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The temperature-jump problem based on the CES model of the linearized Boltzmann equation

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Abstract. An analytical discrete-ordinates method is used to solve the temperature-jump problem as defined by a synthetic-kernel model of the linearized Boltzmann equation. In particular, the temperature and density perturbations and the temperature-jump coefficient defined by the CES model equation are obtained (essentially) analytically in terms of a modern version of the discrete-ordinates method. The developed algorithms are implemented for general values of the accommodation coefficient to yield numerical results that compare well with solutions derived from more computationally intensive techniques.

Keywords. Boltzmann equation, Rarefied gas dynamics, Temperature-jump problem.

1. Introduction

In a recent work [1] concerning the linearized Boltzmann equation for rigid-sphere interactions, a synthetic-kernel [2,3] model equation (the CES model) and the ADO (analytical discrete-ordinates) method [4] were used to solve Kramers' problem and the thermal-creep problem in the general area of rarefied gas dynamics. Since the CES model, which is defined in terms of solutions to the Chapman-Enskog integral equations for heat conduction and viscosity, yielded good results for the two mentioned half-space flow problems, we continue here our investigation of the model. In order to evaluate the effectiveness of the model for problems defined by projections of the balance equation different from those required for flow problems, we develop and evaluate in this work our solution of the classical temperature-jump problem [5–7].

To start this work, we consider the homogeneous and linearized Boltzmann equation written for rigid-sphere collisions as [8]

$$c\mu\frac{\partial}{\partial\tau}h(\tau,\boldsymbol{c}) = \varepsilon L\{h\}(\tau,\boldsymbol{c}) \tag{1}$$

where

$$L\{h\}(\tau,\boldsymbol{c}) = -\nu(c)h(\tau,\boldsymbol{c}) + \int_0^\infty \int_{-1}^1 \int_0^{2\pi} \mathrm{e}^{-c'^2}c'^2 h(\tau,\boldsymbol{c}')K(\boldsymbol{c}',\boldsymbol{c})\,\mathrm{d}\chi'\,\mathrm{d}\mu'\,\mathrm{d}c'. \tag{2}$$

Here the scattering kernel is

$$K(\boldsymbol{c}',\boldsymbol{c})=\frac{1}{4\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(2n+1)(2-\delta_{0,m})P_n^m(\mu')P_n^m(\mu)k_n(c',c)\cos m(\chi'-\chi) \tag{3}$$

where the *normalized* Legendre functions are given (in terms of the Legendre polynomials) by

$$P_n^m(\mu)=\left[\frac{(n-m)\,!}{(n+m)\,!}\right]^{1/2}(1-\mu^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}\mu^m}P_n(\mu),\quad n\geq m. \tag{4}$$

In addition,

$$\varepsilon=\sigma_0^2 n_0\pi^{1/2}l \tag{5}$$

where l is (at this point) an unspecified mean-free path, n_0 is the density and σ_0 is the scattering diameter of the gas particles. In this work, the spatial variable τ is measured in units of the mean-free path l and $c(2kT_0/m)^{1/2}$ is the magnitude of the particle velocity. Also, k is the Boltzmann constant, m is the mass of a gas particle and T_0 is a reference temperature. The basic unknown $h(\tau,\boldsymbol{c})$ in Eq. (1) is a perturbation from a Maxwellian distribution. Continuing, we note that the functions $k_n(c',c)$ in Eq. (3) are the components in an expansion of the scattering law (for rigid-sphere collisions) reported by Pekeris and Alterman [8], and

$$\nu(c)=\frac{2c^2+1}{c}\int_0^c \mathrm{e}^{-x^2}\,\mathrm{d}x+\mathrm{e}^{-c^2} \tag{6}$$

is the collision frequency. And finally, we use spherical coordinates ($c,\arccos\mu,\chi$) to define the (dimensionless) velocity vector $\boldsymbol{c}$.

Due to the presence of a wall located at $\tau=0$, we must supplement Eq. (1) with an appropriate boundary condition. Noting that

$$h(\tau,\boldsymbol{c})\Leftrightarrow h(\tau,c,\mu,\chi), \tag{7}$$

we express the required boundary condition as

$$h(0,c,\mu,\chi)-(1-\alpha)h(0,c,-\mu,\chi)-\alpha\mathcal{I}\{h\}(0)=0 \tag{8}$$

for $\mu\in(0,1]$, $c\in[0,\infty)$ and all χ. Here

$$\mathcal{I}\{h\}(0)=\frac{2}{\pi}\int_0^{\infty}\int_0^1\int_0^{2\pi}\mathrm{e}^{-c^2}c^3h(0,c,-\mu,\chi)\mu\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c \tag{9}$$

and $\alpha\in(0,1]$ is the accommodation coefficient. In this formulation of the temperature-jump problem there is no driving term in Eq. (1), and so in addition to the boundary condition listed as Eq. (8), we will include in our statement of the problem a condition on $h(\tau,\boldsymbol{c})$ as τ tends to infinity. This condition will be seen clearly once we have expressed the quantities of interest in terms of $h(\tau,\boldsymbol{c})$.

