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Poiseuille, thermal creep and Couette flow: results based on the CES model of the linearized Boltzmann equation

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Abstract

A synthetic-kernel model (CES model) of the linearized Boltzmann equation is used along with an analytical discrete-ordinates method (ADO) to solve three fundamental problems concerning flow of a rarefied gas in a plane channel. More specifically, the problems of Couette flow, Poiseuille flow and thermal-creep flow are solved in terms of the CES model equation for an arbitrary mixture of specular and diffuse reflection at the walls confining the flow, and numerical results for the basic quantities of interest are reported. The comparisons made with results derived from solutions based on computationally intensive methods applied to the linearized Boltzmann equation are used to conclude that the CES model can be employed with confidence to improve the accuracy of results available from simpler approximations such as the BGK model or the S model.

Keywords: Boltzmann equation; Rarefied gas dynamics; Couette flow; Poiseuille flow; Thermal-creep flow

1. Introduction

As one of several steps in evaluating the effectiveness of a new kinetic model of the linearized Boltzmann equation or a new computational method it is considered important to study three basic problems defined by flow in a finite plane-parallel channel. These three problems:

(i) Couette flow,
(ii) Poiseuille flow, and
(iii) thermal-creep flow,

allow us to study the flow of a rarefied gas that is driven by

(i) movement parallel to the channel by one or both of the two confining walls,
(ii) a pressure gradient, or
(iii) a temperature gradient in a direction parallel to the boundaries that enclose the gas.

A new book by Cercignani [1] includes complete discussions of these basic flow problems and provides a guide to many works that have contributed to our understanding of these problems. In addition to Ref. [1] we have found a review article by Sharipov and Seleznev [2] to be of fundamental importance to our work here. In addition to providing a comprehensive list of

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important references relative to basic problems in rarefied gas dynamics, Sharipov and Seleznev [2] have provided many good comparisons of numerical results derived from the linearized Boltzmann equation and various kinetic models and based on a diverse collection of computational algorithms. Having seen [3,4] that the CES model of the linearized Boltzmann equation basic to rigid-sphere interactions yielded good results for half-space applications, we use the model here to solve three basic flow problems in a finite channel.

2. The CES model of the linearized Boltzmann equation

In two recent works [3,4] that concern half-space problems in rarefied gas dynamics, the CES model [5] was shown to yield results for the viscous-slip coefficient and the thermal-slip coefficient that can be considered significant improvements to the results obtained from the often-used BGK model and the S model of the linearized Boltzmann equation for rigid-sphere interactions. In this work, we continue our investigation of the CES model by evaluating the merits of the solutions to three basic problems relevant to flow in a plane channel, viz. Couette flow, Poiseuille flow and thermal-creep flow [1,2,6].

To start this work, we follow Pekeris and Alterman [7], who quote Boltzmann [8], Hilbert [9] and Chapman and Cowling [10], and consider the linearized Boltzmann equation written in terms of $h(\tau, \boldsymbol{c})$, a perturbation to the velocity distribution function, for rigid-sphere collisions as

$$S(\boldsymbol{c}) + c\mu \frac{\partial}{\partial \tau} h(\tau, \boldsymbol{c}) = \varepsilon L\{h\}(\tau, \boldsymbol{c}) \tag{1}$$

where

$$L\{h\}(\tau, \boldsymbol{c}) = -\nu(c) h(\tau, \boldsymbol{c}) + \int_0^\infty \int_{-1}^1 \int_0^{2\pi} \mathrm{e}^{-c'^2} h(\tau, \boldsymbol{c}') K(\boldsymbol{c}' : \boldsymbol{c}) c'^2 \, \mathrm{d}\chi' \, \mathrm{d}\mu' \, \mathrm{d}c'. \tag{2}$$

Here the scattering kernel is

$$K(\boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (2n+1)(2 - \delta_{0,m}) P_n^m(\mu') P_n^m(\mu) k_n(c', c) \cos m(\chi' - \chi) \tag{3}$$

where the *normalized* Legendre functions are given (in terms of the Legendre polynomials) by

$$P_n^m(\mu) = \left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} \left(1 - \mu^2\right)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_n(\mu), \quad n \geqslant m. \tag{4}$$

In addition,

$$\varepsilon = \sigma_0^2 n_0 \pi^{1/2} l \tag{5}$$

where l is (at this point) an unspecified mean-free path, n_0 is the density of the gas particles and σ_0 is the scattering diameter of the gas molecules. In this work, the cross-channel spatial variable τ is measured in units of the mean-free path l and $c(2kT_0/m)^{1/2}$ is the magnitude of the particle velocity. Here k is the Boltzmann constant, m is the mass of a gas particle and T_0 is a convenient reference temperature. It should be noted that we have included in Eq. (1) an inhomogeneous driving term $S(\boldsymbol{c})$ that we will specify, along with an appropriate definition of the perturbation $h(\tau, \boldsymbol{c})$, for the three types of flow (Couette, Poiseuille and thermal-creep) we consider in this work. Continuing, we note that the functions $k_n(c', c)$ in Eq. (3) are the components in an expansion of the scattering law (for rigid-sphere collisions) used by Pekeris and Alterman [7] and

$$\nu(c) = \frac{2c^2 + 1}{c} \int_0^c \mathrm{e}^{-x^2} \, \mathrm{d}x + \mathrm{e}^{-c^2} \tag{6}$$

is the collision frequency. While Pekeris and Alterman [7] give explicit expressions for the components $k_n(c', c)$ only for $n = 1$ and 2, explicit results are given (with at least one error) in a later work [11] for larger values of n. Note that we use spherical coordinates $(c, \arccos \mu, \chi)$ to define the (dimensionless) velocity vector $\boldsymbol{c}$.

While the scattering kernel as defined by Pekeris and Alterman [7] for rigid-sphere collisions is given by Eq. (3) in which the component functions $k_n(c', c)$ are required of all n, our work here with the CES model equation is based on using $n = 0$, 1 and 2, along with approximations to the true component functions. To be clear, we list these CES component functions as [5]

$$k_0(c',c) = \varpi_{01}\Delta_{01}(c')\Delta_{01}(c) + \varpi_{02}\Delta_{02}(c')\Delta_{02}(c), \tag{7a}$$

$$k_1(c',c) = \varpi_{11}\Delta_{11}(c')\Delta_{11}(c) + \varpi_{12}\Delta_{12}(c')\Delta_{12}(c), \tag{7b}$$

$$k_2(c',c) = \varpi_2\Delta_2(c')\Delta_2(c) \tag{7c}$$

and

$$k_n(c',c) = 0, \quad n > 2, \tag{7d}$$

where

$$\Delta_{01}(c) = \nu(c), \qquad \Delta_{02}(c) = \nu(c)\big(c^2 - 7/4\big), \qquad \Delta_{11}(c) = c\nu(c), \tag{8a,b,c}$$

$$\Delta_{12}(c) = \nu(c)A(c) - c\big(c^2 - 5/2\big) - a_* c\nu(c) \quad \text{and} \quad \Delta_2(c) = \nu(c)B(c) - c^2. \tag{8d,e}$$

