

Model equations in rarefied gas dynamics: Viscous-slip and thermal-slip coefficients

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Various model equations are used to define the viscous-slip and the thermal-slip coefficients in rarefied gas dynamics. More specifically, the BGK model, the S model, the variable collision model and the CES model are used to establish the slip coefficients basic to Kramers' problem and the half-space problem of thermal creep. While the most general results are developed from use of the Maxwell boundary condition, results for the BGK model and the S model as defined by the Cercignani–Lampis boundary condition are also reported. An analytical discrete-ordinates method is used to establish the reported numerical results, and when available results from a numerical solution of the linearized Boltzmann equation are used as reference values. In addition to the numerical work based on model equations, the important issue of how to define meaningful ways (appropriate mean-free paths) to compare the results for the various models is discussed. © 2002 American Institute of Physics. [DOI: 10.1063/1.1514973]

I. INTRODUCTION

In reviewing numerous papers devoted to model equations in rarefied gas dynamics, we have found no definitive way to compare in a consistent manner the results from different model approximations to the linearized Boltzmann equation. However, as will be seen here for rigid-sphere interactions, we can make use of kernels^{1,2} that approximate the exact scattering kernels from the Pekeris–Alterman analysis² of the linearized Boltzmann equation and follow Loyalka and co-workers^{3–5} to define convenient mean-free paths for the various models considered.

While essentially all of the definitive numerical work in rarefied gas dynamics is based on the use of the classical Maxwell gas–surface interaction law (characterized by a single accommodation coefficient), we have two recent works^{6,7} that include the effects of the accommodation coefficients α_t and α_n on two flow problems in a plane channel. Note that, in contrast to the Maxwell boundary condition which has the unique accommodation coefficient α for all physical properties, the Cercignani–Lampis (CL) condition⁸ allows us to distinguish the accommodation of different properties. Physically, the quantity α_t is the accommodation coefficient of the tangential momentum, while the other quantity α_n describes the accommodation of the kinetic energy corresponding to the normal velocity. Since the CL boundary condition is based on the two mentioned accommodation coefficients, the use of this boundary condition yields the possibility of including better physics in the study of the basic problems of rarefied gas dynamics. In this work we report numerical results for the viscous-slip and the thermal-slip coefficients for both the Maxwell boundary condition and the Cercignani–Lampis boundary condition. The inclusion in this work of the Cercignani–Lampis gas–

surface interaction model is considered to be important contribution since essentially all definitive numerical work available is based on the classical Maxwell gas–surface interaction model.

While the study of approximating models to the linearized Boltzmann equation and the inclusion of various gas–surface interaction models is considered important if we are going to be able to obtain, at a modest computational cost, good results for practical problems, it is also important to be able to solve well the formulated model problem. It is for this reason that we continue here our work with an analytical discrete-ordinates method⁹ that can be implemented to yield what we believe to be excellent numerical results.

To start this work, we consider the linearized Boltzmann equation written for rigid-sphere collisions as²

$$S(\mathbf{c}) + c\mu \frac{\partial}{\partial \tau} h(\tau, \mathbf{c}) = \varepsilon L\{h\}(\tau, \mathbf{c}), \quad (1)$$

where

$$L\{h\}(\tau, \mathbf{c}) = -\nu(c)h(\tau, \mathbf{c}) + \int_0^\infty \int_{-1}^1 \int_0^{2\pi} e^{-c'^2} h(\tau, \mathbf{c}') \times K(\mathbf{c}':\mathbf{c})c'^2 d\chi' d\mu' dc'. \quad (2)$$

Here the scattering kernel is

$$K(\mathbf{c}':\mathbf{c}) = \frac{1}{4\pi} \sum_{n=0}^\infty \sum_{m=0}^n (2n+1)(2-\delta_{0,m})P_n^m(\mu') \times P_n^m(\mu)k_n(c',c)\cos m(\chi'-\chi), \quad (3)$$

where the *normalized* Legendre functions are given (in terms of the Legendre polynomials) by

$$P_n^m(\mu) = \left[\frac{(n-m)!}{(n+m)!} \right]^{1/2} (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_n(\mu), \quad n \geq m. \tag{4}$$