2. Quantities of interest

Following the discussion from Ref. [7], we see that, while our problem is defined in terms of the unknown $h(\tau, \boldsymbol{c})$, we require only two elementary integrals of $h(\tau, \boldsymbol{c})$ in order to establish the temperature and density perturbations [7,9] defined by

$$N(\tau) = \frac{1}{\pi^{3/2}} \int_0^\infty \int_{-1}^1 \int_0^{2\pi} \mathrm{e}^{-c^2} c^2 h(\tau, c, \mu, \chi)\, \mathrm{d}\chi\, \mathrm{d}\mu\, \mathrm{d}c \tag{10}$$

and

$$T(\tau) = \frac{2}{3\pi^{3/2}} \int_0^\infty \int_{-1}^1 \int_0^{2\pi} \mathrm{e}^{-c^2} c^2 (c^2 - 3/2) h(\tau, c, \mu, \chi)\, \mathrm{d}\chi\, \mathrm{d}\mu\, \mathrm{d}c, \tag{11}$$

or

$$N(\tau) = \frac{2}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} c^2 \phi(\tau, c, \mu)\, \mathrm{d}\mu\, \mathrm{d}c \tag{12}$$

and

$$T(\tau) = \frac{4}{3\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} c^2 (c^2 - 3/2) \phi(\tau, c, \mu)\, \mathrm{d}\mu\, \mathrm{d}c \tag{13}$$

where

$$\phi(\tau, c, \mu) = \frac{1}{2\pi} \int_0^{2\pi} h(\tau, c, \mu, \chi)\, \mathrm{d}\chi \tag{14}$$

is an azimuthal average. We can integrate Eqs. (l) and (8) over χ to find

$$c\mu \frac{\partial}{\partial \tau} \phi(\tau, c, \mu) + \varepsilon \nu(c) \phi(\tau, c, \mu) = \varepsilon \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c'^2} c'^2 k(c', \mu' : c, \mu) \phi(\tau, c', \mu')\, \mathrm{d}\mu'\, \mathrm{d}c' \tag{15}$$

for $\tau > 0$, $\mu \in [-1, 1]$ and $c \in [0, \infty)$ and

$$\phi(0, c, \mu) - (1 - \alpha) \phi(0, c, -\mu) - 4\alpha \int_0^\infty \int_0^1 \mathrm{e}^{-c'^2} c'^3 \phi(0, c', -\mu') \mu'\, \mathrm{d}\mu'\, \mathrm{d}c' = 0 \tag{16}$$

for $\mu \in (0, 1]$ and $c \in [0, \infty)$. In regard to Eq. (15), we note that

$$k(c', \mu' : c, \mu) = \int_0^{2\pi} K(\boldsymbol{c}', \boldsymbol{c})\, \mathrm{d}\chi \tag{17}$$

or

$$k(c', \mu' : c, \mu) = \frac{1}{2} \sum_{n=0}^\infty (2n + 1) P_n(\mu') P_n(\mu) k_n(c', c). \tag{18}$$

As Eqs. (1) and (8) are homogeneous, we must specify a driving term for the temperature-jump problem. We do this implicitly by requiring that $h(\tau, \boldsymbol{c})$ diverge as τ tends to infinity. More specifically, we impose the condition that the temperature perturbation satisfies the Welander condition [10]

$$\lim_{\tau \to \infty} \frac{\mathrm{d}}{\mathrm{d}\tau} T(\tau) = K \tag{19}$$

where K is considered specified.

