PERGAMON

Journal of Quantitative Spectroscopy &
Radiative Transfer 72 (2002) 75–88

Journal of
Quantitative
Spectroscopy &
Radiative
Transfer

www.elsevier.com/locate/jqsrt

Generalized boundary conditions for the S-model kinetic equations basic to flow in a plane channel

C.E. Siewert

Mathematics Department, North Carolina State University, Raleigh, NC 27695–8205, USA

Received 5 December 2000; accepted 29 January 2001

Abstract

An analytical version of the discrete-ordinates method is used to solve the classical problems of Poiseuille flow and thermal-creep flow in a plane channel. The kinetic theory for the rarefied-gas flow is based on the S model (a generalization of the BGK model), and in addition to the use of the diffuse–specular reflection model (based on a single accommodation coefficient) for describing particle scattering from the channel walls, the Cercignani–Lampis model defined in terms of normal and tangential accommodation coefficients is implemented. The established solution is tested numerically, and results for the velocity and heat-flow profiles, the particle-flow rate and the heat-flow rate thought to be correct to many significant figures are reported for various values of the channel width. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Rarefied gas dynamics; Discrete ordinates

1. Introduction

Two recent papers, one by Valougeorgis [1] and the other by Sharipov [2], have discussed the flow, as described by the so-called S model [3], of a rarefied gas in a plane channel. The Valougeorgis paper [1] makes use of an analytical version [4] of the discrete-ordinates method [5] and deals with Poiseuille and thermal-creep flow for the general case of diffuse–specular reflection by the channel walls that confine the flow. On the other hand, Sharipov [2] used an optimized discrete velocity method [6] to solve these same two problems (with an error estimated to be less than 0.1%) by considering that the reflection by the channel walls is described by the Cercignani–Lampis model [7]. In this work these two classical problems are resolved in a concise and accurate way, and some comparisons are made between the two models for describing the reflections by the channel walls. While the books of Cercignani [8,9]

PII: S0022-4073(01)00057-7

and Williams [10] provide excellent material relevant to this field, a comprehensive review recently reported by Sharipov and Seleznev [11] is also a useful up-to-date source that pays much attention to comparing different computational methods as well as different mathematical formulations basic to rarefied-gas dynamics.

2. Defining equations

We start with the basic equation relevant to S-model applications in plane geometry written as

$$\boldsymbol{c}\cdot\nabla h(\boldsymbol{r},\boldsymbol{c})+h(\boldsymbol{r},\boldsymbol{c})=\pi^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-c'^2}h(\boldsymbol{r},\boldsymbol{c}')K(\boldsymbol{c}':\boldsymbol{c})\,\mathrm{d}c_x'\,\mathrm{d}c_y'\,\mathrm{d}c_z'+k(\boldsymbol{c}), \tag{1}$$

where the inhomogeneous driving term, relevant to flow in the x direction, is

$$k(\boldsymbol{c})=-c_x[k_1+k_2(c_x^2+c_y^2+c_z^2-5/2)]. \tag{2}$$

Here

$$K(\boldsymbol{c}':\boldsymbol{c})=1+2\boldsymbol{c}'\cdot\boldsymbol{c}+(2/3)(c'^2-3/2)(c^2-3/2)+M(\boldsymbol{c}':\boldsymbol{c}), \tag{3}$$

where

$$M(\boldsymbol{c}':\boldsymbol{c})=(4/15)\boldsymbol{c}'\cdot\boldsymbol{c}(c'^2-5/2)(c^2-5/2) \tag{4}$$

is the term [11] added to the BGK form [12] to yield the scattering kernel for the S model. We note that the problems of Poiseuille flow and thermal-creep flow discussed later in this work are defined, respectively, by the choices $k_1=1, k_2=0$ and $k_1=0, k_2=1$ in the driving term $k(\boldsymbol{c})$. Also, we note that the velocity vector $\boldsymbol{c}$ has the rectangular components $\{c_x, c_y, c_z\}$ and magnitude c. Now with regard to flow (in the x direction) in a plane channel (without variation in the x and z directions) defined by $y\in[-a,a]$ we can rewrite Eq. (1) as

$$c_y\frac{\partial}{\partial y}h(y,\boldsymbol{c})+h(y,\boldsymbol{c})=\pi^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-c'^2}h(y,\boldsymbol{c}')K(\boldsymbol{c}':\boldsymbol{c})\,\mathrm{d}c_x'\,\mathrm{d}c_y'\,\mathrm{d}c_z'+k(\boldsymbol{c}), \tag{5}$$

where we have taken account of the fact that the distribution function h depends on one spatial variable only. In regard to boundary conditions we consider that

$$h(-a,c_x,c_y,c_z)=\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-c'^2}h(-a,c_x',-c_y',c_z')R(c_x',-c_y',c_z':c_x,c_y,c_z)\,\mathrm{d}c_x'\,\mathrm{d}c_z'\,\mathrm{d}c_y' \tag{6a}$$

and

$$h(a,c_x,-c_y,c_z)=\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{-c'^2}h(a,c_x',c_y',c_z')R(c_x',c_y',c_z':c_x,-c_y,c_z)\,\mathrm{d}c_x'\,\mathrm{d}c_z'\,\mathrm{d}c_y' \tag{6b}$$

