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An analytical discrete-ordinates solution of the S-model kinetic equations for flow in a cylindrical tube

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Abstract

An integro-differential form of the linearized S-model kinetic equations for describing flow in a cylindrical tube is projected in such a way as to yield a pair of coupled transport equations that defines the desired velocity and heat-flow profiles. This system is then solved symbolically to yield a pair of coupled integral equations for the physical quantities required. At this point some transformations are carried out to yield a restatement of the original problem in terms of a "pseudo-problem" defined by plane-geometry variables. An analytical version of the discrete-ordinates method is then used to solve the pseudo-problem, and so, after both MATLAB and FORTRAN versions of the developed algorithm are implemented, results thought to be highly accurate are obtained for the case of diffuse reflection from the walls of a cylindrical tube. In addition to the velocity and heat-flow profiles, for the cases of Poiseuille flow and thermal-creep flow, the velocity slips, the heat-flow profiles evaluated at the wall, the particle-flow rates and the heat-flow rates for these two problems are reported for selected values of the tube radius.

Keywords: Rarefied gas dynamics; Discrete ordinates

1. Introduction

Internal rarefied gas flows define a field of major interest in the general area of rarefied-gas dynamics, and so the contributions to this body of knowledge are many. While the books of

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Cercignani [1,2] and Williams [3] provide excellent material relevant to this field, a comprehensive review recently reported by Sharipov and Seleznev [4] is also a useful up-to-date source that pays much attention to comparing different computational methods as well as different mathematical formulations basic to rarefied-gas dynamics. It is made clear in Ref. [4] that the thermal transpiration phenomena, which exist in internal flows produced by a temperature or pressure gradient, continue to attract the attention of scientists. Moreover there is additional recent interest in these effects due to applications in micro-electro-mechanical systems (MEMS). This is a fast growing industry, and so there is need for further improvement of calculations and modeling for the flow of microfluids. In many cases the flow conditions are in the transition regime and as a result the well known and commonly used Navier–Stokes equations cannot be applied. In these cases the Boltzmann equation or suitable kinetic models should be utilized. And so it follows that we should continue to develop precise and accurate numerical schemes for the computation of thermo-fluid parameters for particle flows, especially flows through capillaries of different cross-sectional area.

It is clear, that in the general theory of particle transport theory, one must deal with the non-linear Boltzmann equation, and in such cases iterative, computationally intensive methods or Monte Carlo methods are ways that are sometimes used in an attempt to obtain results of physical interest. On the other hand in special situations, for example when the density of particles is low, model equations can be used to provide meaningful physical results. Solutions to these model kinetic equations can also be used to establish test results for evaluating solution techniques developed for more exact formulations. In regard to flow in cylindrical tubes, Sharipov and Seleznev [4] report numerical results based on various computational approaches, and while most works available for the cylindrical case are based on the classical BGK model [5], other more general models have also been used. It has been reported [4,6], for example, that in the case of nonisothermal flows the S model of Shakhov [7], as quoted by Ref. [4], offers some improvement over the standard BGK model.

It is clear that the challenges of flow problems defined by cylindrical geometry are significant, but some definite progress has been made, especially in regard to the BGK model. An important work in this regard is that of Ferziger [8] who extended the use of the Mitsis [9] transformations, developed in the context of neutron-transport theory, in order to recast the problem of flow in a cylindrical tube to a much simpler formulation in terms of plane-geometry variables. Subsequent works by Lang and Loyalka [10], Valougeorgis and Thomas [11] and Siewert [12] used what we might call semi-analytical methods to establish and report numerical results for Poiseuille and thermal-creep flow, defined by the BGK model, in a cylindrical tube.

Here, following the work of Sharipov [6], we use the linearized S-model kinetic equations to describe flow in a cylindrical tube. We start with a familiar form of the balance equation, and, following a similar work [13] that was based on the BGK model, we develop an alternative formulation in terms of a system of coupled integral equations. Of course, the formulation of the originally stated problem in terms of an integral equation offers some good possibilities for numerical work, but more importantly here, we use this new formulation to extend the ideas of Mitsis [9]. And so ultimately, we find a so-called "pseudo-problem" defined by plane-geometry variables that we are able to solve with good accuracy using an improved, analytical version [14,15] of the discrete-ordinates method [16].

2. Defining equations

We start with the basic equation relevant to the S model, for applications in cylindrical geometry with no variation in the axial direction, written as

$$\left[c_\perp\left(\cos\phi\frac{\partial}{\partial r}-\frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)+1\right]h(r,c_\perp,c_z,\phi)=\pi^{-3/2}\mathscr{I}[h](r,c_\perp,c_z,\phi)+k(c_\perp,c_z), \tag{1}$$

where the inhomogeneous driving term is

$$k(c_\perp,c_z)=-c_z[k_1+k_2(c_\perp^2+c_z^2-5/2)] \tag{2a}$$

and the integral term is

$$\mathscr{I}[h](r,c_\perp,c_z,\phi)=\int_0^{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\mathrm{e}^{-c'^2}h(r,c'_\perp,c'_z,\phi')K(\boldsymbol{c}'{:}\ \boldsymbol{c})c'_\perp\,\mathrm{d}c'_\perp\,\mathrm{d}c'_z\,\mathrm{d}\phi'. \tag{2b}$$

We note that the problems of Poiseuille flow and thermal-creep flow discussed later in this work are defined, respectively, by the choices $k_1=1$, $k_2=0$ and $k_1=0$, $k_2=1$ in the driving term $k(c_\perp,c_z)$. In addition, the kernel that defines the integral term is

$$K(\boldsymbol{c}':\boldsymbol{c})=1+2[c'_zc_z+c'_\perp c_\perp\cos(\phi'-\phi)]+(2/3)(c'^2-3/2)(c^2-3/2)+M(\boldsymbol{c}':\boldsymbol{c}), \tag{3}$$

where

$$M(\boldsymbol{c}'{:}\ \boldsymbol{c})=(4/15)[c'_zc_z+c'_\perp c_\perp\cos(\phi'-\phi)](c'^2-5/2)(c^2-5/2) \tag{4}$$

is the term [4,6] added to the BGK kernel to yield the kernel for the S model. Also, we note that

$$c^2=c_\perp^2+c_z^2 \tag{5a}$$

and

$$c'^2=c'^2_\perp+c'^2_z. \tag{5b}$$

In regard to boundary conditions, we write our versions, of what Williams [3] has, as

$$h(R,c_\perp,c_z,\phi)=\alpha D+(1-\alpha)h(R,c_\perp,c_z,\phi+\pi),\quad \phi\in[\pi/2,\pi], \tag{6a}$$

and

$$h(R,c_\perp,c_z,\phi)=\alpha D+(1-\alpha)h(R,c_\perp,c_z,\phi-\pi),\quad \phi\in[\pi,3\pi/2], \tag{6b}$$

where the constant D is given by

$$D=\frac{2}{\pi}\left(\int_0^{\pi/2}+\int_{3\pi/2}^{2\pi}\right)\int_{-\infty}^{\infty}\int_0^{\infty}h(R,c_\perp,c_z,\phi)\mathrm{e}^{-c^2}c_\perp^2\cos\phi\,\mathrm{d}c_\perp\,\mathrm{d}c_z\,\mathrm{d}\phi. \tag{7}$$

We consider that the distribution function $h(r, c_\perp, c_z, \phi)$ defined by the basic kinetic equation, written as Eq. (1), and the boundary condition, written as Eqs. (6), depends on the spatial variable $r \in (0, R)$, written in dimensionless units, and the particle velocity vector $\boldsymbol{c}$ expressed, also in dimensionless units, in the cylindrical coordinates, $c_\perp \in [0, \infty)$, $\phi \in [0, 2\pi]$ and $c_z \in (-\infty, \infty)$. To connect with the notation of Refs. [4] and [6], we note that we can write $R = \delta$, where $\delta = a\pi^{1/2}/(2\lambda)$ is the "rarefaction" parameter. Here a is the physical radius of the considered tube and λ is a mean-free path.