3. A model equation

Rather than deal with the foregoing exact version of the linearized Boltzmann equation, we now introduce the approximate CES model. This model is based on replacing the kernel $K(\boldsymbol{c}', \boldsymbol{c})$ defined by Eq. (3) with the simpler form given by

$$F(\boldsymbol{c}', \boldsymbol{c}) = \frac{1}{4\pi} \sum_{n=0}^{2} \sum_{m=0}^{n} (2n+1)(2-\delta_{0,m}) P_n^m(\mu') P_n^m(\mu) f_n(c', c) \cos m(\chi' - \chi) \quad (20)$$

that was reported in Ref. [3]. Here we write

$$f_0(c', c) = \nu(c')\nu(c)[\varpi_{01} + \varpi_{02}({c'}^2 - \omega)(c^2 - \omega)], \quad (21)$$

$$f_1(c', c) = \varpi_{11} c' \nu(c') c \nu(c) + \varpi_{12} \Delta_1(c') \Delta_1(c) \quad (22)$$

and

$$f_2(c', c) = \varpi_2 \Delta_2(c') \Delta_2(c). \quad (23)$$

In regard to Eq. (21), we note that

$$\varpi_{01} = \frac{1}{\nu_2}, \quad \varpi_{02} = \frac{\nu_2}{\nu_2\nu_6 - \nu_4^2} \quad \text{and} \quad \omega = \frac{\nu_4}{\nu_2} \quad (24\text{a,b,c})$$

where

$$\nu_n = \int_0^\infty \mathrm{e}^{-c^2} \nu(c) c^n \, \mathrm{d}c. \quad (25)$$

To complete Eq. (22), we write

$$\varpi_{11} = \frac{1}{\nu_4}, \quad \varpi_{12} = [a_1 - a_2 - a_* a_3]^{-1} \quad \text{and} \quad a_* = a_3/\nu_4 \quad (26\text{a,b,c})$$

where

$$a_1 = \int_0^\infty \mathrm{e}^{-c^2} \nu(c) a^2(c) c^4 \, \mathrm{d}c, \quad (27\text{a})$$

$$a_2 = \int_0^\infty \mathrm{e}^{-c^2} a(c) c^6 \, \mathrm{d}c \quad (27\text{b})$$

and

$$a_3 = \int_0^\infty \mathrm{e}^{-c^2} \nu(c) a(c) c^4 \, \mathrm{d}c. \quad (27\text{c})$$

In addition,

$$\Delta_1(c) = \nu(c)[a_* c - c a(c)] + c(c^2 - 5/2). \quad (28)$$

We note that Eqs. (27) and (28) are defined in terms of the solution to the Chapman-Enskog equation for heat conduction, *viz.*

$$\mathcal{L}_1\{ca\}(c) = c(c^2 - 5/2) \quad (29\text{a})$$

and the (normalization) condition

$$\int_0^\infty \mathrm{e}^{-c^2} a(c) c^4 \, \mathrm{d}c = 0. \quad (29\text{b})$$

And finally, to complete Eq. (23) we note that

$$\varpi_2 = \frac{1}{\nu_*}, \tag{30}$$

with

$$\nu_* = \int_0^\infty \mathrm{e}^{-c^2} b(c)[\nu(c)c^2 b(c) - c^2]c^4\,\mathrm{d}c, \tag{31}$$

and

$$\Delta_2(c) = c^2 - \nu(c)c^2 b(c). \tag{32}$$

Note that $\Delta_2(c)$ is defined in terms of the Chapman-Enskog equation for viscosity, *viz.*

$$\mathcal{L}_2\{c^2 b\}(c) = c^2. \tag{33}$$

Here, in writing Eqs. (29a) and (33), we have made use of the notation

$$\mathcal{L}_n\{\psi\}(c) = \nu(c)\psi(c) - \int_0^\infty \mathrm{e}^{-c'^2}\psi(c')k_n(c',c)c'^2\,\mathrm{d}c' \tag{34}$$

where the components $k_n(c',c)$ are the exact Pekeris-Alterman [8] components of the scattering law.

Having introduced a synthetic-kernel approximation to the true scattering kernel, we now seek a solution of

$$c\mu\frac{\partial}{\partial\tau}\phi(\tau,c,\mu) + \varepsilon\nu(c)\phi(\tau,c,\mu) = \varepsilon\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c'^2}c'^2 f(c',\mu' : c,\mu)\phi(\tau,c',\mu')\,\mathrm{d}\mu'\,\mathrm{d}c' \tag{35}$$

for $\tau > 0$, $\mu \in [-1,1]$ and $c \in [0,\infty)$ and

$$\phi(0,c,\mu) - (1-\alpha)\phi(0,c,-\mu) - 4\alpha\int_0^\infty\int_0^1 \mathrm{e}^{-c'^2}c'^3\phi(0,c',-\mu')\mu'\,\mathrm{d}\mu'\,\mathrm{d}c' = 0 \tag{36}$$

for $\mu \in (0,1]$ and $c \in [0,\infty)$. Here

$$f(c',\mu' : c,\mu) = \frac{1}{2}\sum_{n=0}^{2}(2n+1)P_n(\mu')P_n(\mu)f_n(c',c). \tag{37}$$

In addition, the solution we seek must diverge as τ tends to infinity, but at the same time the solution must be such that Eq. (19) is satisfied.