Here $B(c)$ is defined by the Chapman–Enskog equation for viscosity, viz.:

$$\nu(c)B(c) - \int_0^\infty \mathrm{e}^{-c'^2} B(c')k_2(c',c)c'^2\,\mathrm{d}c' = c^2, \tag{9}$$

and $A(c)$ is determined by the Chapman–Enskog equation for heat conduction

$$\nu(c)A(c) - \int_0^\infty \mathrm{e}^{-c'^2} A(c')k_1(c',c)c'^2\,\mathrm{d}c' = c\big(c^2 - 5/2\big) \tag{10}$$

and the normalizing condition

$$\int_0^\infty \mathrm{e}^{-c^2} A(c)c^3\,\mathrm{d}c = 0. \tag{11}$$

It is important to note that the component functions $k_1(c',c)$ and $k_2(c',c)$ used in Eqs. (9) and (10) are the exact Pekeris–Alterman functions and not the synthetic approximations of these functions as given by Eqs. (7). Continuing to define Eqs. (7), we list the numerical values

$$\varpi_{01} = 0.797884561\ldots, \qquad \varpi_{02} = 0.425538432\ldots, \qquad \varpi_{11} = 0.455934035\ldots, \tag{12a,b,c}$$

$$\varpi_{12} = 0.586873122\ldots, \qquad a_* = 0.221880745\ldots \quad \text{and} \quad \varpi_2 = 2.16400346\ldots, \tag{12d,e,f}$$

that were reported in Ref. [4]. In regard mean-free paths, we note, as discussed (for example) in Refs. [5,7,12], that if we wish to use a mean-free path based on the viscosity, i.e.,

$$l = l_\mathrm{p} = (\mu_*/p_0)(2kT_0/m)^{1/2} \tag{13}$$

where μ_* is the viscosity and $p_0 = n_0 kT_0$ is the pressure, then we should use in Eq. (1)

$$\varepsilon = \varepsilon_\mathrm{p} = \frac{16}{15}\pi^{-1/2}\int_0^\infty \mathrm{e}^{-c^2} B(c)c^4\,\mathrm{d}c. \tag{14}$$

On the other hand, if we wish to use a mean-free path based on heat conduction, then we should use [5,7,13]

$$l = l_\mathrm{t} = (4\lambda_*/5n_0k)(m/2kT_0)^{1/2} \tag{15}$$

where λ_* is the heat-conduction coefficient and where

$$\varepsilon = \varepsilon_\mathrm{t} = \frac{16}{15}\pi^{-1/2}\int_0^\infty \mathrm{e}^{-c^2} A(c)c^5\,\mathrm{d}c. \tag{16}$$

For convenience we list

$$\varepsilon_\mathrm{p} = 0.449027806\ldots \quad \text{and} \quad \varepsilon_\mathrm{t} = 0.679630049\ldots. \tag{17a,b}$$

Continuing to define the problems, we consider that the walls confining the flow are located at $\tau = \pm a$ so that Eq. (1) which is valid for $\tau \in (-a, a)$ is supplemented with the boundary condition

$$h(-a, c, \mu, \chi) - (1-\alpha)h(-a, c, -\mu, \chi) - D = F(\boldsymbol{c}), \tag{18}$$

for $\mu \in (0,1)$, all χ and all c, and a symmetry condition that relates the perturbation $h(\tau, c, \mu, \chi)$ to $h(-\tau, c, -\mu, \chi)$. Here $F(\boldsymbol{c})$ is taken to be specified and

$$D = \frac{2\alpha}{\pi}\int_0^\infty\int_0^1\int_0^{2\pi} \mathrm{e}^{-c^2} h(-a, c, -\mu, \chi)c^3\mu\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c. \tag{19}$$

Since $c\mu$ is the component of the (normalized) velocity in the positive τ direction, we can let

$$c_\eta = c\big(1-\mu^2\big)^{1/2}\cos\chi$$

denote the velocity component in the direction (parallel to the walls) of the flow. It follows that we can express the bulk velocity and heat-flow profiles we intend to compute as

$$u(\tau) = \frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2} h(\tau, \boldsymbol{c})c^3\big(1-\mu^2\big)^{1/2}\cos\chi\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c \tag{20}$$

and

$$q(\tau) = \frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2} h(\tau, \boldsymbol{c})\big(c^2-5/2\big)c^3\big(1-\mu^2\big)^{1/2}\cos\chi\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c. \tag{21}$$

For the problem of Couette flow we also intend to compute a component of the pressure tensor which we write as

$$P_{xy} = \frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2} h(\tau, \boldsymbol{c})c^4\mu\big(1-\mu^2\big)^{1/2}\cos\chi\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c \tag{22}$$

where x and y are the spatial variables (measured in cm) that correspond respectively to η and τ.

It is clear from Eqs. (20)–(22) that the information we seek is expressed in terms of certain moments of $h(\tau, \boldsymbol{c})$, and so we can make a convenient simplification in our formulation. Considering the form of the scattering kernel given by Eq. (3), we introduce

$$\psi(\tau, c, \mu) = \frac{1}{\pi}\big(1-\mu^2\big)^{-1/2}\int_0^{2\pi} h(\tau, \boldsymbol{c})\cos\chi\,\mathrm{d}\chi \tag{23}$$

and rewrite Eqs. (20)–(22) as

$$u(\tau) = \frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\psi(\tau, c, \mu)c^3\big(1-\mu^2\big)\,\mathrm{d}\mu\,\mathrm{d}c, \tag{24}$$

$$q(\tau) = \frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\psi(\tau, c, \mu)\big(c^2-5/2\big)c^3\big(1-\mu^2\big)\,\mathrm{d}\mu\,\mathrm{d}c \tag{25}$$

and

$$P_{xy} = \frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\psi(\tau, c, \mu)c^4\mu\big(1-\mu^2\big)\,\mathrm{d}\mu\,\mathrm{d}c. \tag{26}$$

We can multiply Eq. (1) by $\cos\chi$ and integrate to find, after noting Eq. (3), that $\psi(\tau, c, \mu)$ must satisfy the balance equation

$$S^*(c, \mu) + c\mu\frac{\partial}{\partial\tau}\psi(\tau, c, \mu) = \varepsilon L^*\{\psi\}(\tau, c, \mu) \tag{27}$$

where

$$S^*(c,\mu)=\frac{1}{\pi}\big(1-\mu^2\big)^{-1/2}\int_0^{2\pi} S(\boldsymbol{c})\cos\chi\,\mathrm{d}\chi \tag{28}$$

and

$$L^*\{\psi\}(\tau,c,\mu)=-\nu(c)\psi(\tau,c,\mu)+\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c'^2}\psi(\tau,c'\mu')k(c',\mu':c,\mu)c'^2\,\mathrm{d}\mu'\,\mathrm{d}c'. \tag{29}$$

Here

$$k(c',\mu':c,\mu)=\big(1-\mu'^2\big)\sum_{n=1}^{2}\Pi_n(\mu')\Pi_n(\mu)k_n(c',c) \tag{30}$$

with

$$\Pi_n(\mu)=\left[\frac{2n+1}{2n(n+1)}\right]^{1/2}\frac{\mathrm{d}}{\mathrm{d}\mu}P_n(\mu),\quad n\geqslant 1. \tag{31}$$

Note that

$$\int_{-1}^1\big(1-\mu^2\big)\Pi_n(\mu)\Pi_{n'}(\mu)\,\mathrm{d}\mu=\delta_{n,n'}. \tag{32}$$

In addition to a symmetry relation (to be defined) between $\psi(\tau,c,\mu)$ and $\psi(-\tau,c,-\mu)$, we find from Eq. (18) that $\psi(\tau,c,\mu)$ must satisfy the boundary condition

$$\psi(-a,c,\mu)-(1-\alpha)\psi(-a,c,-\mu)=F^*(c,\mu) \tag{33}$$

for $\mu\in(0,1]$ and all c. Here

$$F^*(c,\mu)=\frac{1}{\pi}\big(1-\mu^2\big)^{-1/2}\int_0^{2\pi} F(\boldsymbol{c})\cos\chi\,\mathrm{d}\chi. \tag{34}$$

Having given a general introduction to the class of flow problems we intend to solve here, we are ready to consider the specific problems of Couette flow, Poiseuille flow and thermal-creep flow.