In this work we seek to compute physical quantities related to particle velocities and heat flow, and since some different notations are used [4,6] and in order to be very clear about our terminology we refer to

$$u(r) = \pi^{-3/2} \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \mathrm{e}^{-c^2} h(r, c_\perp, c_z, \phi) c_z c_\perp \, \mathrm{d}c_\perp \, \mathrm{d}c_z \, \mathrm{d}\phi \tag{8}$$

as the velocity profile and to

$$q(r) = \pi^{-3/2} \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \mathrm{e}^{-c^2} (c^2 - 5/2) h(r, c_\perp, c_z, \phi) c_z c_\perp \, \mathrm{d}c_\perp \, \mathrm{d}c_z \, \mathrm{d}\phi \tag{9}$$

as the heat-flow profile. We note that $u(r)$ and $q(r)$ are the basic quantities of interest, and so we do not (fortunately) actually have to compute the complete distribution function $h(r, c_\perp, c_z, \phi)$. Instead, we can obtain the results we seek from various moments, or integrals, of the distribution function. And so to start our development, we multiply Eq. (1) by

$$\phi_1(c_z) = c_z \exp(-c_z^2), \tag{10}$$

integrate over all c_z and introduce the new variables $\xi = c_\perp$ and $\xi' = c'_\perp$ to find

$$\left[\xi \left(\cos\phi \frac{\partial}{\partial r} - \frac{\sin\phi}{r} \frac{\partial}{\partial \phi}\right) + 1\right] h_1(r, \xi, \phi) = \mathscr{I}[h_1, h_2](r, \xi) + a_1(\xi), \tag{11}$$

where

$$\mathscr{I}[h_1, h_2](r, \xi) = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathrm{e}^{-\xi'^2} [f_{1,1}(\xi', \xi) h_1(r, \xi', \phi') + f_{1,2}(\xi) h_2(r, \xi', \phi')] \xi' \, \mathrm{d}\xi' \, \mathrm{d}\phi' \tag{12}$$

with

$$f_{1,1}(\xi', \xi) = 1 + (2/15)(\xi'^2 - 1)(\xi^2 - 1) \tag{13}$$

and

$$f_{1,2}(\xi) = (1/5)(2/3)^{1/2}(\xi^2 - 1). \tag{14}$$

Here

$$h_1(r, \xi, \phi) = \int_{-\infty}^{\infty} \mathrm{e}^{-c_z^2} h(r, \xi, c_z, \phi) c_z \, \mathrm{d}c_z, \tag{15}$$

$$h_2(r, \xi, \phi) = (2/3)^{1/2} \int_{-\infty}^{\infty} \mathrm{e}^{-c_z^2} (c_z^2 - 3/2) h(r, \xi, c_z, \phi) c_z \, \mathrm{d}c_z \tag{16}$$

and

$$a_1(\xi) = -(1/2)\pi^{1/2}[k_1 + k_2(\xi^2 - 1)]. \tag{17}$$

Continuing, we next multiply Eq. (1) by

$$\phi_2(c_z) = (2/3)^{1/2} c_z (c_z^2 - 3/2)\exp(-c_z^2) \tag{18}$$

and integrate over all c_z to find

$$\left[\xi\left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right) + 1\right] h_2(r,\xi,\phi) = \mathscr{I}[h_1,h_2](r) + a_2, \tag{19}$$

where

$$\mathscr{I}[h_1,h_2](r) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{-\xi'^2}[f_{2,1}(\xi')h_1(r,\xi',\phi') + f_{2,2}h_2(r,\xi',\phi')]\xi'\,\mathrm{d}\xi'\,\mathrm{d}\phi' \tag{20}$$

with

$$f_{2,1}(\xi') = (1/5)(2/3)^{1/2}(\xi'^2 - 1) \tag{21}$$

and

$$f_{2,2} = 1/5. \tag{22}$$

In addition

$$a_2 = -(3\pi/8)^{1/2} k_2. \tag{23}$$

At this point we can rewrite Eqs. (11) and (19) as

$$\left[\xi\left(\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial\phi}\right) + 1\right] \boldsymbol{H}(r,\xi,\phi) = \boldsymbol{\mathscr{I}}(r,\xi) + \boldsymbol{A}(\xi), \tag{24}$$

where

$$\boldsymbol{\mathscr{I}}(r,\xi) = \frac{1}{\pi}\boldsymbol{Q}(\xi)\int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{-\xi'^2}\boldsymbol{Q}^{\mathrm{T}}(\xi')\boldsymbol{H}(r,\xi',\phi')\xi'\,\mathrm{d}\xi'\,\mathrm{d}\phi' \tag{25}$$

with

$$\boldsymbol{Q}(\xi) = \begin{bmatrix} (2/15)^{1/2}(\xi^2 - 1) & 1 \\ (1/5)^{1/2} & 0 \end{bmatrix}. \tag{26}$$

Here the elements of the two-vector $\boldsymbol{H}(r,\xi,\phi)$ are $h_1(r,\xi,\phi)$ and $h_2(r,\xi,\phi)$. In addition the two elements of $\boldsymbol{A}(\xi)$ are $a_1(\xi)$ and a_2 as defined by Eqs. (17) and (23).

Now in regard to boundary conditions, we project Eqs. (6) against $\phi_1(c_z)$ and $\phi_2(c_z)$ to obtain

$$\boldsymbol{H}(R,\xi,\phi) = (1-\alpha)\boldsymbol{H}(R,\xi,\phi+\pi), \quad \phi \in [\pi/2, \pi], \tag{27a}$$

and

$$\boldsymbol{H}(R,\xi,\phi)=(1-\alpha)\boldsymbol{H}(R,\xi,\phi-\pi),\quad \phi\in[\pi,3\pi/2]. \tag{27b}$$