In order to avoid a change of notation, we continue to use $\phi(\tau,c,\mu)$ in Eqs. (35) and (36), but we keep in mind that, in general, solutions of Eqs. (35) and (36) will not be solutions of Eqs. (15) and (16) simply because we have used $f(c',\mu' : c,\mu)$ to approximate the true scattering kernel $k(c',\mu' : c,\mu)$.

In Ref. [3] five exact solutions of Eq. (1) that are independent of τ and three exact solutions linear in τ were listed. However, since Eq. (15) is an azimuthal average of Eq. (1) only four of the mentioned solutions are relevant to Eq. (15).

These four solutions allow us to write *a component* of the complete solution of Eq. (15) as

$$\phi(\tau,c,\mu) = A + Bc\mu + D(c^2 - 5/2) + (K/\varepsilon)[(c^2 - 5/2)\varepsilon\tau - \mu c a(c)] \tag{38}$$

where the constants A, B, D and K are arbitrary. Now, because of the way the synthetic kernel (the CES model) was constructed, the expression given by Eq. (38) is also a solution of Eq. (35). We therefore write

$$\phi(\tau,c,\mu) = (K/\varepsilon)[\phi_*(\tau,c,\mu) + (c^2 - 5/2)\varepsilon\tau - \mu c a(c)] \tag{39}$$

where $\phi_*(\tau,c,\mu)$ is a solution of Eq. (35) that is bounded as τ tends to infinity and that satisfies the boundary condition

$$\phi_*(0,c,\mu) - (1-\alpha)\phi_*(0,c,-\mu) - 4\alpha\int_0^\infty\int_0^1 \mathrm{e}^{-c'^2}c'^3\phi_*(0,c',-\mu')\mu'\,\mathrm{d}\mu'\,\mathrm{d}c' = \mathcal{F}(c,\mu) \tag{40}$$

for $\mu \in (0,1]$ and $c \in [0,\infty)$. Here

$$\mathcal{F}(c,\mu) = (2-\alpha)\mu c a(c). \tag{41}$$

Finally, in regard to the quantities of interest here, we can use Eq. (39) in Eqs. (12) and (13) to obtain

$$N(\tau) = K\big[-\tau + \frac{2}{\varepsilon\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}c^2\phi_*(\tau,c,\mu)\,\mathrm{d}\mu\,\mathrm{d}c\big] \tag{42}$$

and

$$T(\tau) = K\big[\tau + \frac{4}{3\varepsilon\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}c^2(c^2-3/2)\phi_*(\tau,c,\mu)\,\mathrm{d}\mu\,\mathrm{d}c\big]. \tag{43}$$

And so we proceed to use some additional transformations and the ADO method to establish the required integrals of $\phi_*(\tau,c,\mu)$.

4. A reformulation of the temperature-jump problem

To avoid working with three independent variables, as used in Eq. (35), we follow Busbridge [11] and, more explicitly, two recent works [1,7] and introduce into Eq. (35) the composite variable

$$\xi = c\mu/\nu(c). \tag{44}$$

Then, after viewing the resulting form of Eq. (35), we propose a solution written as

$$\phi_*[\tau,c,\xi\nu(c)/c] = \sum_{\beta=1}^{8} f_\beta(c)g_\beta(\tau,\xi)t_\beta(\xi) \tag{45}$$

where the $t_\beta(\xi)$ are the elements of the vector

$$\boldsymbol{T}(\xi) = \begin{bmatrix} 1 & 1 & \xi & \xi & \xi^2 & \xi^2 & 1 & 1 \end{bmatrix} \tag{46}$$

and the f functions are given by

$$f_1(c) = 1, \quad f_2(c) = c^2 - \omega, \quad f_3(c) = \nu(c), \quad f_4(c) = \Delta_1(c)/c \tag{47a,b,c,d}$$

$$f_5(c) = f_6(c) = \Delta_2(c)\nu(c)/c^2 \quad \text{and} \quad f_7(c) = f_8(c) = \Delta_2(c)/\nu(c). \tag{47e,f}$$

