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Two half-space problems based on a synthetic-kernel model of the linearized Boltzmann equation

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Abstract

An analytical discrete-ordinates method is used to solve two basic half-space problems based on a new synthetic-kernel model of the linearized Boltzmann equation. In particular, Kramers' problem and the half-space problem of thermal creep, both basic to the general area of rarefied-gas dynamics, are defined by model equations that are solved (essentially) analytically in terms of a modern version of the discrete-ordinates method. The developed algorithms are implemented to yield numerical results for the slip coefficients and the velocity and heat-flow profiles that compare well with solutions derived from much more computationally intensive techniques. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In a recent work [1], the ideas of Loyalka and Ferziger [2] were used to introduce a new synthetic-kernel approximation to the linearized Boltzmann equation relevant to rigid-sphere collisions, and so in this work we use the analytical discrete-ordinates (ADO) method developed by Barichello and Siewert [3] to solve two basic problems in the general area of rarefied-gas dynamics. In this way, we see well a new application of the ADO method, and we are able to have an initial evaluation of the effectiveness of the developed synthetic-kernel (CES) approximation to the linearized Boltzmann equation.

To start this work, we consider the homogeneous and linearized Boltzmann equation written for rigid-sphere collisions as [1,4]

$$c\mu\frac{\partial}{\partial\tau}h(\tau,\boldsymbol{c}) = \varepsilon L\{h\}(\tau,\boldsymbol{c}), \tag{1}$$

where

$$L\{h\}(\tau,\boldsymbol{c}) = -\nu(c)h(\tau,\boldsymbol{c}) + \int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c'^2}h(\tau,\boldsymbol{c}')K(\boldsymbol{c}',\boldsymbol{c})c'^2\,\mathrm{d}\chi'\,\mathrm{d}\mu'\,\mathrm{d}c'. \tag{2}$$

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Here the scattering kernel is

$$K(\boldsymbol{c}', \boldsymbol{c}) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{2n+1}{2}\right) (2 - \delta_{0,m}) P_n^m(\mu') P_n^m(\mu) k_n(c', c) \cos m(\chi' - \chi), \tag{3}$$

where the *normalized* Legendre functions are given (in terms of the Legendre polynomials) by

$$P_n^m(\mu) = \left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} (1-\mu^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_n(\mu), \quad n \geqslant m. \tag{4}$$

In addition,

$$\varepsilon = \sigma_0^2 n_0 \pi^{1/2} l, \tag{5}$$

where l is (at this point) an unspecified mean-free path and σ_0 is the scattering diameter of the gas molecules. In this work, the spatial variable τ is measured in units of the mean-free path l, $c(2kT_0/m)^{1/2}$ is the magnitude of the particle velocity and $h(\tau, \boldsymbol{c})$ is a perturbation from an absolute Maxwellian distribution. Thus the particle distribution function $f(\tau, \boldsymbol{c})$ can be expressed as

$$f(\tau, \boldsymbol{c}) = f_0(c)[1 + h(\tau, \boldsymbol{c})], \tag{6}$$

where

$$f_0(c) = n_0[m/(2\pi k T_0)]^{3/2} \mathrm{e}^{-c^2}. \tag{7}$$

Here (in the equilibrium distribution) n_0 is the (constant) density of gas particles, each of mass m, k is the Boltzmann constant and T_0 is a (constant) reference temperature. Continuing, we note that the functions $k_n(c', c)$ in Eq. (3) are the components in an expansion of the scattering law (for rigid-sphere collisions) reported by Pekeris and Alterman [4], and

$$\nu(c) = \frac{2c^2 + 1}{c} \int_0^c \mathrm{e}^{-x^2}\, \mathrm{d}x + \mathrm{e}^{-c^2} \tag{8}$$

is the collision frequency. And finally, we use spherical coordinates $(c, \arccos\mu, \chi)$ to define the (dimensionless) velocity vector $\boldsymbol{c}$.

2. Kramers' problem

For this flow problem, since we wish to compute only the velocity and heat-flow profiles, we do not require the complete distribution function $h(\tau, \boldsymbol{c})$. In fact, we require only a certain moment of the distribution function, and so we multiply Eq. (1) by $\cos\chi$ and integrate over χ from 0 to 2π to find

$$c\mu \frac{\partial}{\partial \tau} g(\tau, c, \mu) + \varepsilon \nu(c) g(\tau, c, \mu) = \varepsilon \int_0^{\infty} \int_{-1}^{1} \mathrm{e}^{-c'^2} g(\tau, c', \mu') K(c', \mu' : c, \mu) c'^2\, \mathrm{d}\mu'\, \mathrm{d}c', \tag{9}$$

where

$$g(\tau, c, \mu) = \frac{1}{\pi} \int_0^{2\pi} h(\tau, \boldsymbol{c}) \cos\chi\, \mathrm{d}\chi \tag{10}$$

and

$$K(c',\mu' : c,\mu) = \sum_{n=1}^{\infty} \left(\frac{2n+1}{2}\right) P_n^1(\mu')P_n^1(\mu)k_n(c',c). \tag{11}$$

We now let

$$g(\tau,c,\mu) = (1-\mu^2)^{1/2}\psi(\tau,c,\mu) \tag{12}$$

and rewrite Eq. (9) as

$$c\mu\frac{\partial}{\partial\tau}\psi(\tau,c,\mu) = \varepsilon L^*\{\psi\}(\tau,c,\mu) \tag{13}$$

where

$$L^*\{\psi\}(\tau,c,\mu) = -v(c)\psi(\tau,c,\mu) + \int_0^\infty\int_{-1}^1 \mathrm{e}^{-c'^2}\psi(\tau,c'\mu')k(c',\mu' : c,\mu)c'^2\,\mathrm{d}\mu'\,\mathrm{d}c'. \tag{14}$$

Here

$$k(c',\mu' : c,\mu) = (1-\mu'^2)\sum_{n=1}^{\infty} \Pi_n(\mu')\Pi_n(\mu)k_n(c',c) \tag{15}$$

with

$$\Pi_n(\mu) = \left[\frac{2n+1}{2n(n+1)}\right]^{1/2} \frac{\mathrm{d}}{\mathrm{d}\mu}P_n(\mu), \quad n \geqslant 1. \tag{16}$$

Note that

$$\int_{-1}^1 (1-\mu^2)\Pi_n(\mu)\Pi_{n'}(\mu)\,\mathrm{d}\mu = \delta_{n,n'}. \tag{17}$$