for $c_y \in (0,\infty)$ and all c_x and c_z. Here $R(\boldsymbol{c}':\boldsymbol{c})$ is used to describe the effect of the walls on the particle distribution function, and we have written

$$h(y,\boldsymbol{c}) = h(y,c_x,c_y,c_z). \tag{7}$$

We consider that the distribution function $h(y,c_x,c_y,c_z)$ defined by the basic kinetic equation, written as Eq. (5), and the boundary conditions, written as Eqs. (6), depends on the spatial variable $y \in [-a,a]$, written in dimensionless units, and the three components of velocity $c_x,c_y,c_z \in (-\infty,\infty)$ which are also expressed in dimensionless units. Following Valougeorgis [1] and Sharipov [2], we use the S model to study the effect on the flow resulting from the action of the reflection kernel $R(\boldsymbol{c}':\boldsymbol{c})$, and while various physical constraints must be placed on this function, we use, at this point, only the condition

$$R(c_x',-c_y',c_z':c_x,c_y,c_z) = R(c_x',c_y',c_z':c_x,-c_y,c_z) \tag{8}$$

for c_y and $c_y' \in (0,\infty)$ and all c_x, c_x', c_z and c_z'. It is clear that Eq. (8) implies that the two surfaces of the channel reflect gas particles in the same way. And so we conclude from Eqs. (8) that the basic distribution function satisfies the symmetry condition

$$h(y,c_x,c_y,c_z) = h(-y,c_x,-c_y,c_z) \tag{9}$$

for all y, c_x, c_y and c_z.

In this work we seek to compute physical quantities related to particle velocities and heat flow, and since some different notations are used [1,2] and in order to be very clear about our terminology we refer to

$$u(y) = \pi^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-c^2} h(y,c_x,c_y,c_z)c_x\,dc_x\,dc_y\,dc_z \tag{10}$$

as the velocity profile and to

$$q(y) = \pi^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-c^2} h(y,c_x,c_y,c_z)(c^2-5/2)c_x\,dc_x\,dc_y\,dc_z \tag{11}$$

as the heat-flow profile. We note that $u(y)$ and $q(y)$ are the basic quantities of interest, and so we do not actually have to compute the complete distribution function $h(y,c_x,c_y,c_z)$. Instead, we can obtain the results we seek from various moments, or integrals, of the distribution function. And so to start our development, we multiply Eq. (5) by

$$\phi_1(c_x,c_z) = (1/\pi)c_x e^{-(c_x^2+c_z^2)}, \tag{12}$$

integrate over all c_x and c_z and introduce the new variables $\xi = c_y$ and $\xi' = c_y'$ to find

$$\xi\frac{\partial}{\partial y}h_1(y,\xi) + h_1(y,\xi) = \pi^{-1/2}\int_{-\infty}^{\infty} e^{-\xi'^2}[f_{11}(\xi',\xi)h_1(y,\xi') + f_{12}(\xi)h_2(y,\xi')]\,d\xi' + a_1(\xi) \tag{13}$$

where

$$h_1(y,c_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_1(c_x,c_z)h(y,c_x,c_y,c_z)\,dc_x\,dc_z \tag{14a}$$

and

$$h_2(y,c_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_2(c_x,c_z)h(y,c_x,c_y,c_z)\,\mathrm{d}c_x\,\mathrm{d}c_z. \tag{14b}$$

Here

$$\phi_2(c_x,c_z) = (1/\pi)2^{-1/2}c_x(c_x^2+c_z^2-2)\mathrm{e}^{-(c_x^2+c_z^2)}, \tag{15}$$

$$a_1(\xi) = -(1/2)[k_1 + k_2(\xi^2 - 1/2)], \tag{16}$$

$$f_{11}(\xi',\xi) = 1 + (2/15)(\xi'^2 - 1/2)(\xi^2 - 1/2) \tag{17a}$$

and

$$f_{12}(\xi) = (2/15)2^{1/2}(\xi^2 - 1/2). \tag{17b}$$

Now multiplying Eq. (5) by $\phi_2(c_x,c_z)$ and integrating, we find an equation to go with Eq. (13), viz.

$$\xi\frac{\partial}{\partial y}h_2(y,\xi) + h_2(y,\xi) = \pi^{-1/2}\int_{-\infty}^{\infty} \mathrm{e}^{-\xi'^2}[f_{12}(\xi')h_1(y,\xi') + f_{22}h_2(y,\xi')]\,\mathrm{d}\xi' + a_2, \tag{18}$$

where

$$f_{22} = 4/15 \tag{19}$$

and

$$a_2 = -2^{-1/2}k_2. \tag{20}$$

If we let $\boldsymbol{H}(y,\xi)$ denote the vector-valued function with components $h_1(y,\xi)$ and $h_2(y,\xi)$, then we can write Eqs. (13) and (18) as

$$\xi\frac{\partial}{\partial y}\boldsymbol{H}(y,\xi) + \boldsymbol{H}(y,\xi) = \pi^{-1/2}\boldsymbol{Q}(\xi)\int_{-\infty}^{\infty} \mathrm{e}^{-\xi'^2}\boldsymbol{Q}^{\mathrm{T}}(\xi')\boldsymbol{H}(y,\xi')\,\mathrm{d}\xi' + \boldsymbol{A}(\xi), \tag{21}$$