3. Couette flow

For the Couette-flow problem we follow Ref. [6] and consider that $h(\tau,\boldsymbol{c})$ represents the perturbation from an absolute Maxwellian, and so we express the distribution function as

$$f(\tau,\boldsymbol{c})=f_0(c)\big[1+h(\tau,\boldsymbol{c})\big] \tag{35}$$

where

$$f_0(c)=n_0\big[m/(2\pi kT_0)\big]^{3/2}\mathrm{e}^{-c^2}. \tag{36}$$

For this problem there is no driving term in Eq. (1), and so

$$S^*(c,\mu)=0. \tag{37}$$

In addition, we consider that the two plates (walls) are given velocities $\pm u_{\mathrm{p}}$, and so the known term in the boundary condition listed as Eq. (33) takes the form [6]

$$F^*(c,\mu)=2\alpha c u_{\mathrm{p}}. \tag{38}$$

As a result of the wall velocities, we can make use of the (anti) symmetry condition

$$\psi(-\tau,c,-\mu)=-\psi(\tau,c,\mu) \tag{39}$$

for all τ, c and μ. For the problem of Couette flow we intend to compute, in addition to the quantities listed by Eqs. (20)–(22), the half-channel mass and heat-flow rates

$$U = \frac{1}{2a^2}\int_0^a u(\tau)\,\mathrm{d}\tau \tag{40}$$

and

$$Q = \frac{1}{2a^2}\int_0^a q(\tau)\,\mathrm{d}\tau. \tag{41}$$

Later in this work we discuss our use of the ADO method to deduce the numerical results we report for the problem of Couette flow.

4. Poiseuille and thermal-creep flow

The problems of Poiseuille flow and thermal-creep flow have much in common, and so we find it convenient to consider the two problems formulated together. Here the flow is caused by a constant pressure gradient (Poiseuille flow) and a constant temperature gradient (thermal-creep flow) in a direction parallel to the walls, and so it is helpful to linearize about a local Maxwellian rather than the absolute Maxwellian as was done in Eqs. (35) and (36). We follow Williams [6] and express the distribution function as

$$f(\tau,\eta,\boldsymbol{c}) = f_0(c)\bigl\{1 + \bigl[\bigl(c^2 - 3/2\bigr)K_\eta + R_\eta\bigr]\eta + h(\tau,\boldsymbol{c})\bigr\} \tag{42}$$

where $f_0(c)$ is given by Eq. (36) and we have expressed the imposed temperature and density variations as

$$T(\eta) = T_0(1 + K_\eta \eta) \tag{43a}$$

and

$$n(\eta) = n_0(1 + R_\eta \eta). \tag{43b}$$

We continue to use T_0 and n_0 as convenient reference values of the temperature and density, η is used to define (in terms of the mean-free path l) the direction of flow and K_η and R_η are the constant gradients (in dimensionless units) of the temperature and density. Now, as noted by Williams [6], the problem of Poiseuille flow has $K_\eta = 0$ and R_η arbitrary, while thermal-creep flow is defined by $K_\eta = -R_\eta$, with R_η arbitrary. To distinguish the two problems we use subscript labels P and T, and so we consider that the defining equation for $h(\tau,\boldsymbol{c})$ is the inhomogeneous form given by Eq. (1) with

$$S(\boldsymbol{c}) = c\bigl(1 - \mu^2\bigr)^{1/2}\cos\chi\bigl[\bigl(c^2 - 5/2\bigr)k_{\mathrm{T}} + k_{\mathrm{P}}\bigr], \tag{44}$$

where $k_{\mathrm{T}} = K_\eta$ and $k_{\mathrm{P}} = R_\eta + K_\eta$. In this work Poiseuille flow is defined by $k_{\mathrm{P}} = 1$ and $k_{\mathrm{T}} = 0$, while $k_{\mathrm{T}} = 1$ and $k_{\mathrm{P}} = 0$ defines the case of thermal-creep flow. Making use again of the definition introduced in Eq. (23), we find that here we must solve

$$S^*(c,\mu) + c\mu\frac{\partial}{\partial\tau}\psi(\tau,c,\mu) = \varepsilon L^*\{\psi\}(\tau,c,\mu) \tag{45}$$

where the operator L^* is defined by Eq. (29) and where

$$S^*(c,\mu) = c\bigl[\bigl(c^2 - 5/2\bigr)k_{\mathrm{T}} + k_{\mathrm{P}}\bigr]. \tag{46}$$

For the Poiseuille and thermal-creep problems the wall velocity u_{p} is zero, and so we seek a solution of Eq. (45) that satisfies the symmetry condition

$$\psi(-\tau,c,-\mu) = \psi(\tau,c,\mu) \tag{47}$$

and the boundary condition

$$\psi(-a,c,\mu) - (1-\alpha)\psi(-a,c,-\mu) = 0 \tag{48}$$

for $\mu \in (0,1]$ and all c. For the two flow problems defined in this section, we compute the velocity and heat-flow profiles

$$u(\tau) = \frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\psi(\tau,c,\mu)c^3\bigl(1-\mu^2\bigr)\,\mathrm{d}\mu\,\mathrm{d}c, \tag{49}$$

and

$$q(\tau)=\frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\psi(\tau,c,\mu)\bigl(c^2-5/2\bigr)c^3\bigl(1-\mu^2\bigr)\,\mathrm{d}\mu\,\mathrm{d}c. \tag{50}$$

We also seek the full-channel mass and heat-flow rates

$$U=\frac{1}{2a^2}\int_{-a}^{a} u(\tau)\,\mathrm{d}\tau \tag{51}$$

and

$$Q=\frac{1}{2a^2}\int_{-a}^{a} q(\tau)\,\mathrm{d}\tau. \tag{52}$$

Now, since we wish to base our ADO solution on the homogeneous form of Eq. (45), we write

$$\psi(\tau,c,\mu)=\psi_*(\tau,c,\mu)+\psi_{\mathrm{ps}}(\tau,c,\mu) \tag{53}$$

where $\psi_{\mathrm{ps}}(\tau,c,\mu)$ is a particular solution (that has the correct symmetry) of Eq. (45) and $\psi_*(\tau,c,\mu)$ is a solution of the homogeneous version of Eq. (45) that has the symmetry property

$$\psi_*(-\tau,c,-\mu)=\psi_*(\tau,c,\mu) \tag{54}$$

and that satisfies the boundary condition

$$\psi_*(-a,c,\mu)-(1-\alpha)\psi_*(-a,c,-\mu)=R(c,\mu) \tag{55}$$

for $\mu\in(0,1]$ and all c. Considering that we have found the required particular solution [5], we can write the known term in Eq. (55) as

$$R(c,\mu)=(1-\alpha)\psi_{\mathrm{ps}}(-a,c,-\mu)-\psi_{\mathrm{ps}}(-a,c,\mu). \tag{56}$$

Before proceeding with our solutions to the Couette, Poiseuille and thermal-creep problems, we discuss the particular solutions we require for Poiseuille and thermal-creep flow.

5. Particular solutions

Making use of results from the work of Loyalka and Hickey [12], who considered the Poiseuille-flow problem in terms of the linearized Boltzmann equation (LBE) for rigid-sphere interactions, Barichello and Siewert [5] expressed the particular solutions (relevant to the LBE) as

$$\psi_{\mathrm{ps}}(\tau,c,\mu)=\bigl\{c(\varepsilon\tau)^2-2B(c)\varepsilon\tau\mu+D(c)/5+E(c)\bigl(5\mu^2-1\bigr)/5\bigr\}/(\varepsilon\varepsilon_{\mathrm{p}}) \tag{57a}$$

for Poiseuille flow and

$$\psi_{\mathrm{ps}}(\tau,c,\mu)=-A(c)/\varepsilon \tag{57b}$$

for thermal-creep flow. Here $A(c)$ and $B(c)$ are the solutions of the Chapman–Enskog equations for heat conduction and viscosity and $D(c)$ and $E(c)$ are solutions of the so-called [12,5] Burnett equations. More specifically $D(c)$ is a solution of

$$\nu(c)D(c)-\int_0^\infty \mathrm{e}^{-c'^2}D(c')k_1(c',c)c'^2\,\mathrm{d}c'=2cB(c)-5c\varepsilon_{\mathrm{p}} \tag{58a}$$

subject to the normalization condition

$$\int_0^\infty \mathrm{e}^{-c^2}D(c)c^3\,\mathrm{d}c=0, \tag{58b}$$

and $E(c)$ is a solution of

$$\nu(c)E(c) - \int_0^\infty e^{-c'^2} E(c')k_3(c',c)c'^2\,dc' = 2cB(c). \tag{59}$$