In regard to the boundary condition to go with Eq. (13), we start with

$$h(0,c,\mu,\chi) - (1-\alpha)h(0,c,-\mu,\chi) - D = 0, \quad \chi \in [0,2\pi], \tag{18}$$

for $\mu \in (0,1]$ and $c \in [0,\infty)$. Here α is the accommodation coefficient and

$$D = \frac{2\alpha}{\pi}\int_0^\infty\int_0^1\int_0^{2\pi} \mathrm{e}^{-c^2}h(0,c,-\mu,\chi)c^3\mu\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c \tag{19}$$

is the term that results from diffuse reflection at the wall. Now multiplying Eq. (18) by $\cos\chi$ and integrating, we find the boundary condition we require, viz.

$$\psi(0,c,\mu) - (1-\alpha)\psi(0,c,-\mu) = 0 \tag{20}$$

for $\mu \in (0,1]$ and $c \in [0,\infty)$. In addition to the boundary condition written as Eq. (20), we must define a condition on $\psi(\tau,c,\mu)$ as τ tends to infinity. Since there is no source term in the considered form of the Boltzmann equation, the velocity profile

$$u(\tau) = \pi^{-3/2}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2}h(\tau,\boldsymbol{c})c^3(1-\mu^2)^{1/2}\cos\chi\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c \tag{21}$$

or

$$u(\tau) = \pi^{-1/2} \int_0^\infty \int_{-1}^1 e^{-c^2} \psi(\tau, c, \mu) c^3 (1 - \mu^2) \, d\mu \, dc \tag{22}$$

must diverge as τ tends to infinity. It follows that $\psi(\tau, c, \mu)$ must diverge as τ becomes unbounded. In Ref. [1] some solutions of Eq. (1) that are linear in the spatial variable were discussed, and so here we can use $h_3^*(\tau, \boldsymbol{c})$ from Ref. [1] to write

$$g_3^*(\tau, c, \mu) = \frac{1}{\pi} \int_0^{2\pi} h_3^*(\tau, \boldsymbol{c}) \cos\chi \, d\chi, \tag{23}$$

which yields

$$g_3^*(\tau, c, \mu) = (1 - \mu^2)^{1/2} [c\varepsilon\tau - c^2 b(c)\mu]. \tag{24}$$

Here $c^2 b(c)$ satisfies the Chapman–Enskog equation for viscosity [4], viz.

$$\mathscr{L}_2\{c^2 b\}(c) = c^2, \tag{25}$$

where, in general, we consider the class of (Chapman–Enskog type) integral equations

$$\mathscr{L}_n\{f\}(c) = r(c) \tag{26}$$

with $r(c)$ given, and with

$$\mathscr{L}_n\{f\}(c) = v(c) f(c) - \int_0^\infty e^{-x^2} f(x) k_n(x, c) x^2 \, dx. \tag{27}$$

We now let $\psi_*(\tau, c, \mu)$ denote a bounded (as τ tends to infinity) solution of Eq. (13) and write, after noting Eq. (24), the complete solution as

$$\psi(\tau, c, \mu) = \psi_*(\tau, c, \mu) + \frac{2}{\varepsilon}[c\varepsilon\tau - c^2 b(c)\mu]. \tag{28}$$

Using Eq. (28), we find from Eq. (20) the boundary condition

$$\psi_*(0, c, \mu) - (1 - \alpha)\psi_*(0, c, -\mu) = \frac{2}{\varepsilon}(2 - \alpha) c^2 b(c) \mu \tag{29}$$

for $\mu \in (0, 1]$ and $c \in [0, \infty)$. The velocity profile we seek follows from Eqs. (22) and (28), and so we can write

$$u(\tau) = \tau + \pi^{-1/2} \int_0^\infty \int_{-1}^1 e^{-c^2} \psi_*(\tau, c, \mu) c^3 (1 - \mu^2) \, d\mu \, dc. \tag{30}$$

Should it be required, the heat-flux profile

$$q(\tau) = \pi^{-3/2} \int_0^\infty \int_{-1}^1 \int_0^{2\pi} e^{-c^2} h(\tau, \boldsymbol{c}) c^3 (c^2 - 5/2)(1 - \mu^2)^{1/2} \cos\chi \, d\chi \, d\mu \, dc \tag{31}$$

can be expressed as

$$q(\tau) = \pi^{-1/2} \int_0^\infty \int_{-1}^1 e^{-c^2} \psi_*(\tau, c, \mu) c^3 (c^2 - 5/2)(1 - \mu^2) \, d\mu \, dc. \tag{32}$$

To continue in a rigorous manner, we must solve the Chapman–Enskog equation for viscosity to establish $c^2 b(c)$, and then we should solve Eq. (13) subject to Eq. (29) to find the complete solution. However, at this point we follow an idea of Loyalka and Ferziger [2] and replace the true scattering kernel as listed in Eq. (3) with the synthetic (and truncated) approximation

$$F(\boldsymbol{c}',\boldsymbol{c}) = \frac{1}{2\pi}\sum_{n=0}^{2}\sum_{m=0}^{n}\left(\frac{2n+1}{2}\right)(2-\delta_{0,m})P_n^m(\mu')P_n^m(\mu)f_n(c',c)\cos m(\chi'-\chi) \tag{33}$$

that was reported in Ref. [1]. Here we write

$$f_0(c',c) = v(c')v(c)[\varpi_{01} + \varpi_{02}(c'^2-\omega)(c^2-\omega)], \tag{34}$$

$$f_1(c',c) = \varpi_{11}c'v(c')cv(c) + \varpi_{12}\Delta_1(c')\Delta_1(c) \tag{35}$$

and

$$f_2(c',c) = \varpi_2\Delta_2(c')\Delta_2(c). \tag{36}$$

In regard to Eq. (34), we note that

$$\varpi_{01} = \frac{1}{v_2}, \quad \varpi_{02} = \frac{v_2}{v_2v_6 - v_4^2} \quad \text{and} \quad \omega = \frac{v_4}{v_2}, \tag{37a,b,c}$$

where

$$v_n = \int_0^\infty \mathrm{e}^{-c^2}v(c)c^n\,\mathrm{d}c. \tag{38}$$

To complete Eq. (35), we write

$$\varpi_{11} = \frac{1}{v_4}, \quad \varpi_{12} = [a_1 - a_2 - a_*a_3]^{-1} \quad \text{and} \quad a_* = a_3/v_4, \tag{39a,b,c}$$

where

$$a_1 = \int_0^\infty \mathrm{e}^{-c^2}v(c)a^2(c)c^4\,\mathrm{d}c, \tag{40a}$$

$$a_2 = \int_0^\infty \mathrm{e}^{-c^2}a(c)c^6\,\mathrm{d}c \tag{40b}$$

and

$$a_3 = \int_0^\infty \mathrm{e}^{-c^2}v(c)a(c)c^4\,\mathrm{d}c. \tag{40c}$$

In addition,

$$\Delta_1(c) = v(c)[a_*c - ca(c)] + c(c^2 - 5/2). \tag{41}$$

We note that Eqs. (40) and (41) are defined in terms of the solution to the Chapman–Enskog equation for heat flow, viz.