where we use the superscript T to denote the transpose operation,

$$\boldsymbol{Q}(\xi) = \begin{bmatrix} (2/15)^{1/2}(\xi^2 - 1/2) & 1 \\ 2(15)^{-1/2} & 0 \end{bmatrix} \tag{22}$$

and the two elements of $\boldsymbol{A}(\xi)$ are $a_1(\xi)$ and a_2 as defined by Eqs. (16) and (20). If we now let

$$\boldsymbol{H}(y,\xi) = \boldsymbol{Q}(\xi)\boldsymbol{G}(y,\xi) \tag{23}$$

and

$$\boldsymbol{A}(\xi) = \boldsymbol{Q}(\xi)\boldsymbol{\Gamma} \tag{24}$$

then we can rewrite Eq. (21) as

$$\xi\frac{\partial}{\partial y}\boldsymbol{G}(y,\xi)+\boldsymbol{G}(y,\xi)=\int_{-\infty}^{\infty}\boldsymbol{\Psi}(\xi')\boldsymbol{G}(y,\xi')\,\mathrm{d}\xi'+\boldsymbol{\Gamma}, \tag{25}$$

where

$$\boldsymbol{\Psi}(\xi)=\pi^{-1/2}\mathrm{e}^{-\xi^2}\boldsymbol{Q}^{\mathrm{T}}(\xi)\boldsymbol{Q}(\xi). \tag{26}$$

To be explicit, we note that

$$\boldsymbol{\Gamma}=-(1/2)\begin{bmatrix}(15/2)^{1/2}k_2\\ k_1\end{bmatrix}. \tag{27}$$

Now with regard to the boundary conditions listed as Eqs. (6), we follow Sharipov [2] and use

$$R(c_x',c_y',c_z':c_x,c_y,c_z)=\mathrm{e}^{c^2}T(c_x':c_x)S(c_y':c_y)T(c_z':c_z), \tag{28}$$

where

$$T(x:y)=[\pi\alpha_{\mathrm{t}}(2-\alpha_{\mathrm{t}})]^{-1/2}\mathrm{e}^{-[y-(1-\alpha_{\mathrm{t}})x]^2/[\alpha_{\mathrm{t}}(2-\alpha_{\mathrm{t}})]} \tag{29a}$$

and

$$S(x:y)=\frac{2|x|}{\alpha_{\mathrm{n}}}\mathrm{e}^{-[y^2+(1-\alpha_{\mathrm{n}})x^2]/\alpha_{\mathrm{n}}}I_0[2(1-\alpha_{\mathrm{n}})^{1/2}|xy|/\alpha_{\mathrm{n}}]. \tag{29b}$$

Here $I_0(z)$ is used to denote a modified Bessel function, viz.

$$I_0(z)=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{z\cos\phi}\,\mathrm{d}\phi. \tag{30}$$

In writing Eqs. (28)–(30), Sharipov [2] based his development of the boundary condition on the gas–surface interaction model introduced by Cercignani and Lampis [7]. We note that α_{t} is the accommodation coefficient of tangential momentum and that α_{n} is the accommodation coefficient of energy corresponding to the normal component of velocity [2,7].

Since we have the symmetry conditions listed as Eqs. (8) and (9), we need consider only one of Eqs. (6). And so if we now multiply Eq. (6b) by $\phi_1(c_x,c_y)$ or $\phi_2(c_x,c_y)$ and integrate over all c_x and c_z, we find, after making use of Eqs. (28)–(30),

$$\boldsymbol{H}(a,-\xi)=\boldsymbol{A}\int_0^{\infty}\mathrm{e}^{-\xi'^2}\boldsymbol{H}(a,\xi')f(\xi',\xi)\,\mathrm{d}\xi',\quad \xi\in(0,\infty), \tag{31}$$

where

$$\boldsymbol{A}=\mathrm{diag}\{1-\alpha_{\mathrm{t}},(1-\alpha_{\mathrm{t}})^3\} \tag{32}$$

and

$$f(\xi':\xi)=(2\xi'/\alpha_{\mathrm{n}})\mathrm{e}^{-(1-\alpha_{\mathrm{n}})(\xi'^2+\xi^2)/\alpha_{\mathrm{n}}}I_0[2(1-\alpha_{\mathrm{n}})^{1/2}\xi'\xi/\alpha_{\mathrm{n}}] \tag{33}$$

for $\xi,\xi'\in(0,\infty)$. As a special case, we note that

$$\lim_{\alpha_{\mathrm{n}}\to 0}\mathrm{e}^{-\xi'^2}f(\xi':\xi)=\Delta(\xi-\xi'), \tag{34}$$

where

$$\Delta(\xi-\xi') = \lim_{\alpha\to 0}\frac{1}{(\alpha\pi)^{1/2}}\mathrm{e}^{-(\xi-\xi')^2/\alpha} \tag{35}$$

is a "representation" of the generalized function $\delta(\xi-\xi')$. So Eq. (31) yields

$$\boldsymbol{H}(a,-\xi) = \boldsymbol{A}\boldsymbol{H}(a,\xi), \quad \xi\in(0,\infty), \tag{36}$$