Since the solution we seek has the symmetry property

$$\boldsymbol{H}(r,\xi,2\pi-\phi)=\boldsymbol{H}(r,\xi,\phi),\quad \phi\in[0,\pi], \tag{28}$$

for all r and ξ, we let $\mu=\cos\phi$, for $\phi\in[0,\pi]$, and

$$\boldsymbol{I}(r,\xi,\mu)=\boldsymbol{H}(r,\xi,\arccos\mu),\quad \mu\in[-1,1], \tag{29}$$

and so we can rewrite Eq. (24) as

$$\left[\xi\left(\mu\frac{\partial}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}\right)+1\right]\boldsymbol{I}(r,\xi,\mu)=\mathscr{I}(r,\xi)+\boldsymbol{A}(\xi), \tag{30}$$

where

$$\mathscr{I}(r,\xi)=\frac{2}{\pi}\boldsymbol{Q}(\xi)\int_{-1}^{1}\int_{0}^{\infty}\mathrm{e}^{-\xi'^2}\boldsymbol{Q}^{\mathrm{T}}(\xi')\boldsymbol{I}(r,\xi',\mu')\xi'\,\mathrm{d}\xi'\frac{\mathrm{d}\mu'}{(1-\mu'^2)^{1/2}}. \tag{31}$$

We can also rewrite Eqs. (27) as

$$\boldsymbol{I}(R,\xi,-\mu)=(1-\alpha)\boldsymbol{I}(R,\xi,\mu),\quad \mu\in[0,1]. \tag{32}$$

To conclude this section, we multiply Eq. (30) by $\boldsymbol{Q}^{-1}(\xi)$ and define

$$\boldsymbol{G}(r,\xi,\mu)=\boldsymbol{Q}^{-1}(\xi)\boldsymbol{I}(r,\xi,\mu) \tag{33}$$

to obtain

$$\left[\xi\left(\mu\frac{\partial}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}\right)+1\right]\boldsymbol{G}(r,\xi,\mu)=\int_{-1}^{1}\int_{0}^{\infty}\boldsymbol{\Psi}(\xi',\mu')\boldsymbol{G}(r,\xi',\mu')\,\mathrm{d}\xi'\,\mathrm{d}\mu'+\boldsymbol{\Gamma}, \tag{34}$$

where the elements of $\boldsymbol{\Gamma}$ are

$$\gamma_1=-(15\pi/8)^{1/2}k_2 \tag{35a}$$

and

$$\gamma_2=-(1/2)k_1\pi^{1/2}. \tag{35b}$$

In addition

$$\boldsymbol{\Psi}(\xi,\mu)=\frac{2}{\pi(1-\mu^2)^{1/2}}\boldsymbol{Q}^{\mathrm{T}}(\xi)\boldsymbol{Q}(\xi)\xi\,\mathrm{e}^{-\xi^2}, \tag{36}$$

and finally we can rewrite Eq. (32) as

$$\boldsymbol{G}(R,\xi,-\mu)=(1-\alpha)\boldsymbol{G}(R,\xi,\mu), \quad \mu\in[0,1]. \tag{37}$$

3. A reformulation as an integral equation

It is clear that Eq. (34) provides a significant challenge to workers who wish to proceed directly from that equation with numerical methods. It is for this reason that we avoid such an approach and so wish to extend the ideas of Mitsis [9], as they were employed by Ferziger [8] for the BGK model. While to pursue the intended path requires a great deal of work, we can benefit greatly from a paper by Barichello et al. [13] that reported, in detail for the BGK model, the work we extend here to the case of the S model where we must deal with a system of coupled equations. And, more importantly, we arrive ultimately at a computational problem much simpler than the one defined by Eq. (34).

In order to have our development here closely follow Ref. [13], we restate our problem in slightly different terms. We consider

$$\left[\xi\left(\mu\frac{\partial}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}\right)+1\right]\boldsymbol{G}(r,\xi,\mu)=\int_{-1}^{1}\int_0^\infty \boldsymbol{\Psi}(\xi',\mu')\boldsymbol{G}(r,\xi',\mu')\,\mathrm{d}\xi'\,\mathrm{d}\mu'+\boldsymbol{\Gamma} \tag{38}$$

for $\mu\in[-1,1]$, $\xi\in[0,\infty)$ and $r\in(0,R)$ and

$$\boldsymbol{G}(R,\xi,-\mu)=\boldsymbol{F}(\xi,\mu), \quad \mu\in[0,1] \quad \text{and} \quad \xi\in[0,\infty), \tag{39}$$

where we assume $\boldsymbol{\Gamma}$ and, for the moment, $\boldsymbol{F}(\xi,\mu)$ to be known. To obtain the required integral equation, we start by thinking of the ξ variable as a parameter, and then we introduce

$$\hat{\boldsymbol{G}}(r)=\boldsymbol{G}[r,\xi,\eta(r)]. \tag{40}$$

We thus can write

$$\frac{\mathrm{d}}{\mathrm{d}r}\hat{\boldsymbol{G}}(r)=\frac{\partial}{\partial r}\boldsymbol{G}(r,\xi,\eta)+\left(\frac{\mathrm{d}\eta}{\mathrm{d}r}\right)\frac{\partial}{\partial\eta}\boldsymbol{G}(r,\xi,\eta). \tag{41}$$

We now let

$$\frac{\mathrm{d}\eta}{\mathrm{d}r}=\frac{1-\eta^2}{r\eta} \tag{42}$$

and rewrite Eq. (41) as

$$\frac{\mathrm{d}}{\mathrm{d}r}\hat{\boldsymbol{G}}(r)=\frac{1}{\eta}\left[\eta\frac{\partial}{\partial r}\boldsymbol{G}(r,\xi,\eta)+\frac{1-\eta^2}{r}\frac{\partial}{\partial\eta}\boldsymbol{G}(r,\xi,\eta)\right] \tag{43}$$

or, after we note Eq. (38),

$$\frac{\mathrm{d}}{\mathrm{d}r}\hat{\boldsymbol{G}}(r)+\frac{1}{\xi\eta(r)}\hat{\boldsymbol{G}}(r)=\boldsymbol{H}(r), \tag{44}$$

where now

$$\boldsymbol{H}(r) = \frac{1}{\xi\eta(r)}\left[\int_{-1}^{1}\int_{0}^{\infty}\boldsymbol{\Psi}(\xi',\mu')\boldsymbol{G}(r,\xi',\mu')\,\mathrm{d}\xi'\,\mathrm{d}\mu' + \boldsymbol{\Gamma}\right]. \tag{45}$$

We can solve Eq. (42) to obtain

$$\eta(r) = \pm\, v(r), \tag{46}$$

where

$$v(r) = \frac{1}{r}(r^2 - a^2)^{1/2} \tag{47}$$

and where a is (for the moment) an arbitrary constant. Finding the integrating factor, we write Eq. (44) as

$$\frac{\mathrm{d}}{\mathrm{d}r}(\hat{\boldsymbol{G}}(r)\exp\{\pm rv(r)/\xi\}) = \boldsymbol{H}(r)\exp\{\pm rv(r)/\xi\}, \tag{48}$$

and so, after noting Eq. (40), we can rewrite Eq. (48) as

$$\frac{\mathrm{d}}{\mathrm{d}r}(\boldsymbol{G}[r,\xi,\pm v(r)]\exp\{\pm rv(r)/\xi\}) = \pm\frac{1}{\xi v(r)}\boldsymbol{S}(r)\exp\{\pm rv(r)/\xi\}, \tag{49}$$

where

$$\boldsymbol{S}(r) = \int_{-1}^{1}\int_{0}^{\infty}\boldsymbol{\Psi}(\xi',\mu')\boldsymbol{G}(r,\xi',\mu')\,\mathrm{d}\xi'\,\mathrm{d}\mu' + \boldsymbol{\Gamma}. \tag{50}$$