$$\mathscr{L}_1\{ca\}(c) = c(c^2 - 5/2) \tag{42a}$$

and the (normalization) condition

$$\int_0^\infty \mathrm{e}^{-c^2} a(c)c^4\,\mathrm{d}c = 0. \tag{42b}$$

And finally, to complete Eq. (36) we note [1] that

$$\varpi_2 = \frac{1}{v_*} \tag{43}$$

with

$$v_* = \int_0^\infty \mathrm{e}^{-c^2} b(c)[v(c)c^2 b(c) - c^2]c^4\,\mathrm{d}c \tag{44}$$

and

$$\varDelta_2(c) = c^2 - v(c)c^2 b(c). \tag{45}$$

Having introduced a synthetic-kernel approximation to the true scattering kernel, we now seek a bounded (as τ tends to infinity) solution $\psi_*(\tau, c, \mu)$ of

$$c\mu\frac{\partial}{\partial\tau}\psi(\tau,c,\mu) + \varepsilon v(c)\psi(\tau,c,\mu) = \varepsilon\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c'^2}\psi(\tau,c',\mu')f(c',\mu' : c,\mu)c'^2\,\mathrm{d}\mu'\,\mathrm{d}c' \tag{46}$$

that satisfies the boundary condition listed as Eq. (29). Here

$$f(c',\mu' : c,\mu) = (1-\mu'^2)\sum_{n=1}^{2} \varPi_n(\mu')\varPi_n(\mu)f_n(c',c). \tag{47}$$

Since the Chapman–Enskog functions for viscosity and heat flow are used to define our synthetic-kernel approximation to the true kernel, we refer to the balance equation we use here as the CES model of the linearized Boltzmann equation for rigid-sphere collisions. The CES model should not be confused with the variable collision frequency (CLF) model of Cercignani [5] and Loyalka and Ferziger [6] that has been used [7,8] recently, with the ADO method, to solve with good accuracy Kramers' problem and the temperature-jump problem. While there are some similarities between these two (CLF and CES) model formulations, the differences are considerable.

3. A reformulation of Kramers' problem

To avoid working with three independent variables, as used in Eq. (46), we follow Busbridge [9] and, more explicitly, a recent work of Barichello et al. [8], introduce the new variable

$$\xi = c\mu/v(c) \tag{48}$$

and substitute

$$\psi_*[\tau, c, \xi v(c)/c] = c g_1(\tau, \xi) + [\Delta_1(c)/v(c)] g_2(\tau, \xi) + (\xi/c)\Delta_2(c) g_3(\tau, \xi) \tag{49}$$

into Eq. (46) to find

$$\xi \frac{\partial}{\partial \tau} \boldsymbol{G}(\tau, \xi) + \varepsilon \boldsymbol{G}(\tau, \xi) = \varepsilon \int_{-\gamma}^{\gamma} \boldsymbol{\Psi}(\xi') \boldsymbol{G}(\tau, \xi') \, \mathrm{d}\xi', \tag{50}$$

where $\gamma = \pi^{-1/2}$. Here $\boldsymbol{G}(\tau, \xi)$ has components $g_1(\tau, \xi)$, $g_2(\tau, \xi)$ and $g_3(\tau, \xi)$. In addition, the components of $\boldsymbol{\Psi}(\xi)$ are given by

$$\psi_{11}(\xi) = (3/4)\varpi_{11} \int_{M_\xi} \mathrm{e}^{-c^2} v^2(c) c [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c, \tag{51a}$$

$$\psi_{12}(\xi) = (3/4)\varpi_{11} \int_{M_\xi} \mathrm{e}^{-c^2} v(c) \Delta_1(c) [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c, \tag{51b}$$

$$\psi_{13}(\xi) = (3\xi/4)\varpi_{11} \int_{M_\xi} \mathrm{e}^{-c^2} v^2(c) [\Delta_2(c)/c] [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c, \tag{51c}$$

$$\psi_{21}(\xi) = (3/4)\varpi_{12} \int_{M_\xi} \mathrm{e}^{-c^2} v(c) \Delta_1(c) [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c, \tag{51d}$$

$$\psi_{22}(\xi) = (3/4)\varpi_{12} \int_{M_\xi} \mathrm{e}^{-c^2} [\Delta_1^2(c)/c] [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c, \tag{51e}$$

$$\psi_{23}(\xi) = (3\xi/4)\varpi_{12} \int_{M_\xi} \mathrm{e}^{-c^2} v(c) [\Delta_1(c)/c] [\Delta_2(c)/c] [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c, \tag{51f}$$

$$\psi_{31}(\xi) = (15\xi/4)\varpi_2 \int_{M_\xi} \mathrm{e}^{-c^2} v^2(c) [\Delta_2(c)/c] [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c, \tag{51g}$$

$$\psi_{32}(\xi) = (15\xi/4)\varpi_2 \int_{M_\xi} \mathrm{e}^{-c^2} v(c) [\Delta_1(c)/c] [\Delta_2(c)/c] [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c \tag{51h}$$

and

$$\psi_{33}(\xi) = (15\xi^2/4)\varpi_2 \int_{M_\xi} \mathrm{e}^{-c^2} v^2(c) [\Delta_2^2(c)/c^3] [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c, \tag{51i}$$

where

$$c \in M_\xi \quad \text{if} \ \frac{v(c)|\xi|}{c} \leqslant 1. \tag{52}$$

Now to find the boundary conditions to go with Eq. (50), we use Eq. (49) to deduce from Eq. (29) that

$$\boldsymbol{G}(0,\xi) - (1-\alpha)\boldsymbol{D}\boldsymbol{G}(0,-\xi) = (2/\varepsilon)(2-\alpha)\boldsymbol{D}\boldsymbol{F}(\xi), \quad \xi \in (0,\gamma], \tag{53}$$

where

$$\boldsymbol{D} = \mathrm{diag}\{1,1,-1\} \tag{54}$$

and

$$\boldsymbol{F}(\xi) = \begin{bmatrix} \xi \\ 0 \\ 1 \end{bmatrix}. \tag{55}$$

We can now express the desired velocity profile, as given by Eq. (30), in a convenient way, viz.