In addition

$$\varepsilon = \sigma_0^2 n_0 \pi^{1/2} l, \tag{5}$$

where l is (at this point) an unspecified mean-free path, n_0 is the density of the gas particles and σ_0 is the scattering diameter of the gas molecules. In this work, the spatial variable τ is measured in units of the mean-free path l and $c(2kT_0/m)^{1/2}$ is the magnitude of the particle velocity. Here k is the Boltzmann constant, m is the mass of a gas particle and T_0 is a convenient reference temperature. It should be noted that we have included in Eq. (1) an inhomogeneous driving term $S(\mathbf{c})$ that we will specify, along with an appropriate definition of the perturbation $h(\tau, \mathbf{c})$, for the two types of flow (the Kramers problem and the problem of thermal-creep flow) we consider in this work. Continuing, we note that the functions $k_n(c', c)$ in Eq. (3) are the components in an expansion of the scattering law (for rigid-sphere collisions) reported by Pekeris and Alterman² and

$$\nu(c) = \frac{2c^2 + 1}{c} \int_0^c e^{-x^2} dx + e^{-c^2}, \tag{6}$$

is the collision frequency. Note that we use spherical coordinates $(c, \arccos \mu, \chi)$ to define the (dimensionless) velocity vector \mathbf{c} .

While the scattering kernel as defined by Pekeris and Alterman² for rigid-sphere collisions is given by Eq. (3) in which the component functions $k_n(c', c)$ are required for all n , our work here with model equations is based on using at most $n=0, 1$, and 2 . And so we list the exact forms for these component functions only for these same values of n . Noting that $k_n(c', c) = k_n(c, c')$, we consult other works^{2,3} and write for $c' < c$

$$-(1/2)c'ck_0(c', c) = (2/3)c'^3 + 2c'c^2 - 4P(c'), \tag{7a}$$

$$-(1/2)c'^2c^2k_1(c', c) = (2/15)c'^5 - 4c' - (2/3)c'^3c^2 - 4(c'^2 - 1)P(c'), \tag{7b}$$

and

$$-(1/2)c'^3c^3k_2(c', c) = a_2(c', c) + b_2(c', c)P(c'), \tag{7c}$$

where

$$a_2(c', c) = (2/35)c'^7 - 3c'^3 + 18c' - [(2/15)c'^5 - 3c']c^2 \tag{8a}$$

and

$$b_2(c', c) = -6c'^4 + 15c'^2 - 18 + [2c'^2 - 3]c^2. \tag{8b}$$

Here

$$P(c) = e^{c^2} \int_0^c e^{-x^2} dx. \tag{9}$$

Now, as discussed in other works,^{1,2,5} if we wish to use a mean-free path based on the viscosity, i.e.,

$$l = l_p = (\mu_* / p_0)(2kT_0/m)^{1/2}, \tag{10}$$

where μ_* is the viscosity and $p_0 = n_0 k T_0$ is the pressure, then we should use in Eq. (1)

$$\varepsilon = \varepsilon_p = \frac{16}{15\pi^{1/2}} \int_0^\infty e^{-c^2} b(c) c^6 dc, \tag{11}$$

where $b(c)$ is defined by the Chapman–Enskog equation for viscosity, viz.

$$\nu(c)c^2b(c) - \int_0^\infty e^{-c'^2} b(c')k_2(c', c)c'^4 dc' = c^2. \tag{12}$$

On the other hand, if we wish to use a mean-free path based on heat conduction, then we should use^{1,2,4}

$$l = l_t = [4\lambda_* / (5n_0k)][m/(2kT_0)]^{1/2}, \tag{13}$$

where λ_* is the heat-conduction coefficient and where

$$\varepsilon = \varepsilon_t = \frac{16}{15\pi^{1/2}} \int_0^\infty e^{-c^2} a(c) c^6 dc. \tag{14}$$

Here $a(c)$ is determined by the Chapman–Enskog equation for heat conduction

$$\nu(c)ca(c) - \int_0^\infty e^{-c'^2} a(c')k_1(c', c)c'^3 dc' = c(c^2 - 5/2) \tag{15}$$

and the normalizing condition

$$\int_0^\infty e^{-c^2} a(c) c^4 dc = 0. \tag{16}$$

Continuing to quote from another work,¹ we note that the component functions for $n=0$ and 1 satisfy the integral conditions (which result from the conditions of conservation of mass, momentum and energy)

$$\nu(c) = \int_0^\infty e^{-c'^2} k_0(c', c) c'^2 dc', \tag{17a}$$

$$\nu(c)c = \int_0^\infty e^{-c'^2} k_1(c', c) c'^3 dc', \tag{17b}$$

and

$$\nu(c)c^2 = \int_0^\infty e^{-c'^2} k_0(c', c) c'^4 dc'. \tag{17c}$$

As we intend to solve both the viscous-slip problem and the problem of thermal creep in a semi-infinite half-space, we will specify conditions on $h(\tau, \mathbf{c})$ as τ tends to infinity differently for each of these two problems. However, the boundary condition at the wall for these two problems is the same, viz.