We now can integrate Eq. (49), use $\eta(r) = \mu$ and

$$a = r(1-\mu^2)^{1/2} \tag{51}$$

and follow Ref. [13] to find ultimately that we can write

$$\boldsymbol{G}(r,\xi,\mu) = \boldsymbol{B}(r,\xi,\mu) + \int_{0}^{s_0(r,\xi,\mu)}\boldsymbol{S}[(r^2+s^2\xi^2-2rs\xi\mu)^{1/2}]\mathrm{e}^{-s}\,\mathrm{d}s \tag{52a}$$

and

$$\boldsymbol{G}(r,\xi,-\mu) = \boldsymbol{B}(r,\xi,-\mu) + \int_{0}^{s_0(r,\xi,-\mu)}\boldsymbol{S}[(r^2+s^2\xi^2+2rs\xi\mu)^{1/2}]\mathrm{e}^{-s}\,\mathrm{d}s \tag{52b}$$

for $\mu \in [0,1]$. Here

$$s_0(r,\xi,\mu) = [(R^2 - r^2 + r^2\mu^2)^{1/2} + r\mu]/\xi \tag{53}$$

and

$$\boldsymbol{B}(r,\xi,\mu) = \boldsymbol{F}[\xi,\mu_0(R,r,\mu)]\exp\{-[R\mu_0(R,r,\mu) + r\mu]/\xi\}, \tag{54}$$

where, in general,

$$\mu_0(x,r,\mu) = \frac{1}{x}(x^2 - r^2 + r^2\mu^2)^{1/2}. \tag{55}$$

Seeking to derive an integral equation for

$$\boldsymbol{G}(r) = \int_{-1}^{1}\int_{0}^{\infty} \boldsymbol{\Psi}(\xi,\mu)\boldsymbol{G}(r,\xi,\mu)\,\mathrm{d}\xi\,\mathrm{d}\mu, \tag{56}$$

we multiply Eqs. (52) by $\boldsymbol{\Psi}(\xi,\mu)$ and integrate over all ξ and μ to find, after some extensive calculations analogous to those reported in Ref. [13],

$$\boldsymbol{G}(r) = \boldsymbol{B}(r) + \frac{2}{\pi}\int_{0}^{\infty} \mathrm{e}^{-\xi^2}\boldsymbol{Q}^{\mathrm{T}}(\xi)\boldsymbol{Q}(\xi)\int_{0}^{R} x\boldsymbol{S}(x)\int_{-1}^{1}\frac{\exp\{-p(x,r,\mu)/\xi\}}{p(x,r,\mu)(1-\mu^2)^{1/2}}\,\mathrm{d}\mu\,\mathrm{d}x\,\mathrm{d}\xi, \tag{57}$$

where

$$p(x,r,\mu) = (x^2 + r^2 - 2xr\mu)^{1/2} \tag{58}$$

and where the contribution from the boundary term is

$$\boldsymbol{B}(r) = \int_{-1}^{1}\int_{0}^{\infty} \boldsymbol{\Psi}(\xi,\mu)\boldsymbol{F}[\xi,\mu_0(R,r,\mu)]\exp\{-s_0(r,\xi,\mu)\}\,\mathrm{d}\xi\,\mathrm{d}\mu. \tag{59}$$

Continuing, we make use of various Bessel function identities and changes of variables to find, again after much work closely related to what was reported in Ref. [13],

$$\boldsymbol{G}(r) = \boldsymbol{B}(r) + \int_{0}^{R} x\boldsymbol{K}(x \to r)[\boldsymbol{G}(x) + \boldsymbol{\Gamma}]\,\mathrm{d}x, \tag{60}$$

where the kernel of the integral equation is

$$\boldsymbol{K}(x \to r) = \frac{2}{\pi^{1/2}}\int_{0}^{\infty} \mathrm{e}^{-\tau^2} F_0(x/\tau, r/\tau)\boldsymbol{\Delta}(\tau)\frac{\mathrm{d}\tau}{\tau^2}. \tag{61}$$

Here

$$\boldsymbol{\Delta}(\tau) = \boldsymbol{\Delta}_0 + \boldsymbol{\Delta}_2\tau^2 + \boldsymbol{\Delta}_4\tau^4, \tag{62}$$

where

$$\boldsymbol{\Delta}_0 = \begin{bmatrix} 3/10 & -(1/30)^{1/2} \\ -(1/30)^{1/2} & 1 \end{bmatrix}, \tag{63a}$$

$$\boldsymbol{\Delta}_2 = \begin{bmatrix} -2/15 & (2/15)^{1/2} \\ (2/15)^{1/2} & 0 \end{bmatrix} \tag{63b}$$

and

$$\boldsymbol{\Delta}_4 = \begin{bmatrix} 2/15 & 0 \\ 0 & 0 \end{bmatrix}. \tag{63c}$$

In addition

$$F_0(x,r) = \begin{cases} I_0(x)K_0(r), & x < r, \\ K_0(x)I_0(r), & x > r, \end{cases} \tag{64}$$

where $I_0(x)$ and $K_0(x)$ are used to denote modified Bessel functions. If we now let

$$\boldsymbol{G}(r) = \boldsymbol{Z}(r) - \boldsymbol{\Gamma} \tag{65}$$

then the integral equation to be solved is

$$\boldsymbol{Z}(r) = \boldsymbol{B}(r) + \boldsymbol{\Gamma} + \int_0^R x\boldsymbol{K}(x \to r)\boldsymbol{Z}(x)\,\mathrm{d}x \tag{66}$$

where $\boldsymbol{B}(r)$ is given by Eq. (59) and

$$\boldsymbol{\Gamma} = -\frac{\pi^{1/2}}{2}\begin{bmatrix} (15/2)^{1/2}k_2 \\ k_1 \end{bmatrix}. \tag{67}$$

4. A pseudo-problem

Looking back to Eqs. (8) and (9), we find that the quantities of physical interest here, *viz.* the velocity profile and the heat-flow profile, can be written as

$$u(r) = \pi^{-1/2}[0 \quad 1]\boldsymbol{G}(r) \tag{68}$$

and

$$q(r) = [15/(2\pi)]^{1/2}[1 \quad 0]\boldsymbol{G}(r), \tag{69}$$

where $\boldsymbol{G}(r)$ is expressed in terms of $\boldsymbol{Z}(r)$ by Eq. (65) and where $\boldsymbol{Z}(r)$ is a solution of the integral equation listed as Eq. (66). However, rather than attempting to solve Eq. (66), we will make use of a transformation that is based on the work of Mitsis [9] and which allows a convenient reformulation in terms of a "pseudo-problem" that can be solved with a variation of the discrete-ordinates method. And so we introduce

$$\boldsymbol{\Xi}(r,\xi) = \frac{1}{\xi^2}\int_0^R xF_0(x/\xi, r/\xi)\boldsymbol{Z}(x)\,\mathrm{d}x \tag{70}$$

which can be differentiated to yield

$$\left[\xi^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) - 1\right]\boldsymbol{\Xi}(r,\xi) + \boldsymbol{Z}(r) = \boldsymbol{0}. \tag{71}$$