$$u(\tau) = \tau + \int_{-\gamma}^{\gamma} \boldsymbol{\Upsilon}^{\mathrm{T}}(\xi)\boldsymbol{G}(\tau,\xi)\,\mathrm{d}\xi, \tag{56}$$

where the vector $\boldsymbol{\Upsilon}(\xi)$ has components

$$u_1(\xi) = \pi^{-1/2}\int_{M_\xi} \mathrm{e}^{-c^2} c v(c)[c^2 - \xi^2 v^2(c)]\,\mathrm{d}c, \tag{57a}$$

$$u_2(\xi) = \pi^{-1/2}\int_{M_\xi} \mathrm{e}^{-c^2} \Delta_1(c)[c^2 - \xi^2 v^2(c)]\,\mathrm{d}c \tag{57b}$$

and

$$u_3(\xi) = \xi\pi^{-1/2}\int_{M_\xi} \mathrm{e}^{-c^2} v(c)[\Delta_2(c)/c][c^2 - \xi^2 v^2(c)]\,\mathrm{d}c. \tag{57c}$$

In a similar manner, we can rewrite Eq. (32) as

$$q(\tau) = \int_{-\gamma}^{\gamma} \boldsymbol{\Gamma}^{\mathrm{T}}(\xi)\boldsymbol{G}(\tau,\xi)\,\mathrm{d}\xi, \tag{58}$$

where the vector $\boldsymbol{\Gamma}(\xi)$ has components

$$\gamma_1(\xi) = \pi^{-1/2}\int_{M_\xi} \mathrm{e}^{-c^2} c v(c)(c^2 - 5/2)[c^2 - \xi^2 v^2(c)]\,\mathrm{d}c, \tag{59a}$$

$$\gamma_2(\xi) = \pi^{-1/2}\int_{M_\xi} \mathrm{e}^{-c^2} \Delta_1(c)(c^2 - 5/2)[c^2 - \xi^2 v^2(c)]\,\mathrm{d}c \tag{59b}$$

and

$$\gamma_3(\xi) = \xi\pi^{-1/2}\int_{M_\xi} \mathrm{e}^{-c^2} v(c)[\Delta_2(c)/c](c^2 - 5/2)[c^2 - \xi^2 v^2(c)]\,\mathrm{d}c. \tag{59c}$$

At this point we can use the ADO method [3] to establish the vector-valued function $\boldsymbol{G}(\tau,\xi)$ as defined by Eqs. (50) and (53). In this way, the desired viscous-slip coefficient and the velocity and heat-flow profiles will be available. We note that this "three-component" problem is very similar to the recently reported [8] formulation of the temperature-jump problem for the CLF model, and so much of our previously reported discrete-ordinates solution can be used to solve the "***G*** problem" formulation of Kramers' problem.

4. The thermal-creep problem

For this problem, instead of having the flow driven by an imposed condition as τ tends to infinity, we have an explicit inhomogeneous driving term in the balance equation. And so we follow Loyalka [10] and consider the linearized Boltzmann equation written as

$$c(1-\mu^2)^{1/2}\cos\chi(c^2-5/2)k_2 + c\mu\frac{\partial}{\partial\tau}h(\tau,\boldsymbol{c}) = \varepsilon L\{h\}(\tau,\boldsymbol{c}), \tag{60}$$

where k_2 is a constant (considered given) related to the slope of the temperature as τ tends to infinity, and where the collision operator is defined by Eq. (2). For this problem we also have a boundary condition which is a mixture of diffuse and specular reflection, and so to go with Eq. (60) we write

$$h(0,c,\mu,\chi) - (1-\alpha)h(0,c,-\mu,\chi) - D = 0, \quad \chi\in[0,2\pi], \tag{61}$$

for $\mu\in(0,1]$ and $c\in[0,\infty)$, where (still) α is the accommodation coefficient and

$$D = \frac{2\alpha}{\pi}\int_0^\infty\int_0^1\int_0^{2\pi} \mathrm{e}^{-c^2}h(0,c,-\mu,\chi)c^3\mu\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c. \tag{62}$$

As we did for Kramers' problem, we now multiply Eqs. (60) and (61) by $\cos\chi$ and integrate to find, after imposing the normalizing condition $k_2=1$,

$$c(c^2-5/2) + c\mu\frac{\partial}{\partial\tau}\psi(\tau,c,\mu) = \varepsilon L^*\{\psi\}(\tau,c,\mu) \tag{63}$$

and the boundary condition

$$\psi(0,c,\mu) - (1-\alpha)\psi(0,c,-\mu) = 0 \tag{64}$$

for $\mu\in(0,1]$ and $c\in[0,\infty)$. Here

$$L^*\{\psi\}(\tau,c,\mu) = -\nu(c)\psi(\tau,c,\mu) + \int_0^\infty\int_{-1}^1 \mathrm{e}^{-c'^2}\psi(\tau,c'\mu')k(c',\mu':c,\mu)c'^2\,\mathrm{d}\mu'\,\mathrm{d}c', \tag{65}$$

where $k(c',\mu':c,\mu)$ is given by Eq. (15) and where

$$\psi(\tau,c,\mu) = (1-\mu^2)^{-1/2}\frac{1}{\pi}\int_0^{2\pi} h(\tau,\boldsymbol{c})\cos\chi\,\mathrm{d}\chi. \tag{66}$$

Continuing, we now write

$$\psi(\tau,c,\mu) = \psi_*(\tau,c,\mu) + \psi_{\mathrm{ps}}(\tau,c,\mu), \tag{67}$$

where

$$\psi_{\mathrm{ps}}(\tau,c,\mu) = -ca(c)/\varepsilon \tag{68}$$

is a particular solution of Eq. (63) and where $\psi_*(\tau,c,\mu)$ is a bounded (as τ tends to infinity) solution of the homogeneous version of Eq. (63). Substituting Eq. (67) into Eq. (64), we find the boundary condition

$$\psi_*(0,c,\mu)-(1-\alpha)\psi_*(0,c,-\mu)=\alpha c a(c)/\varepsilon \tag{69}$$

for $\mu\in(0,1]$ and $c\in[0,\infty)$. The velocity profile

$$u(\tau)=\pi^{-3/2}\int_0^\infty\int_{-1}^1\int_0^{2\pi}\mathrm{e}^{-c^2}h(\tau,\boldsymbol{c})c^3(1-\mu^2)^{1/2}\cos\chi\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c \tag{70}$$

and the heat-flow profile

$$q(\tau)=\pi^{-3/2}\int_0^\infty\int_{-1}^1\int_0^{2\pi}\mathrm{e}^{-c^2}h(\tau,\boldsymbol{c})c^3(c^2-5/2)(1-\mu^2)^{1/2}\cos\chi\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c \tag{71}$$

can, after we note Eq. (42b), now be expressed as

$$u(\tau)=\pi^{-1/2}\int_0^\infty\int_{-1}^1\mathrm{e}^{-c^2}\psi_*(\tau,c,\mu)c^3(1-\mu^2)\,\mathrm{d}\mu\,\mathrm{d}c \tag{72}$$

and

$$q(\tau)=-5\varepsilon_{\mathrm{t}}/(4\varepsilon)+\pi^{-1/2}\int_0^\infty\int_{-1}^1\mathrm{e}^{-c^2}\psi_*(\tau,c,\mu)c^3(c^2-5/2)(1-\mu^2)\,\mathrm{d}\mu\,\mathrm{d}c, \tag{73}$$

where

$$\varepsilon_{\mathrm{t}}=\frac{16}{15}\pi^{-1/2}\int_0^\infty\mathrm{e}^{-c^2}a(c)c^6\,\mathrm{d}c. \tag{74}$$