Here, as with Eqs. (9) and (10), the component functions $k_1(c',c)$ and $k_3(c',c)$ are the exact functions from the Pekeris–Alterman theory [7]. Since the CES model has component kernel functions that yield the exact Chapman–Enskog functions $A(c)$ and $B(c)$ the particular solution listed as Eq. (57b) can be used as written. However, for the case of Poiseuille flow, the particular solution listed as Eq. (57a), while it is correct for the LBE, is not a particular solution for the CES model. We can make the required modifications to Eq. (57a) simply by replacing the functions $D(c)$ and $E(c)$ with the approximations $D_0(c)$ and $E_0(c)$ defined by using the CES model versions of the component kernel functions $k_1(c'c)$ and $k_3(c'c,)$, instead of the exact Pekeris–Alterman functions, in Eqs. (58a) and (59). Since the CES model has $k_3(c',c) = 0$, we see at once that

$$E_0(c) = 2c\nu^{-1}(c)B(c). \tag{60}$$

Using Eq. (7b) in Eq. (58a), we can solve Eqs. (58) to find the CES solution of the first Burnett equation. We write this result as

$$D_0(c) = \nu^{-1}(c)\big[2cB(c) - 5c\varepsilon_p + \varpi_{11}c_{11}\Delta_{11}(c) + \varpi_{12}c_{12}\Delta_{12}(c)\big] \tag{61}$$

where

$$c_{11} = -\frac{1}{N_{11}}\int_0^\infty e^{-c^2}\nu^{-1}(c)\big[2cB(c) - 5c\varepsilon_p + \varpi_{12}c_{12}\Delta_{12}(c)\big]c^3\,dc \tag{62a}$$

and

$$c_{12} = \frac{1}{N_{12}}\int_0^\infty e^{-c^2}\nu^{-1}(c)\Delta_{12}(c)\big[2cB(c) - 5c\varepsilon_p\big]c^2\,dc \tag{62b}$$

with

$$N_{11} = (3/8)\varpi_{11}\pi^{1/2} \tag{63a}$$

and

$$N_{12} = 1 - \varpi_{12}\int_0^\infty e^{-c^2}\nu^{-1}(c)\Delta_{12}^2(c)c^2\,dc. \tag{63b}$$

To be clear, we note that Eq. (57b) is the particular solution we use, with the CES model, for the thermal-creep problem, but for the CES model and Poiseuille flow we use

$$\psi_{ps}(\tau,c,\mu) = \big\{c(\varepsilon\tau)^2 - c(\varepsilon a)^2 - 2B(c)\varepsilon\tau\mu + D_0(c)/5 + E_0(c)\big(5\mu^2 - 1\big)/5\big\}/(\varepsilon\varepsilon_p) \tag{64}$$

instead of Eq. (57a). Note that in addition to using, in Eq. (64), $D_0(c)$ and $E_0(c)$ in place of $D(c)$ and $E(c)$, we have also added a solution (a constant multiple of c) of the homogeneous version of the balance equation. It is clear that the particular solutions given by Eqs. (57b) and (64) are such that

$$\psi_{ps}(-\tau,c,-\mu) = \psi_{ps}(\tau,c,\mu). \tag{65}$$

Having established the particular solutions we require, we proceed to reformulate the considered problems so that we can take advantage of previous work [3] with the ADO method of solution.

6. A reformulation of the problems

Since the particular solutions we require are now available, we turn our attention to the homogeneous balance equation

$$c\mu\frac{\partial}{\partial\tau}\psi(\tau,c,\mu) = \varepsilon L^*\{\psi\}(\tau,c,\mu) \tag{66}$$

where the operator L^* is defined by Eq. (29). To avoid working with three independent variables, as used in Eq. (66), we follow Busbridge [14] and, more explicitly, Refs. [3] and [15] and introduce the new variable

$$\xi = c\mu/\nu(c) \tag{67}$$

and write

$$\psi_*\big[\tau, c, \xi\nu(c)/c\big] = \nu^{-1}(c)\sum_{\alpha=1}^{4} T_\alpha(c,\xi)g_\alpha(\tau,\xi) \tag{68}$$

where

$$T_1(c,\xi) = \Pi_1\big[\xi\nu(c)/c\big]\Delta_{11}(c), \tag{69a}$$

$$T_2(c,\xi) = \Pi_1\big[\xi\nu(c)/c\big]\Delta_{12}(c), \tag{69b}$$

$$T_3(c,\xi) = \Pi_2\big[\xi\nu(c)/c\big]\Delta_2(c) \tag{69c}$$

and $T_4(c,\xi)$ is to be determined from the boundary condition relevant to each of the three considered problems. We can now substitute Eq. (68) into Eq. (66) to find

$$\xi\frac{\partial}{\partial\tau}\boldsymbol{G}(\tau,\xi) + \varepsilon\boldsymbol{G}(\tau,\xi) = \varepsilon\int_{-\gamma}^{\gamma}\boldsymbol{\Psi}(\xi')\boldsymbol{G}(\tau,\xi')\,\mathrm{d}\xi' \tag{70}$$

where the four components of $\boldsymbol{G}(\tau,\xi)$ are $g_\alpha(\tau,\xi)$, $\alpha = 1,2,3,4$. In addition, $\gamma = \pi^{-1/2}$ and the components of $\boldsymbol{\Psi}(\xi)$ are given by

$$\psi_{i,j} = \beta_i\int_{m(\xi)}^{\infty}\mathrm{e}^{-c^2}\big[c^2 - \xi^2\nu^2(c)\big](1/c)T_i(c,\xi)T_j(c,\xi)\,\mathrm{d}c, \tag{71}$$

where

$$\beta_1 = \varpi_{11},\qquad \beta_2 = \varpi_{12},\qquad \beta_3 = \varpi_2,\qquad \beta_4 = 0. \tag{72a,b,c,d}$$

Here the integration interval is defined as it was in Ref. [15], viz. we let

$$f(c) = \frac{c}{\nu(c)} \tag{73}$$

and note that $f'(c) \geqslant 0$, for $c \geqslant 0$, and so the inverse function

$$m(\xi) = f^{-1}(|\xi|),\quad \xi\in[-\gamma,\gamma], \tag{74}$$

exists. We therefore can, in general, write

$$P(\xi) = \int_{M_\xi} p(c)\,\mathrm{d}c, \tag{75a}$$

where $c\in M_\xi$ if $\nu(c)|\xi| \leqslant c$, as

$$P(\xi) = \int_{m(\xi)}^{\infty} p(c)\,\mathrm{d}c \tag{75b}$$

which can be (easily) integrated numerically once $m(\xi)$ is available. At this point, we consider the three problems explicitly.