And so, making use of Eq. (44), we substitute Eq. (45) into Eqs. (35) and (40) to find a eight-component problem defined by

$$\xi \frac{\partial}{\partial \tau} \boldsymbol{G}(\tau, \xi) + \varepsilon \boldsymbol{G}(\tau, \xi) = \varepsilon \int_{-\gamma}^{\gamma} \boldsymbol{\Psi}(\xi') \boldsymbol{G}(\tau, \xi') \, \mathrm{d}\xi' \tag{48}$$

and the boundary condition

$$\boldsymbol{G}(0, \xi) - (1 - \alpha) \boldsymbol{D} \boldsymbol{G}(0, -\xi) - \alpha \boldsymbol{V} \int_0^{\gamma} \boldsymbol{R}^{\mathrm{T}}(\xi') \boldsymbol{G}(0, -\xi') \xi' \, \mathrm{d}\xi' = \boldsymbol{F}(\xi), \quad \xi \in (0, \gamma), \tag{49}$$

where we use the superscript T to denote the transpose operation,

$$\boldsymbol{D} = \mathrm{diag}\{1, 1, -1, -1, 1, 1, 1, 1\}, \tag{50}$$

$$\boldsymbol{V} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \tag{51}$$

and

$$\boldsymbol{F}(\xi) = (2 - \alpha) \begin{bmatrix} -(3/4)\xi & \xi & a_* & -1 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{52}$$

In addition, the vector $\boldsymbol{R}(\xi)$ has components

$$r_\beta(\xi) = 4 t_\beta(-\xi) \int_{M_\xi} \mathrm{e}^{-c^2} \nu^2(c) f_\beta(c) c \, \mathrm{d}c, \quad \beta = 1, 2, \ldots, 8, \tag{53}$$

$\gamma = \pi^{-1/2}$, the vector-valued function $\boldsymbol{G}(\tau, \xi)$ has $g_i(\tau, \xi)$, for $i = 1, 2, \ldots, 8$, as components and the 8×8 matrix-valued function $\boldsymbol{\Psi}(\xi)$ has elements

$$\psi_{i,j}(\xi) = (1/2) s_i(\xi) t_j(\xi) \int_{M_\xi} \mathrm{e}^{-c^2} h_i(c) f_j(c) c^2 \, \mathrm{d}c. \tag{54}$$

Here the s functions are

$$s_1(\xi) = \varpi_{01}, \quad s_2(\xi) = \varpi_{02}, \quad s_3(\xi) = 3\varpi_{11}\xi, \tag{55a,b,c}$$

$$s_4(\xi) = 3\varpi_{12}\xi, \quad s_5(\xi) = (45/4)\varpi_2\xi^2, \quad s_6(\xi) = -(15/4)\varpi_2, \tag{55d,e,f}$$

$$s_7(\xi) = -(15/4)\varpi_2\xi^2 \quad \text{and} \quad s_8(\xi) = (5/4)\varpi_2, \tag{55g,h}$$

and the h functions are given by

$$h_\beta(c) = \nu^2(c) f_\beta(c)/c, \quad \beta = 1, 2, 3, 4, \tag{56a,b,c,d}$$

$$h_5(c) = h_7(c) = \nu^2(c) f_5(c)/c \quad \text{and} \quad h_6(c) = h_8(c) = \nu^2(c) f_8(c)/c. \tag{56e,f}$$

And finally

$$c \in M_\xi \quad \text{if} \quad \frac{\nu(c)|\xi|}{c} \leq 1. \tag{57}$$

In concluding this section, we note that instead of the 8-component decomposition listed as Eq. (45) we could have used a representation with only six elements. We have elected to use the 8-component form in order to have our final reformulation, as defined by Eqs. (48), (49) and (54), of the temperature-jump problem resemble as much as possible our reformulations of Kramers' problem and the half-space thermal-creep problem that were based on the CES model and solved in Ref. [1].

5. The discrete-ordinates solution

To start our ADO solution of Eq. (48), we look for solutions of the form

$$\boldsymbol{G}_\nu(\tau,\xi) = \boldsymbol{\Phi}(\nu,\xi)\mathrm{e}^{-\varepsilon\tau/\nu}, \tag{58}$$

and so substituting Eq. (58) into Eq. (48) we find

$$\big(1-\xi/\nu\big)\boldsymbol{\Phi}(\nu,\xi) = \int_0^\gamma [\boldsymbol{\Psi}(\xi')\boldsymbol{\Phi}(\nu,\xi') + \boldsymbol{\Psi}(-\xi')\boldsymbol{\Phi}(\nu,-\xi')]\,\mathrm{d}\xi'. \tag{59}$$