Now rewrite Eq. (66) as

$$\boldsymbol{Z}(r) = \boldsymbol{B}(r) + \boldsymbol{\Gamma} + \int_0^\infty \boldsymbol{\Psi}(\xi)\boldsymbol{\Xi}(r,\xi)\,\mathrm{d}\xi, \tag{72}$$

where

$$\mathbf{\Psi}(\xi) = \frac{2}{\pi^{1/2}} \mathrm{e}^{-\xi^2} \mathbf{\Delta}(\xi). \tag{73}$$

And so we can rewrite Eq. (71) as

$$\left[\xi^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) - 1\right] \mathbf{\Xi}(r,\xi) + \int_0^\infty \mathbf{\Psi}(\xi')\mathbf{\Xi}(r,\xi')\,\mathrm{d}\xi' + \boldsymbol{B}(r) + \mathbf{\Gamma} = \mathbf{0}. \tag{74}$$

Seeking a boundary condition subject to which we must solve Eq. (74), we can differentiate Eq. (70) to find

$$K_1(\xi)\mathbf{\Xi}(R,\xi) + \xi K_0(\xi)\frac{\partial}{\partial r}\mathbf{\Xi}(r,\xi)\bigg|_{r=R} = \mathbf{0}, \quad \xi \in [0,\infty). \tag{75}$$

In regard to the terms $\boldsymbol{B}(r)$ and $\mathbf{\Gamma}$ that appear in Eq. (74), we note that $\mathbf{\Gamma}$ is a true inhomogeneous term since it is known, as can be seen from Eq. (67). The other term, however, is not a true inhomogeneous term unless, as can be seen from Eq. (59), the vector-valued function $\boldsymbol{F}(\xi,\mu)$ introduced in Eq. (39) is known. But in fact, for the general problem considered here, we see from Eq. (37) that $\boldsymbol{F}(\xi,\mu)$ should be used to describe particles reflected specularly from the wall of the tube that confines the flow. For this reason, we consider, in the remainder of this work, only the case of diffuse reflection ($\alpha = 1$). We therefore drop $\boldsymbol{B}(r)$ from Eq. (74) and consider our pseudo-problem to be defined by

$$\left[\xi^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) - 1\right] \mathbf{\Xi}(r,\xi) + \int_0^\infty \mathbf{\Psi}(\xi')\mathbf{\Xi}(r,\xi')\,\mathrm{d}\xi' + \mathbf{\Gamma} = \mathbf{0}, \tag{76}$$

for $r \in (0,R)$ and $\xi \in [0,\infty)$, and

$$K_1(\xi)\mathbf{\Xi}(R,\xi) + \xi K_0(\xi)\frac{\partial}{\partial r}\mathbf{\Xi}(r,\xi)\bigg|_{r=R} = \mathbf{0} \tag{77}$$

for $\xi \in [0,\infty)$. We find that

$$\mathbf{\Xi}_{\mathrm{p}}(r,\xi) = \frac{\pi^{1/2}}{4}\begin{bmatrix} (6/5)^{1/2}k_1 - 3(15/2)^{1/2}k_2 \\ (r^2 - R^2 + 4\xi^2)k_1 \end{bmatrix} \tag{78}$$

is a particular solution of Eq. (76), and so we write

$$\mathbf{\Xi}(r,\xi) = \mathbf{\Xi}_{\mathrm{h}}(r,\xi) + \mathbf{\Xi}_{\mathrm{p}}(r,\xi), \tag{79}$$

where the homogeneous component $\mathbf{\Xi}_{\mathrm{h}}(r,\xi)$ is defined by

$$\left[\xi^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) - 1\right] \mathbf{\Xi}_{\mathrm{h}}(r,\xi) + \int_0^\infty \mathbf{\Psi}(\xi')\mathbf{\Xi}_{\mathrm{h}}(r,\xi')\,\mathrm{d}\xi' = \mathbf{0}, \tag{80}$$

for $r \in (0,R)$ and $\xi \in [0,\infty)$, and the boundary condition

$$\mathbf{\Xi}_{\mathrm{h}}(R,\xi) + \xi\,\Upsilon(\xi)\frac{\partial}{\partial r}\mathbf{\Xi}_{\mathrm{h}}(r,\xi)\bigg|_{r=R} = \boldsymbol{R}(\xi) \tag{81}$$

for $\xi \in [0, \infty)$. Here

$$\boldsymbol{R}(\xi) = -\boldsymbol{\Xi}_{\mathrm{p}}(R, \xi) - \xi \Upsilon(\xi) \frac{\partial}{\partial r} \boldsymbol{\Xi}_{\mathrm{p}}(r, \xi) \Big|_{r=R} \tag{82}$$

and

$$\Upsilon(\xi) = \frac{K_0(R/\xi)}{K_1(R/\xi)}. \tag{83}$$

Before developing our discrete-ordinates solution to the problem defined by Eqs. (80) and (81), we note again that once we have a solution to the considered pseudo-problem, the basic quantities we seek will be available from

$$\boldsymbol{G}(r) = \int_0^{\infty} \boldsymbol{\Psi}(\xi) \boldsymbol{\Xi}(r, \xi) \, \mathrm{d}\xi, \tag{84}$$

and so we proceed to develop an algorithm for establishing the discrete-ordinates solution we require to complete this work.

5. The discrete-ordinates solution

It is usual when working in cylindrical coordinates to anticipate Bessel function solutions, and so even though Eq. (80) defines a pseudo-problem, that equation is related to our problem of flow in a cylindrical tube. Therefore it is reasonable to seek solutions of Eq. (80) of the form

$$\boldsymbol{\Xi}_{\mathrm{h}}(r, \xi) = \boldsymbol{\Phi}(v, \xi) I_0(r/v). \tag{85}$$

Clearly, because $r = 0$ is included in our domain, solutions based on the related Bessel function $K_0(r/v)$ are not appropriate here. And so we substitute Eq. (85) into Eq. (80) to find

$$(v^2 - \xi^2) \boldsymbol{\Phi}(v, \xi) = v^2 \int_0^{\infty} \boldsymbol{\Psi}(\xi') \boldsymbol{\Phi}(v, \xi') \, \mathrm{d}\xi' \tag{86}$$

which we should solve to find the elementary vectors $\boldsymbol{\Phi}(v, \xi)$. At this point we introduce a quadrature scheme and rewrite Eq. (86) as

$$(v^2 - \xi^2) \boldsymbol{\Phi}(v, \xi) = v^2 \sum_{k=1}^{N} w_k \boldsymbol{\Psi}(\xi_k) \boldsymbol{\Phi}(v, \xi_k) \tag{87}$$

where the N weights and nodes $\{w_k, \xi_k\}$ are defined for use on the integration interval $[0, \infty)$. We now evaluate Eq. (87) at the quadrature points and write the resulting equations as

$$(1/\xi_i^2) \left[\boldsymbol{\Phi}(v, \xi_i) - \sum_{k=1}^{N} w_k \boldsymbol{\Psi}(\xi_k) \boldsymbol{\Phi}(v, \xi_k) \right] = (1/v^2) \boldsymbol{\Phi}(v, \xi_i) \tag{88}$$