6.1. Boundary and symmetry conditions: Couette flow

For this problem no particular solution is required, and so we can take $T_4(\tau,\xi) = 0$ and consider that Eq. (70) is a three-vector problem. We substitute Eq. (68) into Eq. (33) and note Eq. (38) to find the boundary condition for the three-vector $\boldsymbol{G}(\tau,\xi)$, viz.

$$\boldsymbol{G}(-a,\xi) - (1-\alpha)\boldsymbol{D}\boldsymbol{G}(-a,-\xi) = \boldsymbol{R} \tag{76}$$

where

$$\boldsymbol{D} = \mathrm{diag}\big\{1, 1, -1\big\} \tag{77}$$

and

$$\boldsymbol{R} = (2/p_1)\alpha u_{\mathrm{p}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{78}$$

Here

$$p_1 = 3^{1/2}/2. \tag{79}$$

We note also that the condition listed as Eq. (39) yields, once we consider Eq. (68), the condition

$$\boldsymbol{G}(-\tau, -\xi) = -\boldsymbol{D}\boldsymbol{G}(\tau, \xi). \tag{80}$$

6.2. Boundary and symmetry conditions: Poiseuille flow

While three components of the vector $\boldsymbol{G}(\tau, \xi)$ are sufficient for decomposing Eq. (66) into three reduced equations, the boundary conditions for the Poiseuille and thermal-creep problems cannot be satisfied with a three-component decomposition. For this reason, we use here

$$T_4(c, \xi) = \nu(c)\big[D_0(c)/5 + E_0(c)\big\{5\big[\xi\nu(c)/c\big]^2 - 1\big\}/5\big] \tag{81}$$

to find

$$\boldsymbol{G}(-a, \xi) - (1-\alpha)\boldsymbol{D}\boldsymbol{G}(-a, -\xi) = \boldsymbol{R}(\xi) \tag{82}$$

where

$$\boldsymbol{D} = \mathrm{diag}\{1, 1, -1, 1\} \tag{83}$$

and

$$\boldsymbol{R}(\xi) = (1/\varepsilon_{\mathrm{p}}) \begin{bmatrix} -(2/p_1)(2-\alpha)\xi a \\ 0 \\ -(2/p_2)(2-\alpha)a \\ -\alpha/\varepsilon \end{bmatrix} \tag{84}$$

with

$$p_2 = 15^{1/2}/2. \tag{85}$$

Here the symmetry condition listed as Eq. (54) yields, once we consider Eq. (68), the condition

$$\boldsymbol{G}(-\tau, -\xi) = \boldsymbol{D}\boldsymbol{G}(\tau, \xi). \tag{86}$$

6.3. Boundary and symmetry conditions: thermal-creep flow

The decomposition for the thermal-creep problem also requires four components, but here we take

$$T_4(c, \xi) = \nu(c)A(c) \tag{87}$$

to find

$$\boldsymbol{G}(-a, \xi) - (1-\alpha)\boldsymbol{D}\boldsymbol{G}(-a, -\xi) = \boldsymbol{R}(\xi) \tag{88}$$

where, again,

$$\boldsymbol{D} = \mathrm{diag}\{1, 1, -1, 1\} \tag{89}$$

and here

$$\boldsymbol{R}(\xi) = \alpha/\varepsilon \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{90}$$

And again the symmetry condition listed as Eq. (54) yields, also for the thermal-creep problem,

$$\boldsymbol{G}(-\tau, -\xi) = \boldsymbol{D}\boldsymbol{G}(\tau, \xi). \tag{91}$$

Having defined our three "$\boldsymbol{G}$" problems, we are ready to construct the required solutions.

7. An analytical discrete-ordinates method

Our use of the discrete-ordinates method for solving problems in transport theory was first reported [16] in regard to an application in the field of radiative transfer, but we have also found our analytical version of the discrete-ordinates method, to which we refer as the ADO method, to be especially useful for applications in rarefied gas dynamics. And so this is the approach we follow here.

We note that the three-component $\boldsymbol{G}$ problem defined for Couette flow and the two four-component $\boldsymbol{G}$ problems developed for Poiseuille flow and thermal-creep flow are very similar; however to be very clear we report our solutions separately.

7.1. Couette flow

To start our ADO solution of a three-vector version of Eq. (70), we look for solutions of the form

$$\boldsymbol{G}_\nu(\tau,\xi)=\boldsymbol{\Phi}(\nu,\xi)\mathrm{e}^{-\varepsilon\tau/\nu}, \tag{92}$$

and so substituting Eq. (92) into Eq. (70) we find

$$(1-\xi/\nu)\boldsymbol{\Phi}(\nu,\xi)=\int_0^\gamma\big[\boldsymbol{\Psi}(\xi')\boldsymbol{\Phi}(\nu,\xi')+\boldsymbol{\Psi}(-\xi')\boldsymbol{\Phi}(\nu,-\xi')\big]\,\mathrm{d}\xi'. \tag{93}$$

Now if we use an N-point quadrature scheme to evaluate the integral in Eq. (93), then we can write

$$(1-\xi/\nu)\boldsymbol{\Phi}(\nu,\xi)=\sum_{k=1}^{N}w_k\big[\boldsymbol{\Psi}(\xi_k)\boldsymbol{\Phi}(\nu,\xi_k)+\boldsymbol{\Psi}(-\xi_k)\boldsymbol{\Phi}(\nu,-\xi_k)\big] \tag{94}$$

where $\{\xi_k, w_k\}$ are the nodes and weights of the quadrature scheme. Evaluating Eq. (94) at $\xi=\pm\xi_i$, we find

$$(1\mp\xi_i/\nu)\boldsymbol{\Phi}(\nu,\pm\xi_i)=\sum_{k=1}^{N}w_k\big[\boldsymbol{\Psi}(\xi_k)\boldsymbol{\Phi}(\nu,\xi_k)+\boldsymbol{\Psi}(-\xi_k)\boldsymbol{\Phi}(\nu,-\xi_k)\big] \tag{95}$$

for $i=1,2,\dots,N$. Following the development given in detail for the "three-component" temperature-jump problem in Ref. [15], we can convert the system of equations listed as Eq. (95) to a $3N\times 3N$ eigenvalue problem which we can solve numerically to yield $3N$ plus-minus pairs of separation constants $\pm\nu_j$ and the corresponding elementary vectors $\boldsymbol{\Phi}(\pm\nu_j,\xi_i)$. And so, we let $\{\nu_j\}$ denote the set of positive separation constants and express the desired solution as

$$\boldsymbol{G}(\tau,\pm\xi_i)=\sum_{j=1}^{3N}A_j\boldsymbol{\Phi}(\nu_j,\pm\xi_i)\mathrm{e}^{-(a+\tau)\varepsilon/\nu_j}+B_j\boldsymbol{D}\boldsymbol{\Phi}(\nu_j,\mp\xi_i)\mathrm{e}^{-(a-\tau)\varepsilon/\nu_j} \tag{96}$$

where the constants $\{A_j, B_j\}$ are to be determined from the symmetry condition listed as Eq. (80) and a discrete-ordinates version of the boundary condition given by Eq. (76) and where $\boldsymbol{D}$ is given by Eq. (77). In writing Eq. (96), we have made use of the fact that

$$\boldsymbol{\Phi}(-\nu_j,\pm\xi_i)=\boldsymbol{D}\boldsymbol{\Phi}(\nu_j,\mp\xi_i). \tag{97}$$

It can be shown, say from Eqs. (93) and (94), that there is only one positive value of ν, say ν_1, that tends to infinity as N increases without bound. We choose to take this fact into account explicitly by ignoring in Eq. (96) the largest separation constant and by using instead the corresponding exact solutions. And so we rewrite Eq. (96) as

$$\boldsymbol{G}(\tau,\pm\xi_i)=\boldsymbol{G}_0(\tau,\pm\xi_i)+\sum_{j=2}^{3N}A_j\boldsymbol{\Phi}(\nu_j,\pm\xi_i)\mathrm{e}^{-(a+\tau)\varepsilon/\nu_j}+B_j\boldsymbol{D}\boldsymbol{\Phi}(\nu_j,\mp\xi_i)\mathrm{e}^{-(a-\tau)\varepsilon/\nu_j} \tag{98}$$

where

$$\boldsymbol{G}_0(\tau,\xi)=A_1\boldsymbol{\Phi}_++B_1\boldsymbol{\Phi}_-(\tau,\xi) \tag{99}$$

is an exact solution of a three-vector version of Eq. (70) deduced from known [5] exact solutions of the homogeneous version of Eq. (1). Here

$$\boldsymbol{\Phi}_+=(1/p_1)\begin{bmatrix}1\\0\\0\end{bmatrix}. \tag{100a}$$

and

$$\boldsymbol{\Phi}_{-}(\tau,\xi) = (1/p_1)\begin{bmatrix} \varepsilon\tau - \xi \\ 0 \\ -p_1/p_2 \end{bmatrix}, \tag{100b}$$

