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Inverse solutions to radiative-transfer problems based on the binomial or the Henyey–Greenstein scattering law

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Abstract

Analytical techniques are used to solve two inverse radiative-transfer problems, for a finite plane-parallel medium, that are (i) based on the binomial scattering law and (ii) based on the Henyey–Greenstein scattering law. In addition, previously reported analytical results (valid for isotropic scattering) that yield an analytical inverse solution for the unknown optical thickness of the medium are extended to the case of anisotropic scattering. The algorithms for the inversions are verified numerically, and some effects of noise on the simulated experimental data are observed. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

While in this brief work we do not attempt to review the many existing variations of inverse problems in the general area of radiative transfer, we can make some general comments. Perhaps the most important inverse problem is the one where we consider the radiation intensity to be known (or available from experimental observations) on the boundaries, and we then attempt to deduce the albedo for single scattering and the scattering law. For other applications, we might consider that the scattering data are known and then attempt to deduce the inhomogeneous source term in the equation of transfer. In another case, some aspect of the internal radiation field is specified and the scattering data are assumed to be known, and we then seek to determine the incoming boundary data. A fourth variation of inverse problems in radiative transfer is defined by known boundary data (incoming and outgoing intensities) and we seek to determine the degree of transparency of the boundary and the reflection coefficients for the two surfaces of a

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finite plane-parallel medium. Of course, we can easily imagine that combinations of these four basic types of inverse problems can yield a very large class of problems of general interest. Also, it is clear that each variation on the basic idea of an inverse radiative-transfer problem poses specific challenges, and so there already exist numerous works devoted to this subject. Good discussions of many of these numerous works are given in McCormick's three review papers [1–3]. It can be seen [1–3] that most of the existing work in regard to inverse problems in radiative transfer has to do with explicit or implicit solutions or iterative algorithms, but there is very little serious work concerning the proof of existence and/or uniqueness of solutions to defined inverse problems. Unfortunately, we also have nothing to say in regard to these important issues.

We choose to introduce our work here with, what we consider to be, one of the first analytical results for an inverse radiative-transfer problem. We found in an early work [4] that the inverse problem based on the equation of transfer

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu)+I(\tau,\mu)=\frac{\varpi}{2}\int_{-1}^{1}[1+b_1\mu\mu'+b_2P_2(\mu)P_2(\mu')]I(\tau,\mu')\,\mathrm{d}\mu', \tag{1}$$

for $\tau\in(0,\tau_0)$ and $\mu\in[-1,1]$, and the boundary conditions

$$I(0,\mu)=F_1(\mu) \tag{2a}$$

and

$$I(\tau_0,-\mu)=F_2(\mu), \tag{2b}$$

for $\mu\in(0,1]$, could be solved in the following sense. If we consider that the quantities ϖ, b_1 and b_2 that define the scattering law used in Eq. (1) are the unknown parameters to be determined, and if we consider that the boundary functions $F_1(\mu)$ and $F_2(\mu)\neq F_1(\mu)$ are given, and noting that $P_2(\xi)$ is the Legendre polynomial of second order, then we can quote from Ref. [4] three algebraic equations that, in principle, allow us to determine the three required unknowns in terms of the exiting intensities $I(0,-\mu)$ and $I(\tau_0,\mu)$, for $\mu\in(0,1]$, that are considered available from experimental data. McCormick, in a subsequent work [5], considered an inverse problem for a more general scattering law and defined algorithms based on assumed knowledge of the exiting components of the nonazimuthally symmetric radiation field. Included in McCormick's work [5] are also results we use here where we assume that measurements only of the azimuthally symmetric component of the exiting radiation are available.

In this work, we investigate a variation of the stated problem. Instead of the three-term scattering law used in Eq. (1), we assume that the scattering is described by one of the two specified forms. We work, first of all, with the binomial form

$$p(\cos\Theta)=\frac{K+1}{2^K}(1+\cos\Theta)^K \tag{3}$$

introduced by Kaper et al. [6]. Here Θ is the scattering angle, and K is a nonnegative parameter. It is known that Eq. (3) can be rewritten in terms of Legendre polynomials as

$$p(\cos\Theta)=\sum_{l=0}^{\infty}\beta_l P_l(\cos\Theta), \tag{4}$$

where, as reported by McCormick and Sanchez [7], the β coefficients can be computed from the recursion formula

$$\beta_l = \left(\frac{2l+1}{2l-1}\right)\left(\frac{K+1-l}{K+1+l}\right)\beta_{l-1}, \tag{5}$$

for $l = 1, 2, \ldots,$ with $\beta_0 = 1$. And so, while we continue to use boundary conditions as given by Eqs. (2a and b), we now work with truncated scattering laws and the equation of transfer written as [8]