At this point we introduce the CES model, and so instead of dealing with the homogeneous version of Eq. (63) we seek a bounded (as τ tends to infinity) solution $\psi_*(\tau,c,\mu)$ of

$$c\mu\frac{\partial}{\partial\tau}\psi(\tau,c,\mu)+\varepsilon v(c)\psi(\tau,c,\mu)=\varepsilon\int_0^\infty\int_{-1}^1\mathrm{e}^{-c'^2}\psi(\tau,c',\mu')f(c',\mu':c,\mu)c'^2\,\mathrm{d}\mu'\,\mathrm{d}c' \tag{75}$$

that satisfies the boundary condition listed as Eq. (69). As for Kramers' problem

$$f(c',\mu':c,\mu)=(1-\mu'^2)\sum_{n=1}^{2}\Pi_n(\mu')\Pi_n(\mu)f_n(c',c). \tag{76}$$

5. A reformulation of the thermal-creep problem

As before, we wish to make use of the composite variable

$$\xi=c\mu/v(c) \tag{77}$$

but for the current problem, because of the form of the right-hand side of Eq. (69), we write

$$\psi_*[\tau,c,\xi v(c)/c]=cg_1(\tau,\xi)+[\Delta_1(c)/v(c)]g_2(\tau,\xi)+(\xi/c)\Delta_2(c)g_3(\tau,\xi)+ca(c)g_4(\tau,\xi). \tag{78}$$

We can now make use of Eq. (77) and substitute Eq. (78) into Eqs. (75) and (69) to find a four-component problem defined by

$$\xi \frac{\partial}{\partial \tau} \boldsymbol{G}(\tau, \xi) + \varepsilon \boldsymbol{G}(\tau, \xi) = \varepsilon \int_{-\gamma}^{\gamma} \boldsymbol{\Psi}(\xi') \boldsymbol{G}(\tau, \xi') \, \mathrm{d}\xi' \tag{79}$$

and the boundary condition

$$\boldsymbol{G}(0, \xi) - (1 - \alpha) \boldsymbol{D} \boldsymbol{G}(0, -\xi) = \boldsymbol{F}(\xi), \quad \xi \in (0, \gamma], \tag{80}$$

where (now)

$$\boldsymbol{D} = \mathrm{diag}\{1, 1, -1, 1\} \tag{81}$$

and

$$\boldsymbol{F}(\xi) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha/\varepsilon \end{bmatrix}. \tag{82}$$

Here the vector-valued function $\boldsymbol{G}(\tau, \xi)$ has $g_i(\tau, \xi)$, for $i = 1, 2, 3, 4$, as components, and the 4×4 matrix-valued function $\boldsymbol{\Psi}(\xi)$ has, in addition to the components listed in Eqs. (51),

$$\psi_{14}(\xi) = (3/4) \varpi_{11} \int_{M_\xi} \mathrm{e}^{-c^2} v^2(c) c a(c) [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c, \tag{83a}$$

$$\psi_{24}(\xi) = (3/4) \varpi_{12} \int_{M_\xi} \mathrm{e}^{-c^2} v(c) [\Delta_1(c)/c] c a(c) [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c, \tag{83b}$$

$$\psi_{34}(\xi) = (15\xi/4) \varpi_2 \int_{M_\xi} \mathrm{e}^{-c^2} v^2(c) [\Delta_2(c)/c^2] c a(c) [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c \tag{83c}$$

and

$$\psi_{41}(\xi) = \psi_{42}(\xi) = \psi_{43}(\xi) = \psi_{44}(\xi) = 0. \tag{83d}$$

If we now consider that $\boldsymbol{\Upsilon}(\xi)$ has four components, the first three of which are given by Eqs. (57), with

$$u_4(\xi) = \pi^{-1/2} \int_{M_\xi} \mathrm{e}^{-c^2} c a(c) v(c) [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c \tag{84}$$

then can express the velocity and heat-flow profiles as

$$u(\tau) = \int_{-\gamma}^{\gamma} \boldsymbol{\Upsilon}^{\mathrm{T}}(\xi) \boldsymbol{G}(\tau, \xi) \, \mathrm{d}\xi, \tag{85}$$

and

$$q(\tau) = -5\varepsilon_{\mathrm{t}}/(4\varepsilon) + \int_{-\gamma}^{\gamma} \boldsymbol{\Gamma}^{\mathrm{T}}(\xi) \boldsymbol{G}(\tau, \xi) \, \mathrm{d}\xi, \tag{86}$$

where the elements of $\boldsymbol{\Gamma}(\xi)$ are given by Eqs. (59) and

$$\gamma_4(\xi) = \pi^{-1/2} \int_{M_\xi} \mathrm{e}^{-c^2} c a(c) v(c) (c^2 - 5/2) [c^2 - \xi^2 v^2(c)] \, \mathrm{d}c. \tag{87}$$

Again, we can use the ADO method [3] to establish the vector-valued function $\boldsymbol{G}(\tau,\xi)$ as defined by Eqs. (79) and (80). In this way, the desired thermal-slip coefficient and velocity and heat-flow profiles will be available from Eqs. (85) and (86). We note that this "four-component" problem is also very similar to the recently reported [8] formulation of the temperature-jump problem for the CLF model, and so we can be brief in the discussion of our discrete-ordinates solution.

6. The discrete-ordinates solution

To start our ADO solution of Eq. (50), we look for solutions of the form

$$\boldsymbol{G}_\nu(\tau,\xi) = \boldsymbol{\Phi}(\nu,\xi)\mathrm{e}^{-\varepsilon\tau/\nu}, \tag{88}$$

and so substituting Eq. (88) into Eq. (50) we find

$$(1-\xi/\nu)\boldsymbol{\Phi}(\nu,\xi) = \int_0^\gamma [\boldsymbol{\Psi}(\xi')\boldsymbol{\Phi}(\nu,\xi') + \boldsymbol{\Psi}(-\xi')\boldsymbol{\Phi}(\nu,-\xi')]\,\mathrm{d}\xi'. \tag{89}$$