where

$$p_1 = 3^{1/2}/2 \quad \text{and} \quad p_2 = 15^{1/2}/2. \tag{101a,b}$$

Because of Eq. (80), we must take $A_1 = 0$ and $B_j = -A_j$ for $j = 2, 3, \ldots, 3N$, and so we rewrite our solution as

$$\boldsymbol{G}(\tau,\pm\xi_i) = \boldsymbol{G}_0(\tau,\pm\xi_i) + \sum_{j=2}^{3N} A_j\big[\boldsymbol{\Phi}(\nu_j,\pm\xi_i)\mathrm{e}^{-(a+\tau)\varepsilon/\nu_j} - \boldsymbol{D\Phi}(\nu_j,\mp\xi_i)\mathrm{e}^{-(a-\tau)\varepsilon/\nu_j}\big] \tag{102}$$

where now

$$\boldsymbol{G}_0(\tau,\xi) = B_1\boldsymbol{\Phi}_{-}(\tau,\xi). \tag{103}$$

To complete the solution of the $\boldsymbol{G}$ problem for Couette flow we substitute Eq. (102) into a discrete-ordinates version of Eq. (76), viz.

$$\boldsymbol{G}(-a,\xi_i) - (1-\alpha)\boldsymbol{DG}(-a,-\xi_i) = \boldsymbol{R}, \tag{104}$$

for $i = 1, 2, \ldots, N$, and solve the resulting system of linear algebraic equations to establish the constants B_1 and A_j. Once the linear system has been solved, we can use Eqs. (102) and (103) in

$$u(\tau) = \int_{-\gamma}^{\gamma} \boldsymbol{\Upsilon}^{\mathrm{T}}(\xi)\boldsymbol{G}(\tau,\xi)\,\mathrm{d}\xi \tag{105}$$

and

$$q(\tau) = \int_{-\gamma}^{\gamma} \boldsymbol{\Gamma}^{\mathrm{T}}(\xi)\boldsymbol{G}(\tau,\xi)\,\mathrm{d}\xi \tag{106}$$

to find

$$u(\tau) = \frac{\varepsilon\tau}{2}B_1 + \sum_{j=2}^{3N} A_j N_j\big[\mathrm{e}^{-(a+\tau)\varepsilon/\nu_j} - \mathrm{e}^{-(a-\tau)\varepsilon/\nu_j}\big] \tag{107}$$

and

$$q(\tau) = \sum_{j=2}^{3N} A_j M_j\big[\mathrm{e}^{-(a+\tau)\varepsilon/\nu_j} - \mathrm{e}^{-(a-\tau)\varepsilon/\nu_j}\big] \tag{108}$$

where

$$N_j = \sum_{k=1}^{N} w_k \boldsymbol{\Upsilon}^{\mathrm{T}}(\xi_k)\big[\boldsymbol{\Phi}(\nu_j,\xi_k) + \boldsymbol{D\Phi}(\nu_j,-\xi_k)\big] \tag{109}$$

and

$$M_j = \sum_{k=1}^{N} w_k \boldsymbol{\Gamma}^{\mathrm{T}}(\xi_k)\big[\boldsymbol{\Phi}(\nu_j,\xi_k) + \boldsymbol{D\Phi}(\nu_j,-\xi_k)\big]. \tag{110}$$

Here $\boldsymbol{\Upsilon}(\xi)$ and $\boldsymbol{\Gamma}(\xi)$ respectively have elements

$$u_\alpha(\xi) = \frac{1}{\pi^{1/2}}\int_{m(\xi)}^{\infty} \mathrm{e}^{-c^2}\big[c^2 - \xi^2\nu^2(c)\big]T_\alpha(c,\xi)\,\mathrm{d}c, \tag{111}$$

and

$$\gamma_\alpha(\xi) = \frac{1}{\pi^{1/2}} \int_{m(\xi)}^{\infty} \mathrm{e}^{-c^2} \big[c^2 - \xi^2 \nu^2(c)\big]\big(c^2 - 5/2\big) T_\alpha(c, \xi)\, \mathrm{d}c, \tag{112}$$

for $\alpha = 1, 2, 3$. In a similar way, we find we can compute the required component of the pressure tensor from

$$P_{xy} = -\frac{1}{4} B_1 \varepsilon_{\mathrm{p}}. \tag{113}$$

Finally, to complete our solution for Couette flow we substitute Eqs. (107) and (108) into Eqs. (40) and (41) to find

$$U = \frac{\varepsilon}{8} B_1 - \frac{1}{2\varepsilon a^2} \sum_{j=2}^{3N} A_j \nu_j N_j \big(1 - \mathrm{e}^{-a\varepsilon/\nu_j}\big)^2 \tag{114}$$

and

$$Q = -\frac{1}{2\varepsilon a^2} \sum_{j=2}^{3N} A_j \nu_j M_j \big(1 - \mathrm{e}^{-a\varepsilon/\nu_j}\big)^2. \tag{115}$$

We are ready now to consider the other problems defined in this work.

7.2. Poiseuille and thermal-creep flow

Here much of the development is the same as for the problem of Couette flow, but to be clear we repeat some material. To start our ADO solution of the four-vector version of Eq. (70), we look for solutions of the form

$$\boldsymbol{G}_\nu(\tau, \xi) = \boldsymbol{\Phi}(\nu, \xi) \mathrm{e}^{-\varepsilon\tau/\nu}, \tag{116}$$

and so substituting Eq. (116) into Eq. (70) we find

$$(1 - \xi/\nu) \boldsymbol{\Phi}(\nu, \xi) = \int_0^{\gamma} \big[\boldsymbol{\Psi}(\xi') \boldsymbol{\Phi}(\nu, \xi') + \boldsymbol{\Psi}(-\xi') \boldsymbol{\Phi}(\nu, -\xi')\big]\, \mathrm{d}\xi'. \tag{117}$$

Now if we use an N-point quadrature scheme to evaluate the integral in Eq. (117), then we can write

$$(1 - \xi/\nu) \boldsymbol{\Phi}(\nu, \xi) = \sum_{k=1}^{N} w_k \big[\boldsymbol{\Psi}(\xi_k) \boldsymbol{\Phi}(\nu, \xi_k) + \boldsymbol{\Psi}(-\xi_k) \boldsymbol{\Phi}(\nu, -\xi_k)\big] \tag{118}$$

where $\{\xi_k, w_k\}$ are the nodes and weights of the quadrature scheme. Evaluating Eq. (118) at $\xi = \pm\xi_i$, we find

$$(1 \mp \xi_i/\nu) \boldsymbol{\Phi}(\nu, \pm\xi_i) = \sum_{k=1}^{N} w_k \big[\boldsymbol{\Psi}(\xi_k) \boldsymbol{\Phi}(\nu, \xi_k) + \boldsymbol{\Psi}(-\xi_k) \boldsymbol{\Phi}(\nu, -\xi_k)\big] \tag{119}$$

for $i = 1, 2, \ldots, N$. We can convert the system of equations listed as Eq. (119) to a $4N \times 4N$ eigenvalue problem which we can solve numerically to yield $4N$ plus-minus pairs of separation constants $\pm\nu_j$ and the corresponding elementary vectors $\boldsymbol{\Phi}(\pm\nu_j, \xi_i)$. And so, we let $\{\nu_j\}$ denote the set of positive separation constants and express the desired solution as

$$\boldsymbol{G}(\tau, \pm\xi_i) = \sum_{j=1}^{4N} A_j \boldsymbol{\Phi}(\nu_j, \pm\xi_i) \mathrm{e}^{-(a+\tau)\varepsilon/\nu_j} + B_j \boldsymbol{D} \boldsymbol{\Phi}(\nu_j, \mp\xi_i) \mathrm{e}^{-(a-\tau)\varepsilon/\nu_j} \tag{120}$$

where the constants $\{A_j, B_j\}$ are to be determined from the symmetry condition listed as Eq. (86) and a discrete-ordinates version of the boundary condition given by Eq. (82) or (88) and where $\boldsymbol{D}$ is given by Eq. (83). In writing Eq. (120), we have made use of the fact that

$$\boldsymbol{\Phi}(-\nu_j, \pm\xi_i) = \boldsymbol{D} \boldsymbol{\Phi}(\nu_j, \mp\xi_i). \tag{121}$$