for $i=1,2,\ldots,N$. If we introduce the $2N\times 1$ vector

$$\boldsymbol{\Phi}(v)=[\boldsymbol{\Phi}^{\mathrm{T}}(v,\xi_1)\quad \boldsymbol{\Phi}^{\mathrm{T}}(v,\xi_2)\quad \cdots\quad \boldsymbol{\Phi}^{\mathrm{T}}(v,\xi_N)]^{\mathrm{T}} \tag{89}$$

we can write Eqs. (88) as

$$(\boldsymbol{D}-\boldsymbol{W})\boldsymbol{\Phi}(v)=\lambda\boldsymbol{\Phi}(v), \tag{90}$$

where $\lambda=1/v^2$,

$$\boldsymbol{D}=\mathrm{diag}\{(1/\xi_1)^2\boldsymbol{I},(1/\xi_2)^2\boldsymbol{I},\ldots,(1/\xi_N)^2\boldsymbol{I}\} \tag{91}$$

and $\boldsymbol{W}$ is a $2N\times 2N$ matrix each $2\times 2N$ row of which is given by

$$\boldsymbol{R}_i=(1/\xi_i)^2[w_1\boldsymbol{\Psi}(\xi_1)\quad w_2\boldsymbol{\Psi}(\xi_2)\quad \cdots\quad w_N\boldsymbol{\Psi}(\xi_N)] \tag{92}$$

for $i=1,2,\ldots,N$. We note also that $\boldsymbol{I}$ is Eq. (91) is used here to denote the 2×2 identity matrix. And so we will solve the eigenvalue problem defined by Eq. (90) and (to start) write our discrete-ordinates solution as

$$\boldsymbol{\Xi}_{\mathrm{h}}(r,\xi_k)=\sum_{j=1}^{2N}A_j\boldsymbol{\Phi}(v_j,\xi_k)\hat{I}_0(r/v_j)\mathrm{e}^{-(R-r)/v_j}, \tag{93}$$

where the 2×1 vectors $\boldsymbol{\Phi}(v_j,\xi_k)$ are the block components of $\boldsymbol{\Phi}(v_j)$. Here the (positive) separation constants $v_j=1/\lambda_j^{1/2}$ and the eigenvectors $\boldsymbol{\Phi}(v_j)$ are available from the eigenvalue problem defined by Eq. (90), and the arbitrary constants A_j are to be determined from the boundary condition. In this work we use, for computational reasons,

$$\hat{I}_n(z)=I_n(z)\mathrm{e}^{-z} \tag{94a}$$

and

$$\hat{K}_n(z)=K_n(z)\mathrm{e}^{z}. \tag{94b}$$

While Eq. (93) is a valid result, one improvement can be made in regard to infinite values of the separation constant v, or equivalently, the eigenvalues of

$$\boldsymbol{A}=\boldsymbol{D}-\boldsymbol{W} \tag{95}$$

that approach zero as N tends to infinity. We first introduce

$$\boldsymbol{\Lambda}(z)=\boldsymbol{I}+z^2\int_0^\infty \boldsymbol{\Psi}(\xi)\frac{\mathrm{d}\xi}{\xi^2-z^2} \tag{96}$$

and note that we consider $\boldsymbol{\Lambda}(z)$ to be the exact version of the discrete-ordinates quantity

$$\boldsymbol{\Omega}(z)=\boldsymbol{I}+z^2\sum_{k=1}^{N}w_k\boldsymbol{\Psi}(\xi_k)\frac{1}{\xi_k^2-z^2}. \tag{97}$$

We can show that the separation constants ν_j defined by the zeros of $\det \boldsymbol{\Omega}(z)$ are the same as those we compute from the eigenvalues of the matrix $\boldsymbol{A}$, and so we base our discussion about the eigenvalues of $\boldsymbol{A}$ (that accumulate at zero as N tends to infinity) on the zeros of $\det \boldsymbol{\Lambda}(z)$ as z tends to infinity. We note that

$$\boldsymbol{\Lambda}(\infty) = \boldsymbol{I} - \int_0^\infty \boldsymbol{\Psi}(\xi)\,\mathrm{d}\xi \tag{98}$$

can be evaluated to yield

$$\boldsymbol{\Lambda}(\infty) = \begin{bmatrix} 2/3 & 0 \\ 0 & 0 \end{bmatrix}. \tag{99}$$

We note also that if

$$\boldsymbol{\Lambda}(z) = \boldsymbol{I} + z^2 \int_0^\infty \boldsymbol{\Psi}(\xi) \frac{\mathrm{d}\xi}{\xi^2 - z^2} \tag{100}$$

then

$$\det \boldsymbol{\Lambda}(z) \sim \frac{1}{3z^2} \tag{101}$$

as z tends to infinity, and so we conclude that, as N tends to infinity, $\boldsymbol{A}$ should have $\lambda = 0$ as a repeated eigenvalue. Therefore instead of using the discrete-ordinates result corresponding to the largest separation constant, say ν_1, we use the exact value $\nu_1 = \infty$ and the exact solution of Eq. (80)

$$\boldsymbol{\Phi}_+ = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{102}$$

in order to write Eq. (93) as

$$\boldsymbol{\Xi}_{\mathrm{h}}(r, \xi_k) = A_1 \boldsymbol{\Phi}_+ + \sum_{j=2}^{2N} A_j \boldsymbol{\Phi}(\nu_j, \xi_k) \hat{I}_0(r/\nu_j) \mathrm{e}^{-(R-r)/\nu_j}. \tag{103}$$

Now we can substitute Eq. (103) into a discrete version of Eq. (81), *viz.*

$$\boldsymbol{\Xi}_{\mathrm{h}}(R, \xi_i) + \xi_i \Upsilon(\xi_i) \frac{\partial}{\partial r} \boldsymbol{\Xi}_{\mathrm{h}}(r, \xi_i)\bigg|_{r=R} = \boldsymbol{R}(\xi_i), \tag{104}$$

for $i = 1, 2, \ldots, N$, to define a linear system that can be solved to find the constants A_j, $j = 1, 2, \ldots, N$, required in Eq. (103). At this point we can use Eqs. (78), (79) and (103) to write our discrete-ordinates version of Eq. (84) as

$$\boldsymbol{G}(r) = \boldsymbol{\Xi}_{\mathrm{p},0}(r) + A_1 \boldsymbol{\Phi}_+ + \sum_{j=2}^{2N} A_j \boldsymbol{N}(\nu_j) \hat{I}_0(r/\nu_j) \mathrm{e}^{-(R-r)/\nu_j}, \tag{105}$$

where

$$\boldsymbol{\Xi}_{\mathrm{p},0}(r) = \frac{\pi^{1/2}}{4}\begin{bmatrix}(6/5)^{1/2}k_1 - (15/2)^{1/2}k_2 \\ (r^2 - R^2 + 2)k_1\end{bmatrix} \tag{106}$$

and

$$\boldsymbol{N}(v_j) = \sum_{k=1}^{N} w_k \boldsymbol{\Psi}(\xi_k)\boldsymbol{\Phi}(v_j, \xi_k). \tag{107}$$