Now if we use an N-point quadrature scheme to evaluate the integral in Eq. (59), then we can write

$$\big(1-\xi/\nu\big)\boldsymbol{\Phi}(\nu,\xi) = \sum_{k=1}^{N} w_k[\boldsymbol{\Psi}(\xi_k)\boldsymbol{\Phi}(\nu,\xi_k) + \boldsymbol{\Psi}(-\xi_k)\boldsymbol{\Phi}(\nu,-\xi_k)] \tag{60}$$

where $\{\xi_k, w_k\}$ are the nodes and weights of the quadrature scheme. Evaluating Eq. (60) at $\xi = \pm\xi_i$, we find

$$\big(1\mp\xi_i/\nu\big)\boldsymbol{\Phi}(\nu,\pm\xi_i) = \sum_{k=1}^{N} w_k[\boldsymbol{\Psi}(\xi_k)\boldsymbol{\Phi}(\nu,\xi_k) + \boldsymbol{\Psi}(-\xi_k)\boldsymbol{\Phi}(\nu,-\xi_k)] \tag{61}$$

for $i = 1, 2, ..., N$. Following the development given in detail for the "three-component" temperature-jump problem in Ref. [7] and the two similar problems solved in Ref. [1], we can convert the system of equations listed as Eq. (61) to an $8N \times 8N$ eigenvalue problem which we can solve numerically to yield $8N$ plus-minus pairs of separation constants $\pm\nu_j$ and the corresponding elementary vectors $\boldsymbol{\Phi}(\pm\nu_j, \xi_i)$. And so, keeping in mind that we seek a bounded (as τ tends to infinity) solution of Eq. (48), we let $\{\nu_j\}$ denote the set of positive separation constants and then express the desired solution as

$$\boldsymbol{G}(\tau,\pm\xi_i) = \sum_{j=1}^{8N} A_j \boldsymbol{\Phi}(\nu_j,\pm\xi_i)\mathrm{e}^{-\varepsilon\tau/\nu_j} \tag{62}$$

where the constants A_j are to be determined from a discrete-ordinates version of Eq. (49) written as

$$\boldsymbol{G}(0,\xi_i) - (1-\alpha)\boldsymbol{D}\boldsymbol{G}(0,-\xi_i) - \alpha\boldsymbol{V}\sum_{k=1}^{N} w_k\xi_k \boldsymbol{R}^{\mathrm{T}}(\xi_k)\boldsymbol{G}(0,-\xi_k) = \boldsymbol{F}(\xi_i), \tag{63}$$

for $i = 1, 2, \ldots, N$. We have found, from numerical studies, that there are only two positive values of the seperation constants ν, say ν_1 and ν_2, that tend to infinity as N increases. We choose to take this fact into account explicitly by ignoring in Eq. (62) the two largest separation constants and by using instead the corresponding exact solution. And so we rewrite Eq.(62) as

$$\boldsymbol{G}(\tau, \pm\xi_i) = A_1\boldsymbol{\Phi}_1 + A_2\boldsymbol{\Phi}_2 + B\boldsymbol{\Phi}_3 + \sum_{j=3}^{8N} A_j\boldsymbol{\Phi}(\nu_j, \pm\xi_i)\mathrm{e}^{-\varepsilon\tau/\nu_j} \tag{64}$$

where

$$\boldsymbol{\Phi_1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \tag{65a}$$

$$\boldsymbol{\Phi_2} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \tag{65b}$$

and

$$\boldsymbol{\Phi_3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{65c}$$

It can be seen that $\boldsymbol{\Phi}_3$ satisfies the homogeneous version of Eq. (63), and so the coefficient B is arbitrary in our solution of the temperature-jump problem. On the other hand, we can now substitute Eq. (64) into Eq. (63) and solve the resulting system of linear algebraic equations to find the constants A_j, $j = 1, 2, \ldots, 8N$ required to complete our solution of the "$\boldsymbol{G}$ problem." In order to establish the required density and temperature perturbations we use Eq. (45) and our discrete-ordinates solution of the $\boldsymbol{G}$ problem in a quadrature version (when the integrals cannot be evaluated analytically) of Eqs. (42) and (43) to find

$$N(\tau) = K\big[-\tau + (1/\varepsilon)(B - A_1/4) + (1/\varepsilon)\sum_{j=3}^{8N} A_j N_j \mathrm{e}^{-\varepsilon\tau/\nu_j}\big] \tag{66}$$

and

$$T(\tau) = K\big[\tau + A_1/\varepsilon + (1/\varepsilon)\sum_{j=3}^{8N} A_j T_j \mathrm{e}^{-\varepsilon\tau/\nu_j}\big] \tag{67}$$