It can be shown, say from Eqs. (117) and (118), that there is only one positive value of ν, say ν_1, that tends to infinity as N increases without bound. We choose to take this fact into account explicitly by ignoring in Eq. (120) the largest separation constant and by using instead the corresponding exact solutions. And so we write

$$\boldsymbol{G}(\tau,\pm\xi_i)=\boldsymbol{G}_0(\tau,\pm\xi_i)+\sum_{j=2}^{4N}A_j\boldsymbol{\Phi}(\nu_j,\pm\xi_i)\mathrm{e}^{-(a+\tau)\varepsilon/\nu_j}+B_j\boldsymbol{D}\boldsymbol{\Phi}(\nu_j,\mp\xi_i)\mathrm{e}^{-(a-\tau)\varepsilon/\nu_j} \tag{122}$$

where

$$\boldsymbol{G}_0(\tau,\xi)=A_1\boldsymbol{\Phi}_+ + B_1\boldsymbol{\Phi}_-(\tau,\xi) \tag{123}$$

is an exact solution of the four-vector version of Eq. (70) deduced from known [5] exact solutions of the homogeneous version of Eq. (1). Here

$$\boldsymbol{\Phi}_+=(1/p_1)\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \tag{124a}$$

and

$$\boldsymbol{\Phi}_-(\tau,\xi)=(1/p_1)\begin{bmatrix}\varepsilon\tau-\xi\\0\\-p_1/p_2\\0\end{bmatrix} \tag{124b}$$

where p_1 and p_2 are given by Eqs. (101). Because of Eq. (86), we must take $B_1=0$ and $B_j=A_j$, and so we rewrite our solution as

$$\boldsymbol{G}(\tau,\pm\xi_i)=\boldsymbol{G}_0(\tau,\pm\xi_i)+\sum_{j=2}^{4N}A_j\big[\boldsymbol{\Phi}(\nu_j,\pm\xi_i)\mathrm{e}^{-(a+\tau)\varepsilon/\nu_j}+\boldsymbol{D}\boldsymbol{\Phi}(\nu_j,\mp\xi_i)\mathrm{e}^{-(a-\tau)\varepsilon/\nu_j}\big] \tag{125}$$

where now

$$\boldsymbol{G}_0(\tau,\xi)=A_1\boldsymbol{\Phi}_+. \tag{126}$$

To complete the solution of the $\boldsymbol{G}$ problem for Poiseuille flow or thermal-creep flow, we substitute Eq. (125) into a discrete-ordinates version of Eq. (82) or Eq. (88) and solve the resulting system of linear algebraic equations to establish the constants A_j, $j=1,2,\ldots,4N$. Once the linear system has been solved, we can use Eqs. (125) and (126) in

$$u(\tau)=u_{\mathrm{ps}}(\tau)+\int_{-\gamma}^{\gamma}\boldsymbol{\Upsilon}^{\mathrm{T}}(\xi)\boldsymbol{G}(\tau,\xi)\,\mathrm{d}\xi \tag{127}$$

and

$$q(\tau)=q_{\mathrm{ps}}(\tau)+\int_{-\gamma}^{\gamma}\boldsymbol{\Gamma}^{\mathrm{T}}(\xi)\boldsymbol{G}(\tau,\xi)\,\mathrm{d}\xi, \tag{128}$$

where

$$u_{\mathrm{ps}}(\tau)=\frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^{1}\mathrm{e}^{-c^2}\psi_{\mathrm{ps}}(\tau,c,\mu)c^3\big(1-\mu^2\big)\,\mathrm{d}\mu\,\mathrm{d}c \tag{129}$$

and

$$q_{\mathrm{ps}}(\tau)=\frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^{1}\mathrm{e}^{-c^2}\psi_{\mathrm{ps}}(\tau,c,\mu)\big(c^2-5/2\big)c^3\big(1-\mu^2\big)\,\mathrm{d}\mu\,\mathrm{d}c, \tag{130}$$

to find, first for Poiseuille flow,

$$u_{\mathrm{P}}(\tau)=\frac{\varepsilon}{2\varepsilon_{\mathrm{p}}}\big(\tau^2-a^2\big)+\frac{1}{2}A_1+\sum_{j=2}^{4N}A_jN_j\big[\mathrm{e}^{-(a+\tau)\varepsilon/\nu_j}+\mathrm{e}^{-(a-\tau)\varepsilon/\nu_j}\big] \tag{131}$$

and

$$q_{\mathrm{P}}(\tau) = \frac{4}{15\varepsilon\varepsilon_{\mathrm{p}}} d_5 + \sum_{j=2}^{4N} A_j M_j \big[\mathrm{e}^{-(a+\tau)\varepsilon/\nu_j} + \mathrm{e}^{-(a-\tau)\varepsilon/\nu_j} \big] \tag{132}$$

where we make use of the subscript P to denote Poiseuille flow, and where

$$d_5 = \frac{1}{\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2} D_0(c) c^5 \, \mathrm{d}c. \tag{133}$$

Here the normalization integrals are given by

$$N_j = \sum_{k=1}^{N} w_k \boldsymbol{\Upsilon}^{\mathrm{T}}(\xi_k) \big[\boldsymbol{\Phi}(\nu_j, \xi_k) + \boldsymbol{D}\boldsymbol{\Phi}(\nu_j, -\xi_k) \big] \tag{134a}$$

and

$$M_j = \sum_{k=1}^{N} w_k \boldsymbol{\Gamma}^{\mathrm{T}}(\xi_k) \big[\boldsymbol{\Phi}(\nu_j, \xi_k) + \boldsymbol{D}\boldsymbol{\Phi}(\nu_j, -\xi_k) \big]. \tag{134b}$$

In addition $\boldsymbol{\Upsilon}(\xi)$ and $\boldsymbol{\Gamma}(\xi)$ have, respectively, elements

$$u_\alpha(\xi) = \frac{1}{\pi^{1/2}} \int_{m(\xi)}^\infty \mathrm{e}^{-c^2} \big[c^2 - \xi^2 \nu^2(c) \big] T_\alpha(c, \xi) \, \mathrm{d}c \tag{135}$$

and

$$\gamma_\alpha(\xi) = \frac{1}{\pi^{1/2}} \int_{m(\xi)}^\infty \mathrm{e}^{-c^2} \big[c^2 - \xi^2 \nu^2(c) \big] \big(c^2 - 5/2 \big) T_\alpha(c, \xi) \, \mathrm{d}c, \tag{136}$$

for $\alpha = 1, 2, 3, 4$. Finally, to complete our solutions for Poiseuille we substitute Eqs. (131) and (132) into Eqs. (51) and (52) to find

$$U_{\mathrm{P}} = -\frac{a\varepsilon}{3\varepsilon_{\mathrm{p}}} + \frac{1}{2a} A_1 + \frac{1}{\varepsilon a^2} \sum_{j=2}^{4N} A_j \nu_j N_j \big(1 - \mathrm{e}^{-2a\varepsilon/\nu_j} \big) \tag{137}$$

and

$$Q_{\mathrm{P}} = \frac{4}{15 a\varepsilon\varepsilon_{\mathrm{p}}} d_5 + \frac{1}{\varepsilon a^2} \sum_{j=2}^{4N} A_j \nu_j M_j \big(1 - \mathrm{e}^{-2a\varepsilon/\nu_j} \big). \tag{138}$$