$$h(0, c, \mu, \chi) = \int_0^\infty \int_0^1 \int_0^{2\pi} e^{-c'^2} h(0, c', -\mu', \chi') \times R(\mathbf{c}':\mathbf{c}) c'^2 d\chi' d\mu' dc', \tag{18}$$

for $\mu \in (0, 1]$ and all c and χ . Here the R function (yet to be specified) describes the manner in which the gas particles interact with the wall.

Having given an exact formulation of the linearized Boltzmann equation for rigid-sphere collisions and expres-

sions required to define mean-free paths in terms of the Chapman–Enskog equations for viscosity and heat conduction, we are essentially ready to investigate some model kinetic equations that can be used to approximate the exact formulation. However, before proceeding to our numerical work, we note that numerous important mathematical issues, such as the existence and uniqueness of solutions, have been addressed by many authors, see (for example) the works by Bardos, Caflisch and Nicolaenko,¹⁰ Coron, Golse and Sulem¹¹ and the important books of Cercignani,^{12,13} and so the basis for numerical work is considered well in place.

II. THE BGK MODEL AND THE S MODEL

Since the BGK model¹⁴ can easily be obtained from a formulation of the *S*-model equation proposed, as mentioned by Sharipov and Seleznev,¹⁵ by Shakhov,¹⁶ we consider these two kinetic models together. And so for the BGK and for the *S* model, we express the scattering kernel required in Eq. (2) as

$$K(\mathbf{c}' : \mathbf{c}) = \pi^{-3/2} [1 + 2(\mathbf{c}' \cdot \mathbf{c}) + (2/3)(c'^2 - 3/2)(c^2 - 3/2) + \beta M(\mathbf{c}' : \mathbf{c})], \quad (19)$$

where

$$M(\mathbf{c}' : \mathbf{c}) = (4/15)(\mathbf{c}' \cdot \mathbf{c})(c'^2 - 5/2)(c^2 - 5/2), \quad (20)$$

and where (in a consistent notation)

$$\mathbf{c}' \cdot \mathbf{c} = c' c \sum_{m=0}^1 (2 - \delta_{0,m}) P_1^m(\mu') P_1^m(\mu) \cos m(\chi' - \chi). \quad (21)$$

We use the parameter β in Eq. (19) as a switch: $\beta = 0$ yields the BGK model, while $\beta = 1$ yields the *S* model. In order to put Eq. (19) in the form of Eq. (3), we take

$$k_0(c', c) = 4 \pi^{-1/2} [1 + (2/3)(c'^2 - 3/2)(c^2 - 3/2)], \quad (22a)$$

$$k_1(c', c) = (8/3) c' c \pi^{-1/2} [1 + (2/15) \beta (c'^2 - 5/2) \times (c^2 - 5/2)], \quad (22b)$$

and

$$k_n(c', c) = 0, \quad n > 1. \quad (22c)$$

If we now use Eqs. (22) in Eqs. (17), we see at once that $\nu(c) = 1$ is the correct collision frequency for the BGK model and for the *S* model. In addition, if we use Eqs. (22) in Eqs. (12), (15), and (16) we find that the Chapman–Enskog functions appropriate to these models are

$$b(c) = 1, \quad (23)$$

for both the BGK model and the *S* model, while

$$a(c) = (c^2 - 5/2), \quad (24a)$$

for the BGK model and

$$a(c) = (3/2)(c^2 - 5/2), \quad (24b)$$

for the *S* model. Using Eqs. (24) in Eq. (14) we find

$$\varepsilon_t = 1, \quad (25a)$$

for the BGK model and

$$\varepsilon_t = 3/2, \quad (25b)$$

for the *S* model, whereas for both models we find from Eq. (11)

$$\varepsilon_p = 1. \quad (26)$$

To conclude this section, we point out that in order to avoid excess notation, we are using, in the various sections of this work based on different kinetic models, the same symbols for the scattering kernel $K(\mathbf{c}' : \mathbf{c})$, the component functions $k_n(c', c)$, the Chapman–Enskog functions $a(c)$ and $b(c)$ and the two constants ε_p and ε_t .