where

$$N_j = \sum_{k=1}^{N} w_k[\boldsymbol{U}^{\mathrm{T}}(\xi_k)\boldsymbol{\Phi}(\nu_j, \xi_k) + \boldsymbol{U}^{\mathrm{T}}(-\xi_k)\boldsymbol{\Phi}(\nu_j, -\xi_k)] \tag{68}$$

and

$$T_j = \sum_{k=1}^{N} w_k[\boldsymbol{V}^{\mathrm{T}}(\xi_k)\boldsymbol{\Phi}(\nu_j, \xi_k) + \boldsymbol{V}^{\mathrm{T}}(-\xi_k)\boldsymbol{\Phi}(\nu_j, -\xi_k)]. \tag{69}$$

In addition the elements of $\boldsymbol{U}(\xi)$ and $\boldsymbol{V}(\xi)$ are

$$u_\beta(\xi) = 2\pi^{-1/2} t_\beta(\xi) \int_{M_\xi} \mathrm{e}^{-c^2}\nu(c) f_\beta(c) c \,\mathrm{d}c \tag{70}$$

and

$$v_\beta(\xi) = (4/3)\pi^{-1/2} t_\beta(\xi) \int_{M_\xi} \mathrm{e}^{-c^2} \nu(c) f_\beta(c)(c^2 - 3/2) c \,\mathrm{d}c \tag{71}$$

for $\beta = 1, 2, \ldots, 8$. At this point we normalize our solution by setting $K = 1$ and by putting the arbitrary constant $B = -(3/4)A_1$. We also introduce the temperature-jump coefficient

$$\zeta = A_1/\varepsilon \tag{72}$$

so that we can write Eqs. (66) and (67) as

$$N(\tau) = -\tau - \zeta + (1/\varepsilon) \sum_{j=3}^{8N} A_j N_j \mathrm{e}^{-\varepsilon\tau/\nu_j} \tag{73}$$

and

$$T(\tau) = \tau + \zeta + (1/\varepsilon) \sum_{j=3}^{8N} A_j T_j \mathrm{e}^{-\varepsilon\tau/\nu_j}. \tag{74}$$

Having completed our ADO solution to the temperature-jump problem, we report some numerical results defined by the CES model.

6. Numerical results and final comments

To define the quadrature scheme to be used with our ADO solutions, we have simply mapped the Gauss-Legendre scheme onto the interval $[0, \gamma]$. In regard to numerical linear-algebra packages, we have used the driver program RG from the EISPACK collection [12] to find the required eigenvalues and eigenvectors, and we used the subroutines DGECO and DGESL from the LINPACK package [13] to solve the linear system that defines the constants $\{A_j\}$.

Considering the important issue of an appropriate mean-free path, we have elected to use the mean-free path based on thermal conductivity, and so have used $\varepsilon = \varepsilon_\mathrm{t}$, where, say from Ref. [3],

$$\varepsilon_\mathrm{t} = 0.679630049\ldots. \tag{75}$$

To complete this work we list in Tables 1, 2 and 3 selected numerical results for the temperature-jump coefficient and the temperature and density perturbations obtained from our FORTRAN implementation of the ADO solution developed here. And, in order to try to evaluate the merits of the CES model and other model equations, we include in Tables 1 and 3 some results obtained from a work [7] based on the CLF model of Cercignani [14] and Loyalka and Ferziger [5]. We also make use of what we take to be reference values, *viz.* Loyalka's results [15] that are based on what appears to be an accurate numerical solution of the linearized Boltzmann equation for rigid-sphere collisions. We note that our ADO results are given with what we believe to be seven, in Table 1, and six, in Table 2, figures of

Table 1. The temperature-jump coefficient ζ

α	BGK	CLF-w	CLF-rs	LBE-v †	CES	LBE-sn †
0.1	21.45012	21.19359	21.24657	21.287	21.32099	21.349
0.2	10.34747	10.10735	10.15704	10.196	10.22670	10.252
0.3	6.630514	6.406417	6.452894	6.4909	6.517910	6.5398
0.4	4.760333	4.551884	4.595202	4.6321	4.655696	4.6747
0.5	3.629125	3.435960	3.476180	3.5118	3.532264	3.5708
0.6	2.867615	2.689383	2.726563	2.7607	2.778342	2.7924
0.7	2.317534	2.153897	2.188095	2.2207	2.235669	2.2476
0.8	1.899741	1.750372	1.781643	1.8123	1.825107	1.8350
0.9	1.570264	1.434848	1.463247	1.4921	1.502689	1.5109
1.0	1.302716	1.180947	1.206526	1.2334	1.242033	1.2486

