta_k} \tau \mathscr{F}_k(\tau) \hat{K}_0(\tau y) \mathrm{e}^{-(\tau-x)y}\, \mathrm{d}\tau, \quad x < \beta_k. \tag{117b}$$

Table 1
Physical data for the different problems

Problem	ε	ρ	Θ_0	ϖ	R	N_c	H
1	0.8	0.2	1.0	0.9	1.0	0.05	1.5
2	0.9	0.1	1.0	0.9	0.5	0.05	100
3	0.9	0.1	1.0	0.9	0.05	0.0005	4000
4	0.9	0.1	1.0	0.9	0.5	0.005	40
5	0.9	0.1	1.0	0.9	5.0	0.5	0.4
6	1.0	0.0	1.0	0.9	1.0	0.1	1.0
7	0.8	0.2	1.0	0.2	1.0	0.5	40
8	0.8	0.2	1.0	0.6	1.0	0.2	200

Now since the spline functions have different definitions on each of two subintervals of $[\alpha_k, \beta_k]$, we use a Gauss–Legendre scheme over each one of these subintervals to evaluate the required integrals. In this way, we can obtain good accuracy for the integrals with a very low-order quadrature scheme.

Finally, we can mention that when implementing Newton's method of iteration we do not use the method as expressed by Eq. (111), but to be more efficient we use

$$\boldsymbol{a}_{j+1} = \boldsymbol{a}_j - \boldsymbol{x}_j \tag{118}$$

where $\boldsymbol{x}_j$ is the solution of the linear system

$$\boldsymbol{J}(\boldsymbol{a}_j)\boldsymbol{x}_j = \boldsymbol{B}(\boldsymbol{a}_j). \tag{119}$$

Once we have found the spline constants $\{a_k\}$ we can combine Eqs. (92) and (94) and then compute the temperature distribution from

$$\Theta(r) = \Gamma(r) + \sum_{k=0}^{K} a_k \Gamma_k(r), \tag{120}$$

where $\Gamma(r)$ and $\Gamma_k(r)$ are available from Eqs. (95) and (96). Continuing, we can express the radiative heat flux as

$$Q_r(r) = \frac{1}{N_c}\left[\gamma f_1(r) + \sum_{k=0}^{K} a_k \psi_{k,1}(r)\right], \tag{121}$$

where $\psi_{k,1}(r)$ is given by Eqs. (75) and (76). Finally, since the total heat flux is

$$Q(r) = \frac{r}{2}H, \tag{122}$$

the conductive heat flux can be computed from

$$Q_c(r) = Q(r) - Q_r(r). \tag{123}$$

In order to test our implementation of the reported algorithm we now consider the eight data cases listed in Table 1. As the first step in evaluating our solution, we solved Problems 1–6 as

Table 2
The temperature distribution and the radiative heat flux

r/R	Problem 3		Problem 7		Problem 8	
	$\Theta(r)$	$Q_r(r)$	$\Theta(r)$	$Q_r(r)$	$\Theta(r)$	$Q_r(r)$
0.0	2.04106	0.0	2.77506	0.0	3.69100	0.0
0.1	2.03634	8.09359	2.77239	1.94626	3.68901	9.96022
0.2	2.02159	15.9445	2.76416	3.88773	3.68300	19.9193
0.3	1.99494	23.2887	2.74953	5.81659	3.67278	29.8757
0.4	1.95313	29.8238	2.72651	7.71596	3.65798	39.8269
0.5	1.89126	35.2049	2.69045	9.54543	3.63774	49.7649
0.6	1.80264	39.0654	2.63011	11.2059	3.60949	59.6554
0.7	1.67883	41.0811	2.51850	12.4645	3.56173	69.3127
0.8	1.51008	41.0863	2.29423	12.8520	3.43350	77.7081
0.9	1.28655	39.2184	1.83907	11.7663	2.91471	80.1743
1.0	1.0	35.9995	1.0	9.31416	1.0	69.6986

listed in Table 1. These six problems were first solved by Siewert and Thomas in Ref. [14]. Since we found exactly the numerical results given for Problems 1, 2, 4–6 in Ref. [14], we do not report them here. However, we did find a few digits different for Problem 3, and for this reason we list our current results for this problem in Table 2.

Since, basic to our solution, there are several computational parameters that can be adjusted to solve a problem defined by a given data set efficiently, we now note some details about what we have actually used in solving the eight problems defined in Table 1. In regard to the iteration schemes and the convergence conditions imposed, we have used $\varepsilon = 10^{-7}$ to check that the value for each of the coefficients $\{a_k\}$ agreed between two successive iterations, and we allowed 100 iterations for the simple iteration scheme and 50 for the Newton scheme. We found little difference in the computational time required by these two methods when both methods converged, but we note that for some cases we investigated the simple iteration scheme available from Eqs. (103) failed the defined convergence conditions. Problems 7 and 8 defined in Table 1 are examples of considered data sets, where the defined simple iteration scheme failed to converge, and so we have used the Newton iteration procedure to solve these problems. Continuing, we note that we used, for all of the considered problems, 4 Gauss points to evaluate the U and V functions defined by Eqs. (113). For Problems 1–6 we have used 40 discrete ordinates and 100 spline functions to find results for the temperature distribution and the radiative heat flux with what we believe to be 7 figures of accuracy. To have an idea about the computational requirements of our FORTRAN implementations of the algorithm, we note that to establish our solution of each of these first six problems required less than 7 s on a 400 MHz Pentium-based PC. We continued with 40 discrete-ordinates, but we used 200 spline functions (and 25 s) for Problem 7 and 300 spline functions (and 65 s) for Problem 8. Our results (thought to be correct to all digits given) for Problems 7 and 8 are given in Table 2. Finally, since the total heat flux and the conductive heat flux are immediately available, once the radiative heat flux is known, from Eqs. (122) and (123), we list in Table 2 only the temperature distribution and the radiative heat flux.

9. Final comments

We can say that the use of this analytical version of the discrete-ordinates method has proved very effective in solving a class of difficult nonlinear coupled conduction–radiation problems in a concise and accurate way. We can also note that the use of Newton's form of iteration has been shown to be a significant improvement when compared to the simple direct iterative approach that has traditionally been used, as in Ref. [14], in studying the considered problems. It is also clear that the particular solution established here in the context of inhomogeneous "pseudo problems" basic to radiative transfer problems in cylindrical geometry will prove to be useful in future studies.

While we consider that we have been able to solve well a good collection of coupled heat-transfer problems, we have to keep in mind the fact that we do not have, for the considered class of problems, a definition of the parameter space for which a solution even exists. In our opinion, this important issue is one that deserves attention, and though to establish conditions for which we can be sure of the existence and uniqueness of a solution could (we believe) prove to be difficult, the paper of Kelley [10] can surely be considered a good starting point for additional work.

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