Now if we use an N-point quadrature scheme to evaluate the integral in Eq. (89), then we can write

$$(1-\xi/\nu)\boldsymbol{\Phi}(\nu,\xi) = \sum_{k=1}^{N} w_k[\boldsymbol{\Psi}(\xi_k)\boldsymbol{\Phi}(\nu,\xi_k) + \boldsymbol{\Psi}(-\xi_k)\boldsymbol{\Phi}(\nu,-\xi_k)], \tag{90}$$

where $\{\xi_k, w_k\}$ are the nodes and weights of the quadrature scheme. Evaluating Eq. (90) at $\xi = \pm\xi_i$, we find

$$(1 \mp \xi_i/\nu)\boldsymbol{\Phi}(\nu,\pm\xi_i) = \sum_{k=1}^{N} w_k[\boldsymbol{\Psi}(\xi_k)\boldsymbol{\Phi}(\nu,\xi_k) + \boldsymbol{\Psi}(-\xi_k)\boldsymbol{\Phi}(\nu,-\xi_k)] \tag{91}$$

for $i = 1,2,\dots,N$. Following the development given in detail for the "three-component" temperature-jump problem in Ref. [8], we can convert the system of equations listed as Eq. (91) to a $3N \times 3N$ eigenvalue problem which we can solve numerically to yield $3N$ plus–minus pairs of separation constants $\pm\nu_j$ and the corresponding elementary vectors $\boldsymbol{\Phi}(\pm\nu_j,\xi_i)$. And so, keeping in mind that we seek a bounded (as τ tends to infinity) solution of Eq. (50), we let $\{\nu_j\}$ denote the set of positive separation constants and then express the desired solution as

$$\boldsymbol{G}(\tau,\pm\xi_i) = \sum_{j=1}^{3N} A_j\boldsymbol{\Phi}(\nu_j,\pm\xi_i)\mathrm{e}^{-\varepsilon\tau/\nu_j}, \tag{92}$$

where the constants A_j are to be determined from a discrete-ordinates version of the boundary condition given by Eq. (53). It can be shown, say from Eqs. (89) and (90), that there is only one positive value of ν, say ν_1, that tends to infinity as N increases without bound. We choose to take this fact into account explicitly by ignoring in Eq. (92) the largest separation constant and by using instead the corresponding exact solution. And so we rewrite Eq. (92) as

$$\boldsymbol{G}(\tau,\pm\xi_i) = A_1\boldsymbol{\Phi}_+ + \sum_{j=2}^{3N} A_j\boldsymbol{\Phi}(\nu_j,\pm\xi_i)\mathrm{e}^{-\varepsilon\tau/\nu_j}, \tag{93}$$

where

$$\boldsymbol{\Phi}_+ = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{94}$$

To complete the solution for Kramers' problem, we substitute Eq. (93) into Eq. (53) evaluated at the quadrature points $\{\xi_i\}$ and solve the resulting system of linear algebraic equations to find the required constants $\{A_j\}$. At this point we can use Eq. (93) in Eq. (56) to obtain the velocity profile which we express as

$$u_{\mathrm{P}}(\tau) = \tau + \zeta_{\mathrm{P}} + \sum_{j=2}^{3N} A_j N(v_j) \mathrm{e}^{-\varepsilon\tau/v_j}, \tag{95}$$

where

$$\zeta_{\mathrm{P}} = A_1/2 \tag{96}$$

is the viscous-slip coefficient and

$$N(v_j) = \sum_{k=1}^{N} w_k[\boldsymbol{\Upsilon}^{\mathrm{T}}(\xi_k)\boldsymbol{\Phi}(v_j,\xi_k) + \boldsymbol{\Upsilon}^{\mathrm{T}}(-\xi_k)\boldsymbol{\Phi}(v_j,-\xi_k)]. \tag{97}$$

In a similar way, we find that the heat-flux profile, from Eq. (58), can be expressed as

$$q_{\mathrm{P}}(\tau) = \sum_{j=2}^{3N} A_j M(v_j) \mathrm{e}^{-\varepsilon\tau/v_j}, \tag{98}$$

where

$$M(v_j) = \sum_{k=1}^{N} w_k[\boldsymbol{\Gamma}^{\mathrm{T}}(\xi_k)\boldsymbol{\Phi}(v_j,\xi_k) + \boldsymbol{\Gamma}^{\mathrm{T}}(-\xi_k)\boldsymbol{\Phi}(v_j,-\xi_k)]. \tag{99}$$

Having completed Kramers' problem, we consider the problem of thermal-creep. As we have it formulated, this problem is a "four-component" problem that is very similar to the "three-component" Kramers' problem. For this reason we do not repeat the discussion of our ADO solution. Instead we note that the relevant version of Eq. (93) is now

$$\boldsymbol{G}(\tau,\pm\xi_i) = A_1\boldsymbol{\Phi}_+ + \sum_{j=2}^{4N} A_j \boldsymbol{\Phi}(v_j,\pm\xi_i) \mathrm{e}^{-\varepsilon\tau/v_j}, \tag{100}$$

where now

$$\boldsymbol{\Phi}_+ = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{101}$$

Table 1
The viscous-slip coefficient ζ_P

$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
17.04462	5.203049	2.815562	1.779429	1.194279	9.864009(−1)

To complete this solution, we substitute Eq. (100) into Eq. (80) evaluated at the quadrature points $\{\xi_i\}$ and solve the resulting system of linear algebraic equations to find the required constants $\{A_j\}$. Having found the constants, we can use Eq. (100) in Eq. (85) to obtain

$$u_T(\tau) = \zeta_T + \sum_{j=2}^{4N} A_j N(v_j) e^{-\varepsilon\tau/v_j}, \tag{102}$$

where now

$$\zeta_T = A_1/2 \tag{103}$$

is the thermal-slip coefficient. We find the heat-flux profile for this problem can be written as

$$q_T(\tau) = -5\varepsilon_t/(4\varepsilon) + \sum_{j=2}^{4N} A_j M(v_j) e^{-\varepsilon\tau/v_j}. \tag{104}$$

To conclude this section, we note that to be (somewhat) consistent with the review paper of Sharipov and Seleznev [11] we have added the subscripts P and T to our results for the slip coefficients and the velocity and heat-flow profiles to distinguish between these quantities for Kramers' problem and for the thermal-creep problem.