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu) + I(\tau,\mu) = \frac{\varpi}{2}\sum_{l=0}^{L}\beta_l P_l(\mu)\int_{-1}^{1}P_l(\mu')I(\tau,\mu')\,\mathrm{d}\mu', \tag{6}$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$.

The inverse problem solved here can now be stated simply. We assume that $F_1(\mu)$ and $F_2(\mu)$, for $\mu \in (0, 1]$, as used in Eqs. (2a and b) along with the exiting intensities

$$G_1(\mu) = I(0, -\mu) \tag{7a}$$

and

$$G_2(\mu) = I(\tau_0, \mu), \tag{7b}$$

for $\mu \in (0, 1]$, are known, say from experimental data, and we seek to determine the albedo for single scattering ϖ, the parameter K of the scattering law, and the optical thickness τ_0.

The second form we consider here is the Henyey–Greenstein scattering law [9] which can be written as

$$p(\cos\Theta) = (1-g^2)(1+g^2-2g\cos\Theta)^{-3/2} \tag{8}$$

or

$$p(\cos\Theta) = \sum_{l=0}^{\infty}(2l+1)g^l P_l(\cos\Theta), \tag{9}$$

for $g \in (-1, 1)$. For the inverse problem defined by the Henyey–Greenstein model, we consider, again, that $F_\alpha(\mu)$, for $\mu \in (0, 1]$, and $G_\alpha(\mu)$, for $\mu \in (0, 1]$, $\alpha = 1, 2$, are known, and so we wish to determine the albedo for single scattering ϖ, the constant g, and the optical thickness τ_0.

2. The scattering parameters

While the inverse problem solved in Ref. [4] was defined in terms of a three-term scattering law, some basic results derived in that work were, as mentioned, extended and generalized by McCormick [5] to a scattering law with $L+1$ terms, as used in Eq. (6). And so we write the two results of McCormick [5] we use here as

$$4S_0 = \varpi\sum_{l=0}^{L}(-1)^l\beta_l[I_l^2(\tau_0) - I_l^2(0)] \tag{10}$$

and

$$4S_2 = \varpi \sum_{l=0}^{L} (-1)^l (2l+1)(\beta_l/h_l)[J_l^2(\tau_0) - J_l^2(0)], \tag{11}$$

where

$$I_l(\tau) = \int_{-1}^{1} I(\tau,\mu) P_l(\mu)\, \mathrm{d}\mu, \tag{12a}$$

$$J_l(\tau) = \int_{-1}^{1} I(\tau,\mu) P_l(\mu) \mu\, \mathrm{d}\mu, \tag{12b}$$

and

$$h_l = 2l + 1 - \varpi\beta_l. \tag{13}$$

In addition,

$$S_0 = \int_0^1 [G_2(\mu)F_2(\mu) - G_1(\mu)F_1(\mu)]\, \mathrm{d}\mu \tag{14a}$$

and

$$S_2 = \int_0^1 [G_2(\mu)F_2(\mu) - G_1(\mu)F_1(\mu)]\mu^2\, \mathrm{d}\mu. \tag{14b}$$

It is clear that Eqs. (10) and (11) are valid for all $\varpi \in (0,1)$. However, while Eq. (10) is also valid for $\varpi = 1$, Eq. (11) clearly contains an indeterminate form for this special (conservative) case since $h_0 = 0$ and $J_0(0) = J_0(\tau_0)$ when $\varpi = 1$. Of course, by comparing $J_0(0)$ to $J_0(\tau_0)$ we can tell immediately if we have an inverse problem for the conservative case.

It should be noted that, while knowledge of the intensity on the two boundaries of the medium is required in Eqs. (10), (11) and (14), the optical thickness τ_0 is not needed, and so, since we intend to develop our inverse solution for the scattering law from these equations, we can delay our discussion of the inverse solution for the optical thickness.