III. THE CLF MODEL

The variable collision frequency model of Cercignani¹⁷ and Loyalka and Ferziger⁴ has the scattering kernel expressed in terms of the collision frequency $\nu(c)$ as

$$K(\mathbf{c}' : \mathbf{c}) = \frac{1}{4\pi} \nu(c') \nu(c) [\varpi_{01} + 3\varpi_{11}(\mathbf{c}' \cdot \mathbf{c}) + \varpi_{02}(c'^2 - \omega)(c^2 - \omega)], \quad (27)$$

where

$$\varpi_{01} = \frac{1}{\nu_2}, \quad (28a)$$

$$\varpi_{11} = \frac{1}{\nu_4}, \quad (28b)$$

$$\varpi_{02} = \frac{\nu_2}{\nu_2 \nu_6 - \nu_4^2}, \quad (28c)$$

and

$$\omega = \frac{\nu_4}{\nu_2}, \quad (28d)$$

with

$$\nu_n = \int_0^\infty e^{-c^2} \nu(c) c^n dc. \quad (29)$$

It follows that we can write the scattering kernel listed as Eq. (27) in the form of Eq. (3) by using

$$k_0(c', c) = \nu(c') \nu(c) [\varpi_{01} + \varpi_{02}(c'^2 - \omega)(c^2 - \omega)], \quad (30a)$$

$$k_1(c', c) = \varpi_{11} c' c \nu(c') \nu(c), \quad (30b)$$

and

$$k_n(c', c) = 0, \quad n > 1. \quad (30c)$$

We can now use Eqs. (30a) and (30b) to confirm that the identities listed as Eqs. (17) are satisfied without specifying the collision frequency. And so for this model we consider $\nu(c)$ to be arbitrary. Now we can use Eqs. (30a) and (30b) in Eqs. (12), (15), and (16) to find that the Chapman–Enskog functions appropriate to the CLF model are

$$b(c) = \nu^{-1}(c) \quad (31a)$$

and

$$a(c) = \nu^{-1}(c)(c^2 - 5/2) + \hat{a}, \tag{31b}$$

where

$$\hat{a} = -\frac{8}{3\pi^{1/2}} \int_0^\infty e^{-c^2} \nu^{-1}(c)(c^2 - 5/2)c^4 dc. \tag{32}$$

At this point we can use Eqs. (31) and (32) in Eqs. (11) and (14) to find that here

$$\varepsilon_p = \frac{16}{15\pi^{1/2}} \int_0^\infty e^{-c^2} \nu^{-1}(c)c^6 dc \tag{33}$$

and

$$\varepsilon_t = \frac{16}{15\pi^{1/2}} \int_0^\infty e^{-c^2} \nu^{-1}(c)(c^2 - 5/2)^2 c^4 dc. \tag{34}$$

To be clear, we note that the CLF model as used here is such that we are free to choose any form of the collision frequency we wish. In two previous works,^{18,19} some basic numerical results were reported for three specific forms of the collision frequency: $\nu(c) = 1$ which yields the BGK model, the Williams model $\nu(c) = c$ and the case of rigid spheres (within the context of the CLF model) where $\nu(c)$ is given by Eq. (6).

IV. THE CES MODEL

For background material relevant to the CES model, we refer to a recent paper by Barichello and Siewert¹ where an explicit development of the model was given. For this model, the exact form of the collision frequency, as given by Eq. (6), is used and the kernel is still in the form of Eq. (3), but now we have

$$k_0(c', c) = \nu(c')\nu(c)[\varpi_{01} + \varpi_{02}(c'^2 - 7/4)(c^2 - 7/4)], \tag{35a}$$

$$k_1(c', c) = \varpi_{11}c'\nu(c')c\nu(c) + \varpi_{12}\Delta_1(c')\Delta_1(c), \tag{35b}$$

$$k_2(c', c) = \varpi_2[c'^2 - \nu(c')c'^2b(c')][c^2 - \nu(c)c^2b(c)], \tag{35c}$$