7. Numerical results

To define the quadrature scheme to be used with our ADO solutions, we have simply mapped the Gauss–Legendre scheme onto the interval $[0, \gamma]$, and in regard to numerical linear-algebra packages, we have used the driver program RG from the EISPACK collection [12] to find the required eigenvalues and eigenvectors, and we used the subroutines DGECO and DGESL from the LINPACK package [13] to solve the linear systems that defines the constants $\{A_j\}$ for each of the two considered problems. We note that for Kramers' problem we have elected to use the mean-free path based on viscosity, and so we have used $\varepsilon = \varepsilon_p$ where, for example from Ref. [1],

$$\varepsilon_p = 0.449027806\ldots \,. \tag{105}$$

On the other hand, for the thermal-creep problem we have used the mean-free path based on thermal conductivity, and so have used $\varepsilon = \varepsilon_t$, where, say from Ref. [1]

$$\varepsilon_t = 0.679630049\ldots \,. \tag{106}$$

To complete our work we list in Tables 1–7 some results obtained from our FORTRAN implementation of the developed solutions. We note that our results are given with what we believe to be seven, in Tables 1 and 4, and six, in Tables 2, 3, 5 and 6, figures of accuracy. While we have no

Table 2
The velocity profile $u_P(\tau)$ for Kramers' problem

τ	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
0.0	1.64798(1)	4.70959	2.39005	1.41876	8.95603(−1)	7.17720(−1)
0.1	1.67830(1)	4.98650	2.64202	1.64707	1.10142	9.12716(−1)
0.2	1.69694(1)	5.16198	2.80717	1.80233	1.24722	1.05393
0.3	1.71254(1)	5.31102	2.94954	1.93831	1.37708	1.18081
0.4	1.72656(1)	5.44624	3.07999	2.06419	1.49856	1.30016
0.5	1.73960(1)	5.57290	3.20304	2.18379	1.61484	1.41483
0.6	1.75198(1)	5.69375	3.32108	2.29912	1.72758	1.52630
0.7	1.76388(1)	5.81043	3.43550	2.41139	1.83777	1.63549
0.8	1.77543(1)	5.92401	3.54724	2.52137	1.94606	1.74296
0.9	1.78671(1)	6.03520	3.65692	2.62960	2.05290	1.84912
1.0	1.79777(1)	6.14452	3.76498	2.73645	2.15859	1.95425
2.0	1.90267(1)	7.18736	4.80202	3.76793	3.18474	2.97781

Table 3
The heat-flow profile $q_P(\tau)$ for Kramers' problem

τ	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
0.0	2.83332(−1)	2.38051(−1)	1.97680(−1)	1.61566(−1)	1.29165(−1)	1.14212(−1)
0.1	2.24002(−1)	1.89667(−1)	1.58739(−1)	1.30769(−1)	1.05383(−1)	9.35580(−2)
0.2	1.89829(−1)	1.61248(−1)	1.35388(−1)	1.11891(−1)	9.04591(−2)	8.04376(−2)
0.3	1.63704(−1)	1.39358(−1)	1.17260(−1)	9.71170(−2)	7.86824(−2)	7.00399(−2)
0.4	1.42510(−1)	1.21514(−1)	1.02412(−1)	8.49556(−2)	6.89392(−2)	6.14154(−2)
0.5	1.24821(−1)	1.06571(−1)	8.99340(−2)	7.47003(−2)	6.06940(−2)	5.41040(−2)
0.6	1.09808(−1)	9.38551(−2)	7.92885(−2)	6.59278(−2)	5.36223(−2)	4.78249(−2)
0.7	9.69236(−2)	8.29194(−2)	7.01141(−2)	5.83518(−2)	4.75025(−2)	4.23852(−2)
0.8	8.57773(−2)	7.34429(−2)	6.21504(−2)	5.17645(−2)	4.21722(−2)	3.76434(−2)
0.9	7.60772(−2)	6.51841(−2)	5.52000(−2)	4.60072(−2)	3.75070(−2)	3.34903(−2)
1.0	6.75962(−2)	5.79544(−2)	4.91083(−2)	4.09550(−2)	3.34082(−2)	2.98393(−2)
2.0	2.20280(−2)	1.89636(−2)	1.61334(−2)	1.35073(−2)	1.10603(−2)	9.89725(−3)

Table 4
The thermal-slip coefficient ζ_T

$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
2.671726(−1)	2.864184(−1)	3.039673(−1)	3.200405(−1)	3.348226(−1)	3.417790(−1)

proof of the accuracy achieved in this work, we have found the results to be stable as the order N of the quadrature scheme is increased. In addition, we have found good agreement with the viscous-slip coefficients and the thermal-slip coefficients reported by Loyalka and Hickey [14] and by Ohwada et al. [15] for the case of diffuse reflection ($\alpha = 1$). Considering first the viscous-slip coefficient ($\alpha = 1$) we note that (to three figures) Refs. [14,15] both have (in our notation) the result $\zeta_P = 0.987$. These results are based on numerical solutions to the linearized Boltzmann equation for rigid-sphere

Table 5
The velocity profile $u_T(\tau)$ for the thermal-creep problem

τ	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
0.0	2.36503(−1)	1.99194(−1)	1.65834(−1)	1.35895(−1)	1.08939(−1)	9.64621(−2)
0.1	2.45367(−1)	2.24056(−1)	2.04661(−1)	1.86935(−1)	1.70671(−1)	1.63032(−1)
0.2	2.50014(−1)	2.37261(−1)	2.25555(−1)	2.14762(−1)	2.04773(−1)	2.00050(−1)
0.3	2.53327(−1)	2.46711(−1)	2.40562(−1)	2.34826(−1)	2.29453(−1)	2.26890(−1)
0.4	2.55849(−1)	2.53920(−1)	2.52037(−1)	2.50198(−1)	2.48402(−1)	2.47519(−1)
0.5	2.57831(−1)	2.59593(−1)	2.61079(−1)	2.62330(−1)	2.63376(−1)	2.63832(−1)
0.6	2.59418(−1)	2.64142(−1)	2.68337(−1)	2.72076(−1)	2.75418(−1)	2.76957(−1)
0.7	2.60706(−1)	2.67834(−1)	2.74233(−1)	2.79999(−1)	2.85216(−1)	2.87639(−1)
0.8	2.61760(−1)	2.70859(−1)	2.79065(−1)	2.86497(−1)	2.93254(−1)	2.96406(−1)
0.9	2.62628(−1)	2.73352(−1)	2.83051(−1)	2.91860(−1)	2.99892(−1)	3.03647(−1)
1.0	2.63348(−1)	2.75420(−1)	2.86356(−1)	2.96308(−1)	3.05400(−1)	3.09657(−1)
2.0	2.66432(−1)	2.84286(−1)	3.00549(−1)	3.15429(−1)	3.29099(−1)	3.35527(−1)