To complete our solution we wish to compute the velocity profile

$$u(r) = \pi^{-3/2}\int \mathrm{e}^{-c^2} h(r, \boldsymbol{c}) c_z \, \mathrm{d}\boldsymbol{c} \tag{108}$$

and the heat-flow profile

$$q(r) = \pi^{-3/2}\int \mathrm{e}^{-c^2}(c^2 - 5/2) h(r, \boldsymbol{c}) c_z \, \mathrm{d}\boldsymbol{c}. \tag{109}$$

We also wish to compute (what we call) the particle-flow rate

$$U = \frac{4}{R^3}\int_0^R u(r) r \, \mathrm{d}r \tag{110}$$

and the heat-flow rate

$$Q = \frac{4}{R^3}\int_0^R q(r) r \, \mathrm{d}r. \tag{111}$$

We find, in terms of our discrete-ordinates solution,

$$u(r) = \pi^{-1/2}[0 \quad 1]\boldsymbol{G}(r) \tag{112}$$

or

$$u(r) = \frac{1}{4}(r^2 - R^2 + 2)k_1 + \pi^{-1/2}\left[A_1 + \sum_{j=2}^{2N} A_j N_2(v_j)\hat{I}_0(r/v_j)\mathrm{e}^{-(R-r)/v_j}\right], \tag{113}$$

where $N_2(v_j)$ is the lower component of $\boldsymbol{N}(v_j)$. We can also find

$$q(r) = [15/(2\pi)]^{1/2}[1 \quad 0]\boldsymbol{G}(r) \tag{114}$$

or

$$q(r) = \frac{1}{4}[3k_1 - (15/2)k_2] + [15/(2\pi)]^{1/2}\sum_{j=2}^{2N} A_j N_1(v_j)\hat{I}_0(r/v_j)\mathrm{e}^{-(R-r)/v_j}, \tag{115}$$

where $N_1(v_j)$ is the upper component of $\boldsymbol{N}(v_j)$. Now using Eq. (113) in Eq. (110) and Eq. (115) in Eq. (111), we find our final results

$$U = \frac{1}{4R}(4 - R^2)k_1 + \pi^{-1/2}\left[\frac{2}{R}A_1 + \frac{4}{R^2}\sum_{j=2}^{2N} A_j v_j N_2(v_j)\hat{I}_1(R/v_j)\right] \tag{116}$$

and

$$Q = \frac{1}{2R}[3k_1 - (15/2)k_2] + \frac{4}{R^2}[15/(2\pi)]^{1/2}\sum_{j=2}^{2N} A_j v_j N_1(v_j)\hat{I}_1(R/v_j). \tag{117}$$

6. Numerical results

Repeating much of the discussion given in Ref. [12], where the version of the discrete-ordinates method used here was used to solve the Poiseuille and thermal-creep problems for flow, as described by the BGK model, in a cylindrical tube, we note that what we must now do is to define the quadrature scheme to be used in our discrete-ordinates solution. In this work we have used both (non-linear) transformations

$$u(\xi) = \exp\{-\xi\} \tag{118a}$$

and

$$u(\xi) = \frac{1}{1+\xi} \tag{118b}$$

to map $\xi \in [0, \infty)$ into $u \in [0, 1]$, and we then used a Gauss–Legendre scheme mapped (linearly) onto the interval $[0, 1]$. Of course other quadrature schemes could be used. In fact we note that recent works by Garcia [17] and Gander and Karp [18] have reported special quadrature schemes for use in the general area of particle transport theory. Such an approach clearly could be used here. In fact the choice of a quadrature scheme based on the integration interval $[0, \infty)$ with a weight function $\exp(-\xi^2)$ is a natural choice for this work. However, we have found the use of a mapping defined by either of Eqs. (118) followed by the use of the Gauss–Legendre integration formulas to be so effective that we have not developed any special-purpose quadrature schemes.

Having defined our quadrature scheme and in developing a FORTRAN implementation of our solution, we found the required separation constants $\{v_j\}$ by using the driver program RG from the EISPACK collection [19]. The required separation constants were then available as the reciprocals of the positive square roots of these eigenvalues. We then used the subroutines DGECO and DGESL from the LINPACK package [20] to solve the linear system obtained when Eq. (103) was substituted into Eq. (104). And in this way our solution was established as a viable algorithm. As an alternative computation our solution was also evaluated using the MATLAB software. Finally, but importantly, we have found, that elements of the matrix-valued function $\boldsymbol{\Psi}(\xi)$ as defined by Eq. (73) can be essentially zero (from a computational point-of-view). In such cases, we found that by defining an element to be precisely zero when that element is less

Table 1
Velocity and heat-flow profiles for the case $R=2$

r/R	$-u_{\text{P}}(r)$	$q_{\text{P}}(r)$	$u_{\text{T}}(r)$	$-q_{\text{T}}(r)$
0.00	2.386147	4.389470(−1)	4.088331(−1)	1.458556
0.05	2.382966	4.384708(−1)	4.084530(−1)	1.457669
0.10	2.373416	4.370367(−1)	4.073086(−1)	1.454995
0.15	2.357468	4.346288(−1)	4.053878(−1)	1.450495
0.20	2.335079	4.312198(−1)	4.026700(−1)	1.444104
0.25	2.306182	4.267700(−1)	3.991252(−1)	1.435726
0.30	2.270689	4.212262(−1)	3.947133(−1)	1.425232
0.35	2.228483	4.145192(−1)	3.893822(−1)	1.412453
0.40	2.179415	4.065609(−1)	3.830660(−1)	1.397174
0.45	2.123296	3.972404(−1)	3.756817(−1)	1.379120
0.50	2.059884	3.864177(−1)	3.671254(−1)	1.357945
0.55	1.988870	3.739155(−1)	3.572657(−1)	1.333204
0.60	1.909852	3.595061(−1)	3.459354(−1)	1.304325
0.65	1.822296	3.428926(−1)	3.329176(−1)	1.270556
0.70	1.725481	3.236767(−1)	3.179239(−1)	1.230881
0.75	1.618383	3.013050(−1)	3.005569(−1)	1.183888
0.80	1.499485	2.749676(−1)	2.802418(−1)	1.127506
0.85	1.366352	2.433855(−1)	2.560820(−1)	1.058480
0.90	1.214573	2.042772(−1)	2.265030(−1)	9.710624(−1)
0.95	1.034121	1.525221(−1)	1.880381(−1)	8.526430(−1)
1.00	7.710133(−1)	6.071916(−2)	1.227525(−1)	6.400572(−1)

than, say, $\varepsilon = 10^{-20}$, we increased the ability of the linear-algebra package to yield the required number of independent eigenvectors when there is a (nearly) repeated eigenvalue.