and

$$k_n(c', c) = 0, \quad n > 2. \tag{35d}$$

Continuing, we note that ϖ_{01} , ϖ_{11} , and ϖ_{02} are still given by Eqs. (28a)–(28c) and (29). Also, quoting from another work,¹ we write

$$\Delta_1(c) = \nu(c)[a_*c - ca(c)] + c(c^2 - 5/2), \tag{36}$$

$$\varpi_{12} = [a_1 - a_2 - a_*a_3]^{-1} \tag{37a}$$

and

$$\varpi_2 = 1/\nu_*, \tag{37b}$$

where

$$a_* = a_3/\nu_4, \tag{38a}$$

$$a_1 = \int_0^\infty e^{-c^2} \nu(c)a^2(c)c^4 dc, \tag{38b}$$

$$a_2 = \int_0^\infty e^{-c^2} a(c)c^6 dc, \tag{38c}$$

$$a_3 = \int_0^\infty e^{-c^2} \nu(c)a(c)c^4 dc, \tag{38d}$$

and

$$\nu_* = \int_0^\infty e^{-c^2} b(c)[\nu(c)c^2b(c) - c^2]c^4 dc. \tag{38e}$$

We note that the terminology CES model derives from the fact that the synthetic kernel used here is based on the exact Chapman–Enskog functions $a(c)$ and $b(c)$. Now, to be complete, we make use of the MAPLE V software package and a numerical algorithm²⁰ for computing the Chapman–Enskog functions and list the following values for the parameters required here:

$$\varpi_{01} = 0.797\,884\,561\dots, \tag{39a}$$

$$\varpi_{02} = 0.425\,538\,432\dots, \tag{39b}$$

$$\varpi_{11} = 0.455\,934\,035\dots, \tag{39c}$$

$$\varpi_{12} = 0.586\,873\,122\dots, \tag{39d}$$

$$a_* = 0.221\,880\,745\dots, \tag{39e}$$

and

$$\varpi_2 = 2.164\,003\,46\dots \tag{39f}$$

For convenience, we also list here

$$\varepsilon_p = 0.449\,027\,806\dots \tag{40a}$$

and

$$\varepsilon_t = 0.679\,630\,049\dots \tag{40b}$$

Having defined the kinetic models we use in this work, we are ready to discuss our computations of the viscous-slip and thermal-slip coefficients.

V. THE VISCOUS-SLIP COEFFICIENT

In regard to the viscous-slip or Kramers' problem, we consider that the required function $h(\tau, \mathbf{c})$ denotes the perturbation from an absolute Maxwellian distribution, i.e.,

$$f(\tau, \mathbf{c}) = f_0(c)[1 + h(\tau, \mathbf{c})], \tag{41}$$

where

$$f_0(c) = n_0[m/(2\pi kT_0)]^{3/2} e^{-c^2}, \tag{42}$$

and so for this problem there is no driving term in Eq. (1). We, therefore, seek a solution (valid for $\tau > 0$) of

$$c\mu \frac{\partial}{\partial \tau} h(\tau, \mathbf{c}) = \varepsilon L\{h\}(\tau, \mathbf{c}), \tag{43}$$

where the collision operator is, in general, given by Eqs. (2) and (3) and the boundary condition at the wall is given by Eq. (18). In addition to the boundary condition at the wall, we must define a condition on $h(\tau, \mathbf{c})$ as τ tends to infinity. We make use of the condition that the bulk velocity

TABLE I. The viscous-slip coefficient ζ_p for the Maxwell boundary condition.

α	BGK	S model	CLF- w	CLF-rs	CES	LBE ^a
0.1	1.710 313(1)	1.711 289(1)	1.700 079(1)	1.701 536(1)	1.704 462(1)	1.7049(1)
0.2	8.224 902	8.233 445	8.128 966	8.142 636	8.169 615	8.1729
0.3	5.255 112	5.262 546	5.165 447	5.178 235	5.203 049	5.2059
0.4	3.762 619	3.769 046	3.679 096	3.691 017	3.713 778	3.7163
0.5	2.861 190	2.866 704	2.783 682	2.794 754	2.815 562	2.8178
0.6	2.255 410	2.260 100	2.183 795	2.194 033	2.212 984	2.2149
0.7	1.818 667	1.822 617	1.752 827	1.762 247	1.779 429	1.7810
0.8	1.487 654	1.490 942	1.427 475	1.436 091	1.451 586	1.4530
0.9	1.227 198	1.229 898	1.172 569	1.180 396	1.194 279	1.1955
1.0	1.016 191	1.018 372	9.670 050(-1)	9.740 570(-1)	9.864 009(-1)	9.8737(-1)