for the case $\alpha_\mathrm{n}\to 0$. Considering another special case of interest, we see that Eq. (31) yields

$$\boldsymbol{H}(a,-\xi) = 2\boldsymbol{A}\int_0^\infty \xi'\mathrm{e}^{-\xi'^2}\boldsymbol{H}(a,\xi')\,\mathrm{d}\xi', \quad \xi\in(0,\infty), \tag{37}$$

for the case $\alpha_\mathrm{n}=1$. Now since we have formulated our basic problem in terms of $\boldsymbol{G}(y,\xi)$ we require the boundary conditions to go with Eq. (25). Noting Eq. (23), we rewrite Eq. (31) as

$$\boldsymbol{G}(a,-\xi) = \int_0^\infty \boldsymbol{T}(\xi':\xi)\boldsymbol{G}(a,\xi')\,\mathrm{d}\xi', \quad \xi\in(0,\infty), \tag{38}$$

where

$$\boldsymbol{T}(\xi':\xi) = \boldsymbol{Q}(\xi)^{-1}\boldsymbol{A}\boldsymbol{Q}(\xi')k(\xi':\xi). \tag{39}$$

For computational reasons we choose to write

$$k(\xi':\xi) = \mathrm{e}^{-\xi'^2}f(\xi':\xi) \tag{40}$$

as

$$k(\xi':\xi) = (2\xi'/\alpha_\mathrm{n})\mathrm{e}^{-(1/\alpha_\mathrm{n})[(1-\alpha_\mathrm{n})^{1/2}\xi-\xi']^2}\hat{I}_0[2(1-\alpha_\mathrm{n})^{1/2}\xi'\xi/\alpha_\mathrm{n}] \tag{41}$$

for $\xi,\xi'\in(0,\infty)$. Here

$$\hat{I}_0(z) = I_0(z)\mathrm{e}^{-z}. \tag{42}$$

To be consistent with Eq. (38), we rewrite the two special cases listed as Eqs. (36) and (37) as

$$\boldsymbol{G}(a,-\xi) = \boldsymbol{Q}^{-1}(\xi)\boldsymbol{A}\boldsymbol{Q}(\xi)\boldsymbol{G}(a,\xi), \quad \xi\in(0,\infty),\ \alpha_\mathrm{n}=0, \tag{43}$$

and

$$\boldsymbol{G}(a,-\xi) = 2\boldsymbol{Q}^{-1}(\xi)\boldsymbol{A}\int_0^\infty \xi'\mathrm{e}^{-\xi'^2}\boldsymbol{Q}(\xi')\boldsymbol{G}(a,\xi')\,\mathrm{d}\xi', \quad \xi\in(0,\infty),\ \alpha_\mathrm{n}=1. \tag{44}$$

To reiterate, we note that the general result of the Cercignani–Lampis model for the wall reflection function is represented here by Eq. (38) and that Eqs. (43) and (44) are two special cases of this model. This Cercignani–Lampis model clearly is based on two accommodation coefficients α_n and α_t. On the other hand, the usual combination of diffuse and specular reflection [8–10], as used by Valougeorgis [1] and by Siewert and Valougeorgis [12] for Poiseuille and thermal-creep flow in cylindrical tubes, corresponds to writing Eq. (38) as

$$\boldsymbol{G}(a,-\xi) = (1-\alpha)\boldsymbol{G}(a,\xi), \quad \xi\in(0,\infty), \tag{45}$$

where $\alpha\in(0,1]$ is the (single) accommodation coefficient. It is clear that the Cercignani–Lampis model does not include the diffuse–specular model as a special case.

So having formulated the $\boldsymbol{G}$ problem we intend to solve, we can express the physical quantities we seek, viz. the velocity profile and the heat-flow profile as defined by Eqs. (10) and (11), as

$$u(y) = [0 \quad 1]\boldsymbol{G}(y) \tag{46}$$

and

$$q(y) = (15/2)^{1/2}[1 \quad 0]\boldsymbol{G}(y), \tag{47}$$

where

$$\boldsymbol{G}(y) = \int_{-\infty}^{\infty} \boldsymbol{\Psi}(\xi)\boldsymbol{G}(y,\xi)\,\mathrm{d}\xi. \tag{48}$$

3. A particular solution and the discrete-ordinates method

Before starting our discrete-ordinates solution we develop an analytical particular solution to account for the inhomogeneous term in Eq. (25). Proposing a solution of the form

$$\boldsymbol{G}^{\mathrm{p}}(y,\xi) = \boldsymbol{B}y^2 + \boldsymbol{C}y\xi + \boldsymbol{D}\xi^2 + \boldsymbol{E}, \tag{49}$$

where $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ and $\boldsymbol{E}$ are constant vectors, we can easily find that

$$\boldsymbol{G}^{\mathrm{p}}(y,\xi) = \frac{1}{2}\begin{bmatrix} (1/10)30^{1/2}k_1 - (3/4)(30)^{1/2}k_2 \\ (y^2 - 2y\xi + 2\xi^2)k_1 \end{bmatrix} \tag{50}$$

is a solution of Eq. (25). So now if we write

$$\boldsymbol{G}(y,\xi) = \boldsymbol{G}^{\mathrm{h}}(y,\xi) + \boldsymbol{G}^{\mathrm{p}}(y,\xi), \tag{51}$$

