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REMARKS ON HALF-SPACE APPLICATIONS REGARDING AN EQUATION OF TRANSFER FOR A PLANETARY ATMOSPHERE

ABSTRACT

The two extreme limits for the generalized Rayleigh-scattering problem are discussed in the light of half-space applications.

In a recent paper Mourad and Siewert (1969) developed the eigenvalue spectrum and the set of normal modes for the equation of transfer,

$$\mu \frac{\partial}{\partial \tau} I(\tau \mu) + I(\tau \mu) = \tfrac{1}{2} c \int_{-1}^{1} K(\mu \mu') I(\tau \mu') d\mu' + \tfrac{1}{2}(1 - c) E \int_{-1}^{1} I(\tau,\mu') d\mu' , \quad (1)$$

formulated by Chandrasekhar (1950) in order to describe the polarized radiation field in a planetary atmosphere. Here μ is the direction cosine (as measured from the *inward* normal to the free surface) of the directed radiation, τ is the optical variable, $I(\tau,\mu)$ is a vector with components $I_l(\tau,\mu)$ and $I_r(\tau,\mu)$,

$$K(\mu,\mu') = \tfrac{3}{4}\begin{vmatrix} 2(1-\mu^2)(1-\mu'^2) + \mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{vmatrix} , \quad \text{and} \quad E = \tfrac{1}{2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} . \quad (2)$$

In addition, the parameter $c \epsilon [0,1]$ is a measure of the relative importance of the two scattering kernels $K(\mu,\mu')$ and E.

Equation (1) for the Rayleigh-scattering limit, $c = 1$, has been thoroughly investigated, and the successes have been numerous (see, e.g., Chandrasekhar 1950; Mullikin 1966; Siewert and Fraley 1967). However, since the dispersion function for the general case $c \epsilon (0,1)$ does not factor (as was pointed out by Chandrasekhar 1950; see § 74), solutions to equation (1) which rigorously satisfy given boundary conditions have been severely limited.

Although Mourad and Siewert (1969) found a set of solutions to equations (1) for general c, their completeness proof is applicable only to full-range problems. There is, therefore, a need to extend their procedure to the half-range, if rigorous solutions to the physically more meaningful half-space or finite-slab problems are to be so constructed.

To date there has been no satisfactory half-range completeness proof for the general case (and it appears likely that closed-form solutions do not exist); however, there are two special cases for which exact results may be obtained. The first is the well discussed Rayleigh-scattering problem, $c = 1$, and the other is the considerably simpler $(c = 0)$-limit. This second limiting case is indeed elementary; in fact, equation (1) for $c = 0$ can be diagonalized, and the ensuing equations may be solved by using any of the classical methods in one-speed or gray transport theory.

In contrast to the above procedure, we should here like to solve a half-space problem for the $(c = 0)$-limit by employing directly the normal modes of equation (1). The theme of this effort is thus (*a*) to demonstrate the manner in which half-space applications for the general case reduce to exact results for the $(c = 0)$-limit, (*b*) to illustrate the small difference between the exact results for the Milne-problem extrapolation distance in the two extreme limits $c = 0$ and $c = 1$, and (*c*) to suggest the possibility of using a perturbation-type procedure to calculate approximately the unknown expansion coefficients for half-space or finite-media problems in the general case $c \epsilon (0,1)$, if rigorous solutions remain elusive.

The solution to the Milne problem is written as

$$I_M(\tau,\mu) = A_+\mathbf{\Phi}_+ + A_-I_-(\tau,\mu) + \int_0^1 A_1(\eta)\,\mathbf{\Phi}_1(\eta,\mu)e^{-\tau/\eta}d\eta + \int_0^1 A_2(\eta)\mathbf{\Phi}_2(\eta,\mu)e^{-\tau/\eta}d\eta\,, \tag{3}$$

where A_+, A_-, $A_1(\eta)$, and $A_2(\eta)$ are the expansion coefficients to be determined. Here $\mathbf{\Phi}_+$, $I_-(\tau,\mu)$ and $\mathbf{\Phi}_2(\eta,\mu)$ are as given explicitly by Mourad and Siewert (1969), and their expression for $\mathbf{\Phi}_1(\eta,\mu)$ has been multiplied by c and the limit $c = 0$ observed to yield

$$\mathbf{\Phi}_1(\eta,\mu) = \tfrac{4}{3}\delta(\eta-\mu)\begin{vmatrix}1\\-1\end{vmatrix}\,. \tag{4}$$