For the case of the binomial scattering law, our algorithm for extracting the desired unknowns ϖ and K is very simple. We first choose some (large) value, say $L = 1000$, and rewrite Eq. (4) as

$$p(\cos\Theta) = \sum_{l=0}^{L} \beta_l P_l(\cos\Theta). \tag{15}$$

Next, we solve Eq. (10) for ϖ and use that result in Eq. (11) to define a function of one unknown (viz., K) a zero of which we then find by Newton's method of iteration with $K = 0$ as an initial estimate. Then, once K has been found in this way, we obtain ϖ from Eq. (10).

For the Henyey–Greenstein case, we again choose some (large) value, say $L = 1000$, and rewrite Eq. (9) as

$$p(\cos\Theta) = \sum_{l=0}^{L} (2l+1) g^l P_l(\cos\Theta). \tag{16}$$

Next, we solve Eq. (10) for ϖ and use that result in Eq. (11) to define a function of one unknown (viz., g) a zero of which we find by Newton's method of iteration with $g=1$ as an initial estimate. Then, once g has been found in this way, we obtain ϖ from Eq. (10).

Of course for either the binomial case or the Henyey–Greenstein case, if some knowledge of K or g is already available, a more precise value of L could (perhaps) be used. In any case, once converged values of K and ϖ or g and ϖ have been obtained with an assumed value of L, more confidence in the results can be obtained by increasing L and repeating the calculation.

For the conservative case, we also use $L=1000$ as the upper summation limit in Eq. (10), and then, with $\varpi=1$, we use Newton's method to find K or g from the resulting equation.

To conclude this section, we note that while we have based our solutions on knowledge of the boundary data (incoming and outgoing intensities), the same procedure could be used to yield similar results if the total intensities were known (or available from experimental observations) at any two points within the medium.

3. The optical thickness

At this point, we assume that we have solved the inverse problem to establish the scattering law, and so now we wish to extend a result, valid only for isotropic scattering, from Ref. [10] in order to find an explicit inverse solution for the optical thickness τ_0. We therefore consider the equation of transfer

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu)+I(\tau,\mu)=\frac{\varpi}{2}\sum_{l=0}^{L}\beta_l P_l(\mu)\int_{-1}^{1}P_l(\mu')I(\tau,\mu')\,\mathrm{d}\mu', \tag{17}$$

for $\tau\in(0,\tau_0)$ and $\mu\in[-1,1]$, and the boundary conditions

$$I(0,\mu)=F_1(\mu) \tag{18a}$$

and

$$I(\tau_0,-\mu)=F_2(\mu), \tag{18b}$$

for $\mu\in(0,1]$. Here we assume that we know ϖ, the order of the scattering law L, and the coefficients $\{\beta_l\}$ that define the scattering law and (from experimental data) the exiting intensities

$$I(0,-\mu)=G_1(\mu) \tag{19a}$$

and

$$I(\tau_0,\mu)=G_2(\mu), \tag{19b}$$

for $\mu\in(0,1]$. We base our solution for the optical thickness τ_0 on the method of elementary solutions [11], and so to start we write our solution to Eq. (17) as

$$I(\tau,\mu)=I_*(\tau,\mu)+\int_0^1[A(v)\phi(v,\mu)\mathrm{e}^{-\tau/v}+B(v)\phi(-v,\mu)\mathrm{e}^{-(\tau_0-\tau)/v}]\,\mathrm{d}v, \tag{20}$$

where

$$I_*(\tau,\mu)=\sum_{j=1}^{\aleph}[A_j\phi(v_j,\mu)^{-\tau/v_j}+B_j\phi(-v_j,\mu)\mathrm{e}^{-(\tau_0-\tau)/v_j}] \tag{21}$$

and

$$\phi(\pm\nu_j,\mu) = \frac{\varpi\nu_j}{2}\sum_{l=0}^{L}\beta_l P_l(\mu) g_l(\pm\nu_j)\frac{1}{\nu_j \mp \mu}. \tag{22}$$