then to complete the desired solution we must solve the homogeneous equation

$$\xi\frac{\partial}{\partial y}\boldsymbol{G}^{\mathrm{h}}(y,\xi) + \boldsymbol{G}^{\mathrm{h}}(y,\xi) = \int_{-\infty}^{\infty} \boldsymbol{\Psi}(\xi')\boldsymbol{G}^{\mathrm{h}}(y,\xi')\,\mathrm{d}\xi' \tag{52}$$

subject to either the Cercignani–Lampis boundary condition

$$\boldsymbol{G}^{\mathrm{h}}(a,-\xi) - \int_0^{\infty} \boldsymbol{T}(\xi' : \xi)\boldsymbol{G}^{\mathrm{h}}(a,\xi')\,\mathrm{d}\xi' = \boldsymbol{R}(\xi), \quad \xi \in (0,\infty), \tag{53}$$

where the known term is

$$\boldsymbol{R}(\xi) = \int_0^{\infty} \boldsymbol{T}(\xi' : \xi)\boldsymbol{G}^{\mathrm{p}}(a,\xi')\,\mathrm{d}\xi' - \boldsymbol{G}^{\mathrm{p}}(a,-\xi) \tag{54}$$

or the diffuse–specular boundary condition

$$\boldsymbol{G}^{\mathrm{h}}(a,-\xi) - (1-\alpha)\boldsymbol{G}^{\mathrm{h}}(a,\xi) = \boldsymbol{R}(\xi), \quad \xi \in (0,\infty), \tag{55}$$

where the known term is

$$\boldsymbol{R}(\xi) = (1-\alpha)\boldsymbol{G}^{\mathrm{p}}(a,\xi) - \boldsymbol{G}^{\mathrm{p}}(a,-\xi). \tag{56}$$

Of course, since the particular solution $\boldsymbol{G}^{\mathrm{p}}(y,\xi)$ already has the appropriate symmetry, we must also insist that the homogeneous component $\boldsymbol{G}^{\mathrm{h}}(y,\xi)$ be such that

$$\boldsymbol{G}^{\mathrm{h}}(y,-\xi)=\boldsymbol{G}^{\mathrm{h}}(-y,\xi) \tag{57}$$

for all y and ξ. Since the development of our discrete-ordinates solution follows very closely Ref. [13], we can be brief here. We replace the integral term in Eq. (52) by a quadrature approximation and look for solutions of the form

$$\boldsymbol{G}^{\mathrm{h}}(y,\xi)=\boldsymbol{\Phi}(v,\xi)\mathrm{e}^{-y/v} \tag{58}$$

to find

$$(v-\xi)\boldsymbol{\Phi}(v,\xi)=v\sum_{k=1}^{N}w_k\boldsymbol{\Psi}(\xi_k)[\boldsymbol{\Phi}(v,\xi_k)+\boldsymbol{\Phi}(v,-\xi_k)], \tag{59}$$

where the N nodes and weights $\{\xi_k,w_k\}$ are defined for evaluating integrals over the interval $[0,\infty)$. If we now let $\boldsymbol{\Phi}_+(v)$ and $\boldsymbol{\Phi}_-(v)$ denote $2N\times 1$ vectors, the 2×1 components of which are, respectively, $\boldsymbol{\Phi}(v,\xi_k)$ and $\boldsymbol{\Phi}(v,-\xi_k)$, then we can let

$$\boldsymbol{U}=\boldsymbol{\Phi}_+(v)+\boldsymbol{\Phi}_-(v) \tag{60}$$

and deduce from Eq. (59) evaluated at $\xi=\pm\xi_k$ the eigenvalue problem

$$(\boldsymbol{D}-2\boldsymbol{W})\boldsymbol{U}=\lambda\boldsymbol{U}, \tag{61}$$

where $\lambda=1/v^2$. In addition

$$\boldsymbol{D}=\mathrm{diag}\{(1/\xi_1)^2\boldsymbol{I},(1/\xi_2)^2\boldsymbol{I},\dots,(1/\xi_N)^2\boldsymbol{I}\} \tag{62}$$

and $\boldsymbol{W}$ is a $2N\times 2N$ matrix each $2\times 2N$ row of which is given by

$$\boldsymbol{R}_i=(1/\xi_i)^2\,[w_1\boldsymbol{\Psi}(\xi_1)\quad w_2\boldsymbol{\Psi}(\xi_2)\quad\cdots\quad w_N\boldsymbol{\Psi}(\xi_N)] \tag{63}$$

for $i=1,2,\dots,N$. We note also that $\boldsymbol{I}$ in Eq. (62) is used to denote the 2×2 identity matrix. If we now let $\{\lambda_j,\boldsymbol{U}_j\}$, $j=1,2,\dots,2N$, denote the eigenvalues and eigenvectors defined by Eq. (61), then we find that we can write

$$\boldsymbol{\Phi}_+(v_j)=\frac{1}{2v_j}\,\mathrm{diag}\{(v_j+\xi_1)\boldsymbol{I},(v_j+\xi_2)\boldsymbol{I},\dots,(v_j+\xi_N)\boldsymbol{I}\}\boldsymbol{U}_j \tag{64a}$$