† Loyalka [15]

Table 2. The temperature and density perturbations

	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.9$	
τ	$T(\tau)$	$-N(\tau)$	$T(\tau)$	$-N(\tau)$	$T(\tau)$	$-N(\tau)$
0.0	5.85804	6.09712	2.96157	3.16242	1.09997	1.23407
0.1	6.22678	6.38942	3.29280	3.43017	1.36145	1.45413
0.2	6.43591	6.56105	3.48727	3.59319	1.52824	1.59998
0.3	6.60432	6.70370	3.64660	3.73083	1.67034	1.72754
0.4	6.75177	6.83205	3.78779	3.85591	1.79963	1.84598
0.5	6.88642	6.95200	3.91789	3.97358	1.92106	1.95901
0.6	7.01254	7.06654	4.04060	4.08648	2.03723	2.06856
0.7	7.13268	7.17741	4.15810	4.19614	2.14972	2.17571
0.8	7.24845	7.28569	4.27182	4.30350	2.25950	2.28118
0.9	7.36096	7.39207	4.38270	4.40918	2.36727	2.38541
1.0	7.47098	7.49706	4.49141	4.51363	2.47349	2.48872
2.0	8.50990	8.51488	5.52529	5.52954	3.49770	3.50063

accuracy. While we have no proof of the accuracy achieved in this work and, of course, we cannot be sure that no programming errors have been made, we have some confidence in the reported values since (for one thing) we found the results to be stable as the order N of the quadrature scheme is increased. To be clear, we note that the accuracy we believe we have achieved here with our ADO solution refers to the solution of the CES model equation and not to the merits of the model itself.

Table 3. The temperature and density perturbations for $\alpha = 0.8$

	$T(\tau)$					$-N(\tau)$				
τ	BGK	CLF-w	CLF-rs	CES	LBE-sn †	BGK	CLF-w	CLF-rs	CES	LBE-sn †
0.00	1.349	1.488	1.414	1.382	1.360	1.472	1.623	1.541	1.531	1.502
0.25	1.811	1.872	1.831	1.909	1.860	1.900	1.939	1.907	1.980	1.930
0.50	2.147	2.168	2.144	2.235	2.190	2.216	2.212	2.198	2.278	2.233
0.75	2.451	2.444	2.431	2.523	2.485	2.506	2.474	2.471	2.549	2.513
1.00	2.739	2.710	2.706	2.793	2.763	2.784	2.732	2.737	2.810	2.782
1.50	3.290	3.228	3.236	3.312	3.296	3.320	3.240	3.255	3.320	3.305
2.00	3.821	3.738	3.753	3.820	3.813	3.843	3.745	3.765	3.823	3.817

† Loyalka [15]

In Table 1 we list values of the temperature-jump coefficient as evaluated from the BGK model, the Williams variation (CLF-w) of the CLF model [has $\nu(c) = c$], the rigid-sphere variation (CLF-rs) of the CLF model [has $\nu(c)$ given by Eq. (6)] and the CES model. We also list Loyalka's results [15] from his variational solution (LBE-v) and his results (LBE-sn) from a numerical (S_N) solution of the linearized Boltzmann equation for rigid-sphere collisions. In comparing the results listed in Table 1, we consider that the order with increasing merit is: CLF-w, BGK, CLF-rs, LBE-v, CES and LBE-sn. It is clear that the variations in these results are the greatest for the case of diffuse reflection ($\alpha = 1$), but the order of merit for the case $\alpha = 0.1$ is still the same as mentioned.

In Table 2 we list what we consider to be essentially exact solutions for the temperature and density perturbations as predicted by the CES model equation. These results are based on using $N = 40$ in our ADO solution.

While we have found here another example where the ADO method can be used effectively to solve basic problems in rarefied gas dynamics, the results listed in Table 3 yield some surprises (at least for the author) in regard to model equations. For example, it is clear that the best model (CES) results in Table 1 and the variational results (LBE-v) both give good improvements over the BGK model for the asymptotic temperature and density results (essentially the temperature-jump coefficient), but the results in Table 3 show that the BGK model is as good as (or better than) the other models for defining the temperature and density perturbations near the wall. Of course, in making these comparisons we must recall that the mean-free paths for the BGK model and the CLF models are based on using approximate values for ε , rather than Eq. (75), to define the mean-free path. On the other hand, since the CES model [3] has the correct Prandtl number, we feel comfortable in suggesting that this model, while more complicated than the BGK model, provides a useful alternative to the more computational intensive S_N solution [15] we have taken for our reference results.

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