Here we use $\aleph$ to denote the number of pairs of discrete eigenvalues $\{\pm\nu_j\}$, and we use $g_l(\xi)$ to denote the Chandrasekhar polynomials [8]. These polynomials are defined by the three-term recursion formula

$$(2l+1-\varpi\beta_l)\xi g_l(\xi) = (l+1)g_{l+1}(\xi) + l g_{l-1}(\xi) \tag{23}$$

and the starting values

$$g_0(\xi) = 1 \quad \text{and} \quad g_1(\xi) = (1-\varpi)\xi. \tag{24a,b}$$

Since the elementary functions $\phi(\pm\nu,\mu)$ based on the continuum, $\nu\in(0,1)$, make no explicit contribution to our calculation of τ_0, we omit these definitions. Of course, to define the complete solution for the radiation intensity, all of Eq. (20) must be used, and to complete that solution, the constants A_j and B_j and the functions $A(\nu)$ and $B(\nu)$ must be determined so that the solution satisfies the boundary conditions given by Eqs. (18). However, simply to determine τ_0 for our inverse problem we do not require the complete solution $I(\tau,\mu)$.

While we do not have, in general, explicit expressions for the eigenvalues $\pm\nu_j$, $j = 1,2,\ldots,\aleph$, as we did have for the isotropic case [10], these discrete eigenvalues can be computed as zeros of the dispersion function

$$\Lambda(z) = 1 + \int_{-1}^{1}\Psi(\xi)\frac{\mathrm{d}\xi}{\xi - z}, \qquad z\notin[-1,1], \tag{25}$$

where the characteristic function is

$$\Psi(\xi) = \frac{\varpi}{2}\sum_{l=0}^{L}\beta_l P_l(\xi) g_l(\xi). \tag{26}$$

The elementary functions $\phi(\xi,\mu)$ are orthogonal (with weight function μ) on the full-range [12], and so we can, for example, evaluate Eq. (20) at $\tau = 0$ and $\tau = \tau_0$ and conclude that

$$A_j N_j = \int_{-1}^{1}\phi(\nu_j,\mu) I(0,\mu)\mu\,\mathrm{d}\mu, \tag{27a}$$

$$B_j \mathrm{e}^{-\tau_0/\nu_j} N_j = -\int_{-1}^{1}\phi(-\nu_j,\mu) I(0,\mu)\mu\,\mathrm{d}\mu, \tag{27b}$$

$$A_j \mathrm{e}^{-\tau_0/\nu_j} N_j = \int_{-1}^{1}\phi(\nu_j,\mu) I(\tau_0,\mu)\mu\,\mathrm{d}\mu, \tag{27c}$$

and

$$B_j N_j = -\int_{-1}^{1}\phi(-\nu_j,\mu) I(\tau_0,\mu)\mu\,\mathrm{d}\mu, \tag{27d}$$

for $j=1,2,\ldots,\aleph$. Here

$$N_j=\int_{-1}^{1}[\phi(\nu_j,\mu)]^2\mu\,\mathrm{d}\mu. \tag{28}$$

Now, from Eqs. (27a) and (27c), we see that

$$\mathrm{e}^{\tau_0/\nu_j}=M(0,\nu_j)/M(\tau_0,\nu_j), \tag{29}$$

where

$$M(\tau,\nu_j)=\int_{-1}^{1}\phi(\nu_j,\mu)I(\tau,\mu)\mu\,\mathrm{d}\mu. \tag{30}$$

Also, from Eqs. (27b) and (27d), it follows that

$$\mathrm{e}^{\tau_0/\nu_j}=M(\tau_0,-\nu_j)/M(0,-\nu_j). \tag{31}$$

We therefore conclude that

$$\tau_0=\nu_j\ln\{M(0,\nu_j)/M(\tau_0,\nu_j)\} \tag{32}$$

and

$$\tau_0=\nu_j\ln\{M(\tau_0,-\nu_j)/M(0,-\nu_j)\}, \tag{33}$$

for any $j=1,2,\ldots,\aleph$. Of course, we can use Eqs. (18) and (19) to write

$$M(0,\pm\nu_j)=\int_0^1\phi(\pm\nu_j,\mu)F_1(\mu)\mu\,\mathrm{d}\mu-\int_0^1\phi(\mp\nu_j,\mu)G_1(\mu)\mu\,\mathrm{d}\mu \tag{34}$$

and

$$M(\tau_0,\pm\nu_j)=\int_0^1\phi(\pm\nu_j,\mu)G_2(\mu)\mu\,\mathrm{d}\mu-\int_0^1\phi(\mp\nu_j,\mu)F_2(\mu)\mu\,\mathrm{d}\mu. \tag{35}$$