and

$$\boldsymbol{\Phi}_-(v_j)=\frac{1}{2v_j}\,\mathrm{diag}\{(v_j-\xi_1)\boldsymbol{I},(v_j-\xi_2)\boldsymbol{I},\dots,(v_j-\xi_N)\boldsymbol{I}\}\boldsymbol{U}_j, \tag{64b}$$

where v_j is the positive square root of $1/\lambda_j$. If we let $\boldsymbol{G}^{\mathrm{h}}_\pm(y)$ denote vectors with the vectors $\boldsymbol{G}^{\mathrm{h}}(y,\pm\xi_k)$ as components, then we can make note of Eq. (57) and write

$$\boldsymbol{G}^{\mathrm{h}}_\pm(y)=\sum_{j=1}^{2N}A_j[\boldsymbol{\Phi}_\pm(v_j)\mathrm{e}^{-(a+y)/v_j}+\boldsymbol{\Phi}_\mp(v_j)\mathrm{e}^{-(a-y)/v_j}], \tag{65}$$

where the $\{A_j\}$ are arbitrary constants to be determined from a discrete-ordinates version of the boundary condition listed as either Eq. (53) or Eq. (55). Now since, as discussed in Ref. [13], one of the separation constants, say v_1, becomes unbounded as N tends to infinity, we choose to use the exact result ($N \to \infty$) corresponding to this separation constant and therefore to rewrite Eq. (65) as

$$\boldsymbol{G}^{\rm h}_{\pm}(y) = A_1\boldsymbol{\Phi} + \sum_{j=2}^{2N} A_j[\boldsymbol{\Phi}_{\pm}(v_j){\rm e}^{-(a+y)/v_j} + \boldsymbol{\Phi}_{\mp}(v_j){\rm e}^{-(a-y)/v_j}], \tag{66}$$

where the $\boldsymbol{\Phi}$ has N vector components given by

$$\boldsymbol{\Phi}_{+} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{67}$$

Considering that we have established the constants $A_j, j = 1, 2, \dots, 2N$, by solving the linear system obtained when Eq. (66) is substituted into a discrete-ordinates version of Eq. (53) or (55), we find that we can now use Eq. (51), with Eqs. (50) and (66), to deduce from Eq. (48) that

$$\boldsymbol{G}(y) = \boldsymbol{G}_{*}(y) + \sum_{j=2}^{2N} A_j[{\rm e}^{-(a+y)/v_j} + {\rm e}^{-(a-y)/v_j}]\boldsymbol{N}(v_j), \tag{68}$$

where

$$\boldsymbol{G}_{*}(y) = \begin{bmatrix} 30^{1/2}[(1/20)k_1 - (1/8)k_2] \\ (1/2)(y^2+1)k_1 + A_1 \end{bmatrix} \tag{69}$$

and

$$\boldsymbol{N}(v_j) = [w_1\boldsymbol{\Psi}(\xi_1) \quad w_2\boldsymbol{\Psi}(\xi_2) \quad \cdots \quad w_N\boldsymbol{\Psi}(\xi_N)][\boldsymbol{\Phi}_{+}(v_j) + \boldsymbol{\Phi}_{-}(v_j)]. \tag{70}$$

Having found $\boldsymbol{G}(y)$, we can use Eq. (68) in Eqs. (46) and (47) to find the velocity and heat-flow profiles we seek. In this way we obtain

$$u(y) = (1/2)(y^2+1)k_1 + A_1 + \sum_{j=2}^{2N} A_j[{\rm e}^{-(a+y)/v_j} + {\rm e}^{-(a-y)/v_j}]N_2(v_j) \tag{71}$$

and

$$q(y) = (3/4)k_1 - (15/8)k_2 + (15/2)^{1/2}\sum_{j=2}^{2N} A_j[{\rm e}^{-(a+y)/v_j} + {\rm e}^{-(a-y)/v_j}]N_1(v_j), \tag{72}$$

where $N_1(v_j)$ and $N_2(v_j)$ are the components of $\boldsymbol{N}(v_j)$. At this point we can complete this section by using Eqs. (71) and (72) in

$$U = \frac{1}{2a^2}\int_{-a}^{a} u(y)\,{\rm d}y \tag{73}$$

and

$$Q = \frac{1}{2a^2}\int_{-a}^{a} q(y)\,{\rm d}y \tag{74}$$

to find expressions for the particle-flow rate U and the heat-flow Q, viz.

$$U = \frac{1}{2a^2}[ak_1(a^2/3+1) + 2aA_1 + 2\sum_{j=2}^{2N} A_j\nu_j(1-\mathrm{e}^{-2a/\nu_j})N_2(\nu_j)] \tag{75}$$

and

$$Q = \frac{1}{2a^2}[(3a/2)k_1 - (a/4)k_2 + 2(15/8)^{1/2}\sum_{j=2}^{2N} A_j\nu_j(1-\mathrm{e}^{-2a/\nu_j})N_1(\nu_j)]. \tag{76}$$

4. Numerical results

Repeating much of the discussion given in Ref. [13], where the version of the discrete-ordinates method used here was used to solve the Poiseuille and thermal-creep problems for flow in a cylindrical tube, we note that what we must now do is to define the quadrature scheme to be used in our discrete-ordinates solution. In this work we have used the (nonlinear) transformation

$$u(\xi) = \exp\{-\xi\} \tag{77}$$

to map $\xi \in [0,\infty)$ into $u \in [0,1]$, and we then used a Gauss–Legendre scheme mapped (linearly) onto the interval $[0,1]$.