Table 6
The heat-flow profile $q_T(\tau)$ for the thermal-creep problem

τ	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
0.0	−1.15473	−9.78709(−1)	−8.19864(−1)	−6.75965(−1)	−5.45149(−1)	−4.84148(−1)
0.1	−1.18551	−1.06552	−9.56161(−1)	−8.56078(−1)	−7.64130(−1)	−7.20900(−1)
0.2	−1.20068	−1.10870	−1.02460	−9.47379(−1)	−8.76196(−1)	−8.42642(−1)
0.3	−1.21107	−1.13836	−1.07176	−1.01048	−9.53872(−1)	−9.27148(−1)
0.4	−1.21874	−1.16030	−1.10669	−1.05729	−1.01160	−9.90009(−1)
0.5	−1.22462	−1.17713	−1.13352	−1.09329	−1.05605	−1.03843
0.6	−1.22922	−1.19033	−1.15458	−1.12158	−1.09100	−1.07652
0.7	−1.23289	−1.20084	−1.17137	−1.14415	−1.11890	−1.10694
0.8	−1.23584	−1.20931	−1.18490	−1.16234	−1.14140	−1.13148
0.9	−1.23824	−1.21619	−1.19590	−1.17713	−1.15971	−1.15145
1.0	−1.24020	−1.22182	−1.20489	−1.18923	−1.17469	−1.16779
2.0	−1.24823	−1.24492	−1.24185	−1.23901	−1.23637	−1.23511

collisions, and so (aside from numerical approximations) we consider these results to be essentially definitive. Our result $\zeta_P = 0.986$, which we consider to be a computationally rigorous solution (if no programming errors have been made!) of the CES model approximation of the linearized Boltzmann equation, agrees well with the quoted results of Refs. [14,15] and is a good improvement over the BGK result $\zeta_P = 1.02$ and over two results [7] based on the variable collision-frequency model, viz. $\zeta_P = 0.967$ for the case $v(c) = c$ and $\zeta_P = 0.974$ for the case where $v(c)$ is given by Eq. (8). In regard to the thermal-slip coefficient, we note that Refs. [10,15] both report (in our notation and for $\alpha = 1$) the result $\zeta_T = 0.336$. Again, considering that the CES equation is just a model of the linearized Boltzmann equation (for rigid-sphere collisions), we believe that our result $\zeta_T = 0.342$ can be considered a good improvement over the BGK result $\zeta_T = 0.383$. In order to have an idea of how well the CES model (and our calculation) can predict the velocity and heat-flow profiles evaluated at the wall, we again compare our results with those obtained from the linearized

Table 7
Slip coefficients and velocity profiles at the wall with $\varepsilon = \varepsilon_{\mathrm{osa}}$

α	ζ_{P}	$u_{\mathrm{P}}(0)$	β_A	$u_{2A}(0)$	ζ_{T}	$u_{\mathrm{T}}(0)$	β_B	$u_{2B}(0)$
0.1	21.647	20.930	21.652	20.921				
0.2	10.376	9.7043	10.380	9.6957	0.5325	0.4177	0.5275	0.4256
0.3	6.6081	5.9814	6.6117	5.9736				
0.4	4.7167	4.1336	4.7198	4.1266	0.5678	0.3500	0.5595	0.3628
0.5	3.5759	3.0355	3.5787	3.0293				
0.6	2.8106	2.3118	2.8130	2.3064	0.6001	0.2892	0.5899	0.3048
0.7	2.2600	1.8019	2.2620	1.7972				
0.8	1.8436	1.4253	1.8454	1.4214	0.6297	0.2347	0.6188	0.2510
0.9	1.5168	1.1375	1.5183	1.1341				
1.0	1.2528	0.9115	1.2540	0.9088	0.6570	0.1854	0.6463	0.2012

Boltzmann equation by Ohwada et al. [15]. For Kramers problem (with $\alpha = 1$) we have $u_{\mathrm{P}}(0) = 0.718$ and $q_{\mathrm{P}}(0) = 0.114$, while Ohwada et al. [15] have (in our notation) $u_{\mathrm{P}}(0) = 0.716$ and $q_{\mathrm{P}}(0) = 0.105$. For the thermal-creep problem, Ref. [15] reports $u_{\mathrm{T}}(0) = 0.105$ and $q_{\mathrm{T}}(0) = -0.527$, while we have found here $u_{\mathrm{T}}(0) = 0.0964$ and $q_{\mathrm{T}}(0) = -0.484$.

To compare our CES model results, for various values of the accommodation coefficient α, with numerical solutions based on the linearized Boltzmann equation, we list in Table 7 our solutions along with results taken from Ref. [16]. Our Kramers results for ζ_{P} and $u_{\mathrm{P}}(0)$ are compared to the equivalent results β_A and $u_{2A}(0)$ deduced from Ref. [16], and our thermal-creep results for ζ_{T} and $u_{\mathrm{T}}(0)$ are compared to the equivalent results β_B and $u_{2B}(0)$ also deduced from Ref. [16]. We note that the results in Table 7 are based on a mean-free path defined by using $\varepsilon = \varepsilon_{\mathrm{osa}}$, where

$$\varepsilon_{\mathrm{osa}} = 2^{1/2}/4 \tag{107}$$

in contrast to our normal use of $\varepsilon = \varepsilon_{\mathrm{p}}$ for Kramers' problem and $\varepsilon = \varepsilon_{\mathrm{t}}$ for the thermal-creep problem.

As a final calculation, we ran our code and evaluated the thermal-slip coefficient for cases of very small α to try to confirm the result

$$\lim_{\alpha \to 0^+} = 0.493704\ldots \tag{108}$$

reported in Ref. [16]. We found with $\alpha = 10^{-7}$ and 10^{-8} the result $\zeta_{\mathrm{T}} = 0.493704$.

We have typically used $N = 30$ to generate the slip coefficients and the velocity and heat-flow profiles given in our tables, good to, say, six or seven significant figures, and since our FORTRAN implementation (no special effort was made to make the code especially efficient) of the ADO solution (with $N = 30$) runs in 2 min on a 400 MHz Pentium-based PC, we believe we can justify our opinion that the combination of the CES model and the ADO method offers a convenient alternative to more computationally intensive approaches. And while, as a matter of convenience, we have recomputed the basic quantities $\boldsymbol{\Psi}(\xi)$, $\boldsymbol{\Upsilon}(\xi)$ and $\boldsymbol{\Gamma}(\xi)$ each time we solved a problem, it is clear that by storing these quantities evaluated at the quadrature points we could evaluate the CES model solutions of the considered problems (and other flow problems) in a matter of some seconds even on a 400 MHz PC.

8. Final comments

We are of the opinion that the CES model can provide quite a good practical alternative to more computational intensive methods based on the linearized Boltzmann equation. In fact we believe there to be only a modest number of engineering applications where accuracy better than what is shown in Table 7 would be required—especially for Kramers' problem. At the same time we are surprised that the results for the thermal-creep problem, while good, do not seem to be quite as good as the equivalent results for Kramers' problem. Soon some finite-media problems (Couette flow, Poiseuille flow and thermal-creep flow) and the temperature-jump problem will be solved in terms of the CES model in order to know better what confidence we can have in the model. And finally, in addition to the merits of the CES model, we have seen here another case where the ADO method [3] can be used to solve well some model problems in the field of rarefied-gas dynamics.

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