To complete our work we list in Tables 1–3 some results obtained from our FORTRAN and MATLAB implementations of the developed solutions for Poiseuille flow (identified by the subscript P) and thermal-creep flow (identified by the subscript T). In Table 1 the complete velocity and heat-flow profiles are given for the case $R=2$. In Tables 2 and 3 the particle-flow rates and the heat-flow rates, accompanied by the velocity slips and the heat-flow profiles evaluated at the wall, are given for selected values of R. It is interesting to observe that the heat-flow profile evaluated at the wall can, for the case of Poiseuille flow, actually have a change of sign as the rarefaction parameter increases from $R=3$ to $R=3.5$. As expected, the Onsager reciprocity relation [6,10], *viz.* $U_{\text{T}} = Q_{\text{P}}$, is clearly verified for all cases listed in Tables 2 and 3. We note that our results are given with what we believe to be seven figures of accuracy. While we have no proof of the accuracy achieved in this work, we have done some things to support the confidence we have. First of all the fact that our results from the FORTRAN implementation and the MATLAB implementation agreed gave us some confidence in the programming aspect of the computations. We also found apparent convergence in our numerical results as we increased N, the number of quadrature points used. Finally, we note that for the case of $R=2$ we found agreement with results communicated by Sharipov [21]. While the agreement was not to as many figures that we believe we have correct, the degree of agreement was significant.

Table 2
Velocity slips and particle-flow rates

R	$-u_{\mathrm{P}}(R)$	$u_{\mathrm{T}}(R)$	$-U_{\mathrm{P}}$	U_{T}
1.0(−3)	5.615159(−4)	2.793541(−4)	1.499564	7.469327(−1)
1.0(−2)	5.484670(−3)	2.657321(−3)	1.477013	7.210237(−1)
2.0(−2)	1.077748(−2)	5.110821(−3)	1.461627	7.019628(−1)
3.0(−2)	1.593887(−2)	7.421781(−3)	1.450166	6.868658(−1)
4.0(−2)	2.099655(−2)	9.618488(−3)	1.441014	6.741078(−1)
5.0(−2)	2.596783(−2)	1.171873(−2)	1.433444	6.629545(−1)
7.0(−2)	3.569677(−2)	1.567695(−2)	1.421555	6.439594(−1)
9.0(−2)	4.519237(−2)	1.936570(−2)	1.412653	6.280139(−1)
1.0(−1)	4.986657(−2)	2.112361(−2)	1.409017	6.208757(−1)
3.0(−1)	1.370961(−1)	4.851826(−2)	1.386792	5.304811(−1)
5.0(−1)	2.179308(−1)	6.729058(−2)	1.400539	4.784350(−1)
7.0(−1)	2.955389(−1)	8.122695(−2)	1.426619	4.404396(−1)
9.0(−1)	3.711649(−1)	9.197151(−2)	1.458860	4.100247(−1)
1.0	4.084491(−1)	9.645221(−2)	1.476445	3.967500(−1)
1.5	5.914445(−1)	1.127909(−1)	1.573028	3.429561(−1)
2.0	7.710133(−1)	1.227525(−1)	1.677914	3.027037(−1)
3.0	1.125576	1.334554(−1)	1.899694	2.450111(−1)
3.5	1.301794	1.364345(−1)	2.014114	2.234706(−1)
4.0	1.477686	1.385637(−1)	2.130089	2.052718(−1)
5.0	1.829008	1.413013(−1)	2.365454	1.762377(−1)
6.0	2.180127	1.429115(−1)	2.604078	1.541579(−1)
7.0	2.531257	1.439333(−1)	2.844990	1.368475(−1)
9.0	3.233786	1.451138(−1)	3.331419	1.115442(−1)
1.0(1)	3.585202	1.454767(−1)	3.576236	1.020442(−1)
1.0(2)	3.527277(1)	1.474937(−1)	2.602506(1)	1.159143(−2)

While higher-order approximations were required to achieve the desired degree of accuracy for the case $R = 0.001$, we have typically used $N = 80$ to generate the results shown in Tables 1–3, and we note that our FORTRAN implementation (no special effort was made to make the code especially efficient) of our discrete-ordinates solution (with $N = 80$) runs in a few seconds on a 400 MHz Pentium-based PC.

7. Final remarks

In successfully extending the use of the Mitsis transformations to find a convenient pseudo-problem to describe flow, as described by the S-model kinetic equations, we have been able to use effectively an analytical version of the discrete-ordinates method to solve especially well (we believe) the important problems of Poiseuille and thermal-creep flow in a cylindrical tube. While the kinetic equations for the S model lead to a system of moment equations, in contrast to the scalar formulation obtained from the classical BGK model, the final computations were successfully implemented in what we consider to be a definitive manner. It can be seen clearly that a great deal of analytical work (calculus) has been used to obtain the forms that defined

Table 3
Heat-flow profiles at the wall and heat-flow rates

R	$q_P(R)$	$-q_T(R)$	Q_P	$-Q_T$
1.0(−3)	2.793537(−4)	1.261629(−3)	7.469327(−1)	3.369503
1.0(−2)	2.657105(−3)	1.219530(−2)	7.210237(−1)	3.284827
2.0(−2)	5.109436(−3)	2.371572(−2)	7.019628(−1)	3.216628
3.0(−2)	7.417747(−3)	3.472991(−2)	6.868658(−1)	3.159537
4.0(−2)	9.609956(−3)	4.532003(−2)	6.741078(−1)	3.109260
5.0(−2)	1.170357(−2)	5.553981(−2)	6.629545(−1)	3.063811
7.0(−2)	1.564123(−2)	7.501621(−2)	6.439594(−1)	2.983242
9.0(−2)	1.929858(−2)	9.338320(−2)	6.280139(−1)	2.912576
1.0(−1)	2.103640(−2)	1.022000(−1)	6.208757(−1)	2.880057
3.0(−1)	4.730447(−2)	2.431888(−1)	5.304811(−1)	2.425120
5.0(−1)	6.345393(−2)	3.420581(−1)	4.784350(−1)	2.136032
7.0(−1)	7.329508(−2)	4.160334(−1)	4.404396(−1)	1.920322
9.0(−1)	7.858345(−2)	4.733153(−1)	4.100247(−1)	1.748587
1.0	7.986698(−2)	4.972763(−1)	3.967500(−1)	1.674548
1.5	7.618417(−2)	5.852983(−1)	3.429561(−1)	1.383987
2.0	6.071916(−2)	6.400572(−1)	3.027037(−1)	1.179408
3.0	9.743156(−3)	7.016543(−1)	2.450111(−1)	9.082233(−1)
3.5	−2.172108(−2)	7.200142(−1)	2.234706(−1)	8.136837(−1)
4.0	−5.564367(−2)	7.338079(−1)	2.052718(−1)	7.365532(−1)
5.0	−1.283763(−1)	7.529442(−1)	1.762377(−1)	6.184590(−1)
6.0	−2.051841(−1)	7.654700(−1)	1.541579(−1)	5.324764(−1)
7.0	−2.844151(−1)	7.742648(−1)	1.368475(−1)	4.672002(−1)
9.0	−4.469509(−1)	7.857839(−1)	1.115442(−1)	3.748535(−1)
1.0(1)	−5.294313(−1)	7.897641(−1)	1.020442(−1)	3.410421(−1)
1.0(2)	−8.177668	8.214638(−1)	1.159143(−2)	3.715566(−2)

our final computational problem, but the end result was a more or less analytical solution for the considered S-model flow problems that yields high quality numerical results at very modest computational expense.

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