Equation (3) may be written more concisely as

$$I_M(\tau,\mu) = \left[A_+ + A_-(\tau-\mu) + \int_0^1 3(1-\eta^2)A_2(\eta)\phi(\eta,\mu)e^{-\tau/\eta}d\eta\right]\begin{vmatrix}1\\1\end{vmatrix} + \int_0^1[\tfrac{4}{3}A_1(\eta) - A_2(\eta)]\delta(\eta-\mu)e^{-\tau/\eta}d\eta\begin{vmatrix}1\\-1\end{vmatrix}\,, \tag{5}$$

where

$$\phi(\eta,\mu) = \tfrac{1}{2}\eta\frac{P}{\eta-\mu} + (1-\eta\tanh^{-1}\eta)\delta(\eta-\mu)\,. \tag{6}$$

Since the solution given by equation (5) satisfies the correct condition at infinity (Chandrasekhar 1950), the remaining Milne-problem condition to be met is that of zero re-entrant radiation. Thus the coefficients A_+, $A_1(\eta)$, and $A_2(\eta)$ are to be determined from the half-range boundary condition,

$$I_M(0,\mu) = \mathbf{0} \quad \text{for} \quad \mu \in (0,1)\,; \tag{7}$$

we normalize the problem by taking $A_- = \frac{3}{8}F$, F being the flux constant.

If we subtract the two components of equation (5) evaluated at $\tau = 0$, there results

$$A_2(\eta) = \tfrac{4}{3}A_1(\eta)\,. \tag{8}$$

The surface boundary condition thus takes the form

$$\mathbf{0} = \left[A_+ - \tfrac{3}{8}F\mu + \int_0^1 3(1-\eta^2)A_2(\eta)\phi(\eta,\mu)d\eta\right]\begin{vmatrix}1\\1\end{vmatrix}\,, \quad \mu \in (0,1)\,, \tag{9a}$$

or alternatively, since these two equations are identical,

$$\tfrac{3}{8}F\mu = A_+ + \int_0^1 3(1-\eta^2)A_2(\eta)\phi(\eta,\mu)d\eta\,, \quad \mu \in (0,1)\,. \tag{9b}$$

We note that the set of functions $\phi(\eta,\mu)$ and 1 are the eigenfunctions introduced by Case (1960) for the conservative gray model. Equation (9b) is thus a valid half-range expansion, and the coefficients may be obtained immediately by utilizing the half-range orthogonality theorem developed by Kuščer, McCormick, and Summerfield (1964). It follows that

$$A_{+} = \tfrac{3}{8}F\frac{\sqrt{3}}{2}\int_0^1 \mu^2 H(\mu)d\mu \triangleq \tfrac{3}{8}F\tau_0 \,, \tag{10a}$$

and

$$3(1-\eta^2)A_2(\eta) = -\tfrac{3}{8}F\left\{(\sqrt{3})H(\eta)\left[(1-\eta\tanh^{-1}\eta)^2 + \left(\frac{\pi\eta}{2}\right)^2\right]\right\}^{-1} , \tag{10b}$$

where $H(\eta)$ is Chandrasekhar's (1950) H-function for characteristic function $\frac{1}{2}$.

Since the unknown coefficients have been determined, the solution is complete:

$$I_M(\tau,\mu) = \tfrac{3}{8}F\left[\tau_0 + \tau - \mu - \int_0^1 \left\{(\sqrt{3})H(\eta)\left[(1-\eta\tanh^{-1}\eta)^2 + \left(\frac{\pi\eta}{2}\right)^2\right]\right\}^{-1}\phi(\eta,\mu)e^{-\tau/\eta}d\eta\right]\begin{vmatrix}1\\1\end{vmatrix} . \tag{11}$$

It is clear that if half-space problems are to be solved in terms of the normal modes established for equation (1) by Mourad and Siewert (1969), then asymptotically, the Milne-problem solution for *any* value of c will take the same form, i.e.,

$$I_M(\tau,\mu) \sim \tfrac{3}{8}F(\tau_0 + \tau - \mu)\begin{vmatrix}1\\1\end{vmatrix} \quad \text{for large } \tau \,; \tag{12}$$

but certainly the extrapolation distance τ_0 will be a function of the parameter c. Equation (10a) for the case $c = 0$ is a result well known from the gray or one-speed model, and it has often been evaluated numerically. In addition, Bond and Siewert (1967) made the analogous calculations for $c = 1$. It is interesting how similar the results are for the two extreme cases:

$$\tau_0 = 0.710446089\ldots \quad \text{for} \quad c = 0 \,, \tag{13a}$$

and

$$\tau_0 = 0.712109761\ldots \quad \text{for} \quad c = 1 \,. \tag{13b}$$

Although fundamental solutions to equation (1) have been established for general c, the theorems necessary to prove that half-range boundary conditions can be met rigorously for all cases are not yet available. On the other hand, rigorous solutions to equation (1) which meet approximately half-range boundary conditions can easily be constructed in many ways. The similarity of the exact asymptotic solutions to the Milne problem for the two extreme cases $c = 0$ and $c = 1$ suggest that a perturbation-type analysis may lead to highly accurate approximate solutions to half-space or finite-media problems.

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