We note that Eqs. (32) and (33) are valid for $\varpi\in(0,1)$. For the conservative case, $\varpi=1$, we find an even simpler result, viz.

$$\tau_0=(3-\beta_1)[J_1(0)-J_1(\tau_0)]/[3J_0(0)], \tag{36}$$

where we have made use of the definitions introduced in Eq. (12b). Since Eq. (36) is written in terms of boundary results, we can use in that equation

$$J_\beta(0)=\int_0^1[F_1(\mu)-(-1)^\beta G_1(\mu)]\mu^{\beta+1}\,\mathrm{d}\mu,\quad \beta=0,1, \tag{37a}$$

and

$$J_\beta(\tau_0)=\int_0^1[G_2(\mu)-(-1)^\beta F_2(\mu)]\mu^{\beta+1}\,\mathrm{d}\mu,\quad \beta=0,1. \tag{37b}$$

While we have made use of the boundary data to find Eqs. (32), (33) and (36), it is clear that were it more convenient, from say an experimental point of view, to know the intensity at any two values of τ would be sufficient to determine, in a manner analogous to what has been done here, the optical distance between the two chosen values of τ. We note that Eqs. (32) and (33) can be immediately generalized so as to be valid for any value of $v \in (0,1)$; however, to be rigorous some ensuing integrals in Eqs. (34) and (35) would have to be evaluated in the Cauchy principal-value sense. Finally, although there exist methods [13,14] for estimating the optical thickness of a finite layer from boundary data, we believe that we are justified in claiming that the results given by Eqs. (32), (33) and (36) are definitive.

4. Concluding comments

The algorithms defined for solving the inverse problems considered in this work were tested numerically by using a version of the discrete-ordinates method [15,16] to solve the direct problem. We found, typically using $F_1(\mu) = 1$ and $F_2(\mu) = 0$, that we could, for the binomial scattering law, then extract ϖ, K and τ_0 with five figures of accuracy with a discrete-ordinates solution of order $N = 40$. Similar results were found for the Henyey–Greenstein scattering law. The solution of the direct problems and the solving of the inverse problems, as implemented in FORTRAN and run on a 400 MHz PC, required, for the (typical) cases tested, less than 1 s of computation time. Of course, such testing only helps confirm the validity of the inversion method and does not, in any serious way, address the important issue of how experimental errors could affect the results. We note that Chalhoub and Campos Velho [17,18] recently used an approach based on optimization techniques and repeated solutions of the direct problem to show the effects of errors [on their algorithm] for a model based on the Henyey–Greenstein scattering law. Of course neither the Henyey–Greenstein nor the binomial scattering law can be expected to describe well the scattering process for all applications, but when the binomial scattering law or the Henyey–Greenstein law can be used, we believe that the method reported here can be used with confidence.

To give some numerical results, we report on what we consider to be two reasonably severe cases. First of all we used our [15,16] discrete-ordinates solution with $N = 40$ to solve the direct problem based on the binomial model with $K = 543.21$, $\varpi = 0.92$, and $\tau_0 = 2$. Using our algorithm for the inverse problem, we found (without adding any noise to the solution of the direct problem) in less than a second for both the direct and the inverse solutions, $K = 543.21$, $\varpi = 0.92000$, and $\tau_0 = 2.0000$. Following this calculation, we added 5% of random noise to the solution of the direct problem and saw that the results (with $N = 200$) from the inverse solution changed to $K = 537$, $\varpi = 0.919$ and $\tau_0 = 1.96$. Our results for the Henyey–Greenstein scattering law are similar. Here the direct problem was defined by $\varpi = 0.9$, $g = 0.97$ and $\tau_0 = 2$. Without noise, we found (with $N = 50$) $\varpi = 0.90000$, $g = 0.97000$ and $\tau_0 = 2.0000$ when solving the inverse problem. With 5% of random noise added to the solution of the direct problem, these results became (with $N = 200$) $\varpi = 0.909$, $g = 0.972$ and $\tau_0 = 2.17$.

To conclude, we note that we have found that the algorithms developed here work well, but we must not forget that the issues of existence and uniqueness of the solutions to the inverse problems considered have not been addressed.

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