Table 1
Profiles ($2a=1$): Cercignani–Lampis model ($\alpha_n = 0.5$ and $\alpha_t = 0.5$)

x/a	$-u_P(x)$	$q_P(x)$	$u_T(x)$	$-q_T(x)$
0.00	1.777590	2.406295(−1)	2.308391(−1)	1.054961
0.05	1.776919	2.404129(−1)	2.306809(−1)	1.054427
0.10	1.774904	2.397614(−1)	2.302050(−1)	1.052822
0.15	1.771536	2.386709(−1)	2.294087(−1)	1.050134
0.20	1.766801	2.371338(−1)	2.282867(−1)	1.046342
0.25	1.760680	2.351396(−1)	2.268318(−1)	1.041417
0.30	1.753144	2.326737(−1)	2.250342(−1)	1.035317
0.35	1.744158	2.297176(−1)	2.228811(−1)	1.027993
0.40	1.733678	2.262477(−1)	2.203564(−1)	1.019378
0.45	1.721646	2.222343(−1)	2.174400(−1)	1.009390
0.50	1.707993	2.176404(−1)	2.141065(−1)	9.979281(−1)
0.55	1.692628	2.124192(−1)	2.103243(−1)	9.848626(−1)
0.60	1.675437	2.065109(−1)	2.060530(−1)	9.700305(−1)
0.65	1.656272	1.998383(−1)	2.012403(−1)	9.532213(−1)
0.70	1.634936	1.922988(−1)	1.958173(−1)	9.341567(−1)
0.75	1.611157	1.837514(−1)	1.896892(−1)	9.124570(−1)
0.80	1.584542	1.739924(−1)	1.827199(−1)	8.875782(−1)
0.85	1.554483	1.627059(−1)	1.746988(−1)	8.586841(−1)
0.90	1.519916	1.493411(−1)	1.652605(−1)	8.243358(−1)
0.95	1.478540	1.327038(−1)	1.536164(−1)	7.814630(−1)
1.00	1.419022	1.068938(−1)	1.358984(−1)	7.153601(−1)

Table 2
Flow rates: Cercignani–Lampis model ($\alpha_n = 0.5$ and $\alpha_t = 0.5$)

$2a$	$-U_P$	$Q_P = U_T$	$-Q_T$
1.0(−2)	5.014219	1.423000	7.460666
2.0(−2)	4.668298	1.249702	6.602193
3.0(−2)	4.474712	1.151878	6.102448
4.0(−2)	4.342079	1.084182	5.748874
5.0(−2)	4.242237	1.032667	5.475141
7.0(−2)	4.097278	9.566156(−1)	5.063181
9.0(−2)	3.993879	9.010436(−1)	4.756102
1.0(−1)	3.951889	8.780440(−1)	4.627533
3.0(−1)	3.573656	6.471470(−1)	3.298498
5.0(−1)	3.447263	5.444746(−1)	2.698543
7.0(−1)	3.388287	4.786409(−1)	2.317748
9.0(−1)	3.359841	4.307185(−1)	2.044533
1.0	3.352483	4.110242(−1)	1.933542
3.0	3.499791	2.272407(−1)	9.539034(−1)
3.5	3.565466	2.056379(−1)	8.481270(−1)
4.0	3.634859	1.879290(−1)	7.634316(−1)
5.0	3.780735	1.604938(−1)	6.361523(−1)
6.0	3.932453	1.401296(−1)	5.450338(−1)
7.0	4.087720	1.243653(−1)	4.765955(−1)
9.0	4.404578	1.014953(−1)	3.807025(−1)
1.0(1)	4.565036	9.292892(−2)	3.458333(−1)
1.0(2)	19.50102	1.065782(−2)	3.720775(−2)

Having defined our quadrature scheme and in developing a FORTRAN implementation of our solution, we found the required separation constants $\{v_j\}$ by using the driver program RG from the EISPACK collection [14] to solve the eigenvalue/eigenvector problem defined by Eq. (61). The required separation constants were then available as the reciprocals of the positive square roots of these eigenvalues, and the eigenvectors were used in Eqs. (64) to establish the elementary vectors $\mathbf{\Phi}_+(v_j)$ and $\mathbf{\Phi}_-(v_j)$. We then used the subroutines DGECO and DGESL from the LINPACK package [15] to solve the linear system that defines the required constants $A_j, j = 1, 2, \ldots, 2N$. And in this way our solution was established as a viable algorithm. Finally, but importantly, we have found that elements of the matrix-valued function $\mathbf{\Psi}(\xi)$ as defined by Eqs. (26) can be essentially zero (from a computational point-of-view). In such cases, we found that by defining an element to be precisely zero when that element is less than, say, $\varepsilon = 10^{-20}$, we increased the ability of the linear-algebra package [14] to yield the required number of independent eigenvectors when there is a (nearly) repeated eigenvalue.