^aWakabayashi, Ohwada, and Golse (Ref. 26).

$$u(\tau) = \frac{1}{\pi^{3/2}} \int_0^\infty \int_{-1}^1 \int_0^{2\pi} e^{-c^2} h(\tau, \mathbf{c}) c^3 (1 - \mu^2)^{1/2} \times \cos \chi d\chi d\mu dc, \tag{44}$$

must diverge as τ tends to infinity, but at the same time

$$\lim_{\tau \rightarrow \infty} \frac{d}{d\tau} u(\tau) = k_p. \tag{45}$$

Here k_p is a normalizing constant. We now follow another work²¹ where this problem was solved for the CES model, make use of a subscript P and write our final solution for the bulk velocity for this (Kramers) problem as

$$u_p(\tau) = k_p [\tau + \zeta_p + u_p^*(\tau)], \tag{46}$$

where $k_p u_p^*(\tau)$ is the component of the bulk velocity that vanishes as τ tends to infinity. It follows that the constant ζ_p is the viscous-slip coefficient.

VI. THE THERMAL-SLIP COEFFICIENT

Considering now the thermal-slip problem, we note that for this problem the flow is caused by a constant temperature gradient in a direction parallel to the wall, and so it is convenient to linearize about a local Maxwellian rather than the absolute Maxwellian as was done in Eqs. (41) and (42). Here we follow Williams²² and express the distribution function as

$$f(\tau, \eta, \mathbf{c}) = f_0(c) [1 + (c^2 - 5/2)k_T \eta + h(\tau, \mathbf{c})], \tag{47}$$

where $f_0(c)$ is given by Eq. (42) and we have expressed the imposed temperature variation as

$$T(\eta) = T_0(1 + k_T \eta). \tag{48}$$

We continue to use T_0 as a convenient reference temperature, η is used to define (in terms of the mean-free path l) the direction of flow and k_T is the constant gradient (in dimensionless units) of the temperature. Now, as noted by Williams,²² the defining equation for $h(\tau, \mathbf{c})$ is in the inhomogeneous form given by Eq. (1), viz.

$$c(1 - \mu^2)^{1/2} \cos \chi (c^2 - 5/2)k_T + c\mu \frac{\partial}{\partial \tau} h(\tau, \mathbf{c}) = \varepsilon L\{h\}(\tau, \mathbf{c}). \tag{49}$$

And so we now seek a bounded (as τ tends to infinity) solution of Eq. (49) that also satisfies the boundary condition

listed as Eq. (18). Again, we omit the details of our mathematical algorithms for solving this problem and list only our final result written as

$$u_T(\tau) = k_T [\zeta_T + u_T^*(\tau)], \tag{50}$$

where $k_T u_T^*(\tau)$ is the component of the bulk velocity that vanishes as τ tends to infinity. Note that we now use the subscript T in order to tag this thermal-creep problem. We consider that the constant ζ_T is the thermal-slip coefficient.

VII. NUMERICAL RESULTS

In his fundamental work in the area of radiative transfer, Chandrasekhar²³ used a numerical quadrature scheme to represent the scattering integral in the equation of transfer. Chandrasekhar subsequently reduced his problem to a system of first-order ordinary differential equations that could be solved essentially analytically. This approach of using a quadrature scheme to evaluate the scattering integral was extended and used, with various numerical improvements, by Barichello and Siewert⁹ to solve a nongray problem in the field of radiative transfer. This approach, now referred to as the ADO method, was then used in the field of rarefied gas dynamics to solve various basic problems based on the BGK model.^{24,25} Here we have used the analytical discrete-ordinates (ADO) method as defined by Barichello and Siewert⁹ to solve Kramers' problem and the half-space thermal-creep problem for all of the models discussed in this work. However, since the solutions for the CES model have been recently reported and since the BGK and the CLF models can be obtained as special cases of the CES model, we omit the details of our ADO solution as implemented for these problems. As the purpose of this work is to evaluate the slip coefficients for the various models, we list in Tables I and II the viscous-slip and the thermal-slip coefficients for the different kinetic models considered and for various values of the accommodation coefficient α basic to the Maxwell boundary condition (a mixture of specular and diffuse reflection). We note that the results listed in Tables I and II under the labels CLF- w and CLF-rs correspond, respectively, to using in the CLF model $\nu(c) = c$ and $\nu(c)$ as given by Eq. (6). We have also included in Tables I and II reference results under the label of LBE. These LBE results²⁶ are based on a numerical solution of the linearized Boltzmann equation (for rigid-sphere collisions) and so are thought to be the best

TABLE II. The thermal-slip coefficient ζ_T for the Maxwell boundary condition.