To complete our work we list in Tables 1–4 some results obtained from our FORTRAN implementations of the developed solutions for Poiseuille flow (identified by the subscript P) and thermal-creep flow (identified by the subscript T) in a plane channel. In Table 1 the velocity and heat-flow profiles are given for a case based on the Cercignani–Lampis boundary condition. In Table 2 the particle-flow rates and the heat-flow rates are given, again relevant to the Cercignani–Lampis boundary condition, for selected values of the channel width $2a$. In Tables 3 and 4 we

Table 3
Profiles ($2a = 1$): diffuse–specular model ($\alpha = 0.5$)

x/a	$-u_P(x)$	$q_P(x)$	$u_T(x)$	$-q_T(x)$
0.00	1.792544	2.766993(−1)	2.637228(−1)	1.288279
0.05	1.791906	2.765059(−1)	2.636069(−1)	1.287935
0.10	1.789989	2.759243(−1)	2.632585(−1)	1.286901
0.15	1.786786	2.749506(−1)	2.626755(−1)	1.285169
0.20	1.782284	2.735782(−1)	2.618543(−1)	1.282727
0.25	1.776464	2.717975(−1)	2.607896(−1)	1.279557
0.30	1.769301	2.695955(−1)	2.594743(−1)	1.275633
0.35	1.760763	2.669553(−1)	2.578994(−1)	1.270925
0.40	1.750808	2.638558(−1)	2.560532(−1)	1.265393
0.45	1.739384	2.602701(−1)	2.539213(−1)	1.258987
0.50	1.726425	2.561643(−1)	2.514854(−1)	1.251644
0.55	1.711849	2.514960(−1)	2.487226(−1)	1.243285
0.60	1.695548	2.462105(−1)	2.456037(−1)	1.233812
0.65	1.677384	2.402368(−1)	2.420909(−1)	1.223094
0.70	1.657173	2.334803(−1)	2.381337(−1)	1.210963
0.75	1.634657	2.258098(−1)	2.336630(−1)	1.197185
0.80	1.609464	2.170352(−1)	2.285787(−1)	1.181426
0.85	1.581011	2.068584(−1)	2.227252(−1)	1.163173
0.90	1.548275	1.947554(−1)	2.158304(−1)	1.141540
0.95	1.509013	1.795760(−1)	2.073022(−1)	1.114627
1.00	1.451987	1.555159(−1)	1.941861(−1)	1.073270

report similar results based on the use of the diffuse–specular boundary condition. As expected the Onsager reciprocity relation, viz. $U_T = Q_P$, was verified for all cases listed in Tables 2 and 4. We note that our results are given with what we believe to be seven figures of accuracy. While we have no proof of the accuracy achieved in this work, we have done some things to support the confidence we have. First of all the fact that we found apparent convergence in our numerical results as we increased N, the number of quadrature points used, was considered a necessary test we passed. In addition, we found agreement with the flow rates reported with four figures of accuracy by Sharipov [2] for the S-model theory with the Cercignani–Lampis boundary condition. We also confirmed the results given with five/six figures of accuracy by Valougeorgis [1] for the S-model theory with the diffuse–specular boundary condition. We note also that Valougeorgis [16] has confirmed the velocity and heat-flow profiles given in Table 3 and that Sharipov [17] has confirmed (with four figures of accuracy) the profiles listed in Table 1.

To conclude this work, we note that we have typically used $N = 60$ to generate the results shown in our tables. Since the FORTRAN implementation of our discrete-ordinates solution (with $N = 60$) runs in a second on a 400 MHz Pentium-based PC, we feel justified in thinking the reported solution to the Poiseuille and thermal-creep problems in a plane-parallel channel is efficient as well as accurate.

Table 4
Flow rates: diffuse–specular model ($\alpha = 0.5$)

$2a$	$-U_P$	$Q_P = U_T$	$-Q_T$
1.0(−2)	7.210007	2.770617	15.50420
2.0(−2)	6.298270	2.311215	13.11035
3.0(−2)	5.808061	2.060542	11.75386
4.0(−2)	5.482139	1.891307	10.81454
5.0(−2)	5.242765	1.765080	10.10061
7.0(−2)	4.905303	1.583132	9.050784
9.0(−2)	4.672567	1.453788	8.289623
1.0(−1)	4.580089	1.401214	7.976847
3.0(−1)	3.806140	9.094469(−1)	4.983364
5.0(−1)	3.571767	7.155058(−1)	3.798980
7.0(−1)	3.464010	6.004731(−1)	3.108895
9.0(−1)	3.409028	5.216112(−1)	2.644884
1.0	3.392769	4.904286(−1)	2.463996
3.0	3.503677	2.349312(−1)	1.061216
3.5	3.569734	2.091089(−1)	9.302355(−1)
4.0	3.640058	1.885784(−1)	8.280755(−1)
5.0	3.788425	1.578737(−1)	6.789398(−1)
6.0	3.942771	1.359157(−1)	5.752755(−1)
7.0	4.100498	1.193805(−1)	4.990293(−1)
9.0	4.421533	9.606501(−2)	3.943911(−1)
1.0(1)	4.583722	8.752351(−2)	3.569419(−1)
1.0(2)	19.53951	9.670226(−3)	3.731918(−2)

Acknowledgements

The author takes this opportunity to thank Felix Sharipov and Dimitris Valougeorgis for several helpful discussions concerning this (and other) work.

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