α	BGK	S model	CLF- w	CLF-rs	CES	LBE ^a
0.1	2.641 783(-1)	2.660 636(-1)	2.777 778(-1)	2.684 888(-1)	2.671 726(-1)	
0.2	2.781 510(-1)	2.816 551(-1)	2.777 778(-1)	2.730 999(-1)	2.770 231(-1)	2.744(-1)
0.3	2.919 238(-1)	2.967 944(-1)	2.777 778(-1)	2.776 571(-1)	2.864 184(-1)	
0.4	3.055 019(-1)	3.115 005(-1)	2.777 778(-1)	2.821 617(-1)	2.953 902(-1)	2.911(-1)
0.5	3.188 906(-1)	3.257 911(-1)	2.777 778(-1)	2.866 147(-1)	3.039 673(-1)	
0.6	3.320 949(-1)	3.396 832(-1)	2.777 778(-1)	2.910 174(-1)	3.121 761(-1)	3.069(-1)
0.7	3.451 195(-1)	3.531 928(-1)	2.777 778(-1)	2.953 707(-1)	3.200 405(-1)	
0.8	3.579 692(-1)	3.663 351(-1)	2.777 778(-1)	2.996 757(-1)	3.275 826(-1)	3.219(-1)
0.9	3.706 483(-1)	3.791 246(-1)	2.777 778(-1)	3.039 336(-1)	3.348 226(-1)	
1.0	3.831 612(-1)	3.915 748(-1)	2.777 778(-1)	3.081 452(-1)	3.417 790(-1)	3.362(-1)

^aWakabayashi, Ohwada and Golse (Ref. 26).

available. To be explicit about the (Maxwell) boundary condition we have used, we make use of the generalized function $\delta(x)$ and write, for this case, the R function used in Eq. (18) as

$$R(\mathbf{c}' : \mathbf{c}) = 2(\alpha/\pi)c' \mu' + (1 - \alpha)e^{c^2}(1/c^2)\delta(\mu' - \mu) \times \delta(c' - c)\delta(\chi' - \chi), \tag{51}$$

where α is the accommodation coefficient.

It should be noted that we have used a mean-free path based on viscosity ($\varepsilon = \varepsilon_p$) for the viscous-slip problem. On the other hand, we have used a mean-free based on heat flow ($\varepsilon = \varepsilon_t$) for the thermal-creep problem. It is clear that while the values of ε_p and ε_t are very different for the various models, the final slip-coefficients are similar when a mean-free path is defined in a way consistent with the model. In other words, we find reasonable comparisons between the various models when values of ε_p and ε_t deduced from the same models are used.

In two previous works^{6,7} the S model was used with the Cercignani–Lampis boundary condition⁸ to analyze Poiseuille flow and thermal-creep flow in a plane channel. Because of our current interest in the CL boundary condition we have added Table III to this work in order to show the effects of the normal (α_n) and the tangential (α_t) accommo-

dation coefficients on the two slip coefficients considered here. Since the BGK results can easily be obtained from the S model, we have also included in Table III slip coefficients for the BGK model. Finally we complete this section by listing the explicit expression for the R function we have used in our numerical with the CL boundary condition, viz.

$$R(\mathbf{c}' : \mathbf{c}) = e^{c^2}T(c'_x, c_x)S(c'_y, c_y)T(c'_z, c_z), \tag{52}$$

where

$$T(x, y) = [\pi\alpha_t(2 - \alpha_t)]^{-1/2}e^{-[y - (1 - \alpha_t)x]^2/[\alpha_t(2 - \alpha_t)]} \tag{53a}$$

and

$$S(x, y) = \frac{2|x|}{\alpha_n}e^{-[y^2 + (1 - \alpha_n)x^2]/\alpha_n}I_0[2(1 - \alpha_n)^{1/2}|xy|/\alpha_n]. \tag{53b}$$

Here $I_0(z)$ is used to denote a modified Bessel function, i.e.,

$$I_0(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{z \cos \phi} d\phi. \tag{54}$$

Note that to be consistent with our previous computations^{6,7} with the CL boundary condition, we have listed the R func-

TABLE III. The slip coefficients for the Cercignani–Lampis boundary condition.

α_n	α_t	ζ_p		ζ_T	
		BGK	S model	BGK	S model
0.25	0.25	6.412 098	6.417 960	3.103 473(-1)	3.179 911(-1)
0.25	0.50	2.840 346	2.844 690	3.351 889(-1)	3.448 837(-1)
0.25	0.75	1.632 538	1.635 681	3.594 527(-1)	3.689 663(-1)
0.25	1.00	1.016 191	1.018 372	3.831 612(-1)	3.915 748(-1)
0.50	0.25	6.383 796	6.388 167	3.350 826(-1)	3.422 777(-1)
0.50	0.50	2.821 884	2.825 386	3.513 446(-1)	3.603 901(-1)
0.50	0.75	1.623 502	1.626 289	3.673 690(-1)	3.764 752(-1)
0.50	1.00	1.016 191	1.018 372	3.831 612(-1)	3.915 748(-1)
0.75	0.25	6.357 833	6.361 016	3.593 360(-1)	3.659 777(-1)
0.75	0.50	2.804 774	2.807 569	3.673 334(-1)	3.757 197(-1)
0.75	0.75	1.615 045	1.617 514	3.752 749(-1)	3.839 875(-1)
0.75	1.00	1.016 191	1.018 372	3.831 612(-1)	3.915 748(-1)
1.00	0.25	6.333 553	6.335 792	3.831 612(-1)	3.891 445(-1)
1.00	0.50	2.788 645	2.790 843	3.831 612(-1)	3.908 681(-1)
1.00	0.75	1.607 009	1.609 192	3.831 612(-1)	3.914 871(-1)
1.00	1.00	1.016 191	1.018 372	3.831 612(-1)	3.915 748(-1)

tion here with the velocity vector expressed in the rectangular coordinates

$$c_y = c\mu, \quad (55a)$$

$$c_x = c(1 - \mu^2)^{1/2} \cos \chi, \quad (55b)$$

and

$$c_z = c(1 - \mu^2)^{1/2} \sin \chi. \quad (55c)$$

VIII. CONCLUDING REMARKS

In this work we have made use of various kinetic models to solve two basic problems in the area of rarefied gas dynamics, viz. Kramers' problem and the problem of thermal-creep flow in a semi-infinite half space. We have also shown how (for rigid-sphere collisions) the solutions of the Chapman–Enskog (CE) equations basic to viscosity and heat conduction and solutions of model versions of the CE equations provide convenient definitions of mean-free paths l_p and l_t in order to compare well the various kinetic models. We have reported extensive numerical results for the case of a Maxwell boundary condition at the wall since there exists reference quality results available for this boundary condition, but we have also extended our previous work^{6,7} to yield the viscous-slip and the thermal-creep slip coefficients as defined by the Cercignani–Lampis boundary condition. Finally, we note two somewhat surprising results we have found: (i) The CLF- w kinetic model yields, for the case of the Maxwell boundary condition, the same thermal-slip coefficient for all values of the accommodation coefficient α , while for the linearized Boltzmann equation and all the other considered model equations the thermal-slip coefficient depends significantly on α , and (ii) the BGK model yields, for the case of the CL boundary condition, the same thermal-slip coefficient for all values of the tangential momentum accommodation α_t when the normal energy accommodation α_n is equal to unity. We note that Loyalka and Cipolla²⁷ have considered a boundary condition that, like the Cercignani–Lampis model, contains two accommodation coefficients. However, the boundary condition defined by Eq. (46) of the mentioned work²⁷ is not the Cercignani–Lampis condition used in this work. Interestingly, Loyalka and Cipolla,²⁷ in regard to kinetic models and to the linearized Boltzmann equation, noted that the thermal-slip coefficient, when based on their Eq. (46), did not depend on the two accommodation coefficients (α_p and α_E) used in that work. Although similar results for some special cases of the Cercignani–Lampis boundary condition may be true, such results, while valid for the BGK model, have not been found in this work for the S model.

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