The equivalent results, with subscript T, for thermal creep flow are

$$u_{\mathrm{T}}(\tau) = \frac{1}{2} A_1 + \sum_{j=2}^{4N} A_j N_j \big[\mathrm{e}^{-(a+\tau)\varepsilon/\nu_j} + \mathrm{e}^{-(a-\tau)\varepsilon/\nu_j} \big], \tag{139}$$

$$q_{\mathrm{T}}(\tau) = -\frac{5\varepsilon_{\mathrm{t}}}{4\varepsilon} + \sum_{j=2}^{4N} A_j M_j \big[\mathrm{e}^{-(a+\tau)\varepsilon/\nu_j} + \mathrm{e}^{-(a-\tau)\varepsilon/\nu_j} \big], \tag{140}$$

$$U_{\mathrm{T}} = \frac{1}{2a} A_1 + \frac{1}{\varepsilon a^2} \sum_{j=2}^{4N} A_j \nu_j N_j \big(1 - \mathrm{e}^{-2a\varepsilon/\nu_j} \big) \tag{141}$$

and

$$Q_{\mathrm{T}} = -\frac{5\varepsilon_{\mathrm{t}}}{4a\varepsilon} + \frac{1}{\varepsilon a^2} \sum_{j=2}^{4N} A_j \nu_j M_j \big(1 - \mathrm{e}^{-2a\varepsilon/\nu_j} \big). \tag{142}$$

Having established the solutions to all three of the considered flow problems, we can now discuss the numerical results obtained from the CES model of the linearized Boltzmann equation and the analytical discrete-ordinates method.

8. Numerical results

To define the quadrature scheme to be used with our ADO solutions, we have simply mapped the Gauss–Legendre scheme onto the interval $[0, \gamma]$. We then used the driver program RG from the EISPACK collection [17] to find the required eigenvalues and eigenvectors, and we used the subroutines DGECO and DGESL from the LINPACK package [18] to solve the linear systems that define the constants for each of the three considered problems. For all three of problems we have elected to use the mean-free path based on viscosity, and so we have used $\varepsilon = \varepsilon_{\mathrm{p}}$ where the numerical value of ε_{p} is given by Eq. (17a). To illustrate our numerical work we list in Tables 1–9 some results obtained from our FORTRAN implementation of the developed solutions. We note that our results are given with many significant figures, and while we have no proof of the accuracy achieved in this work, we have found the results to be stable as the order N of the quadrature scheme is increased. It should be noted that in suggesting that we have many figures of accuracy we are referring to an accurate implementation of the CES model – not to the ability of the model to match more intensive calculations based on the linearized Boltzmann equation.

In regard to Couette flow, we list in Tables 1–3 the newly established results based on our ADO solution of the CES model. We also list in these first three tables, the equivalent results we found for the BGK model. In all of our calculations we have

Table 1
Couette flow with $u_{\mathrm{p}} = 1$: a component P_{xy} of the reduced pressure tensor

	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 1.0$	
$2a$	BGK	CES	BGK	CES	BGK	CES
1.0(−7)	2.96942(−2)	2.96942(−2)	1.88063(−1)	1.88063(−1)	5.64190(−1)	5.64190(−1)
1.0(−3)	2.96928(−2)	2.96926(−2)	1.88008(−1)	1.88001(−1)	5.63692(−1)	5.63636(−1)
1.0(−1)	2.95618(−2)	2.95505(−2)	1.82984(−1)	1.82618(−1)	5.22325(−1)	5.20156(−1)
1.0	2.85930(−2)	2.85847(−2)	1.52354(−1)	1.52351(−1)	3.38925(−1)	3.39977(−1)
1.0(1)	2.26221(−2)	2.26813(−2)	6.36078(−2)	6.39749(−2)	8.31122(−2)	8.35227(−2)
1.0(3)	9.66925(−4)	9.67035(−4)	9.94310(−4)	9.94400(−4)	9.97972(−4)	9.98031(−4)
1.0(7)	9.99997(−8)	9.99997(−8)	9.99999(−8)	9.99999(−8)	1.00000(−7)	1.00000(−7)

Table 2
Couette flow with $u_{\mathrm{p}} = 1$: the flow rate $-U$

	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 1.0$	
$2a$	BGK	CES	BGK	CES	BGK	CES
0.01	8.00320(−2)	9.71553(−2)	4.77540(−1)	5.75403(−1)	1.29070	1.53426
0.10	4.81420(−2)	5.41084(−2)	2.74926(−1)	3.04586(−1)	6.85780(−1)	7.41991(−1)
1.00	2.34756(−2)	2.31248(−2)	1.16120(−1)	1.13676(−1)	2.32188(−1)	2.26777(−1)
10.0	1.17090(−2)	1.15560(−2)	3.26636(−2)	3.24470(−2)	4.22811(−2)	4.21424(−2)

Table 3
Couette flow with $u_{\mathrm{p}} = 1$: the heat-flow rate Q

	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 1.0$	
$2a$	BGK	CES	BGK	CES	BGK	CES
0.01	3.25959(−2)	2.01604(−2)	1.91891(−1)	1.19086(−1)	5.05517(−1)	3.15754(−1)
0.10	1.66805(−2)	1.12938(−2)	9.17172(−2)	6.22276(−2)	2.12309(−1)	1.44794(−1)
1.00	4.58954(−3)	4.34898(−3)	1.99715(−2)	1.81577(−2)	3.13629(−2)	2.69864(−2)
10.0	1.98991(−4)	1.79134(−4)	4.29861(−4)	3.64077(−4)	3.62529(−4)	2.85980(−4)

Table 4
Couette flow with $u_{\mathrm{p}} = 1$: comparison results for the case $\alpha = 1$

	P_{xy}			$-U$			Q		
k_*	BGK	CES	[19]	BGK	CES	[19]	BGK	CES	[19]
0.10	1.0096(−1)	1.0156(−1)	1.015(−1)	5.1825(−2)	5.1519(−2)	5.154(−2)	6.7255(−4)	5.4981(−4)	5.8(−4)
1.00	3.6759(−1)	3.6803(−1)	3.687(−1)	2.6728(−1)	2.6242(−1)	2.626(−1)	4.1742(−2)	3.4921(−2)	2.87(−2)
10.0	5.3027(−1)	5.2828(−1)	5.2890(−1)	7.4439(−1)	8.1533(−1)	8.00(−1)	2.3963(−1)	1.6106(−1)	1.3(−1)
20.0	5.4617(−1)	5.4479(−1)	5.4515(−1)	9.2142(−1)	1.0426	1.02	3.2417(−1)	2.1071(−1)	1.8(−1)

Table 5
Couette flow with $u_\mathrm{P}=1$: velocity and heat-flow profiles (CES model) for the case $2a=1$

	$\alpha=0.1$		$\alpha=0.5$		$\alpha=1.0$	
τ/a	$-u(\tau)$	$q(\tau)$	$-u(\tau)$	$q(\tau)$	$-u(\tau)$	$q(\tau)$
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	4.21138(–3)	7.67912(–4)	2.10475(–2)	3.25143(–3)	4.29400(–2)	4.92200(–3)
0.2	8.45518(–3)	1.54464(–3)	4.22286(–2)	6.53651(–3)	8.60736(–2)	9.88757(–3)
0.3	1.27668(–2)	2.33962(–3)	6.36889(–2)	9.89106(–3)	1.29611(–1)	1.49428(–2)
0.4	1.71885(–2)	3.16366(–3)	8.56013(–2)	1.33558(–2)	1.73802(–1)	2.01395(–2)
0.5	2.17748(–2)	4.03012(–3)	1.08189(–1)	1.69806(–2)	2.18965(–1)	2.55397(–2)
0.6	2.66031(–2)	4.95687(–3)	1.31765(–1)	2.08311(–2)	2.65547(–1)	3.12236(–2)
0.7	3.17943(–2)	5.97051(–3)	1.56821(–1)	2.50037(–2)	3.14241(–1)	3.73061(–2)
0.8	3.75661(–2)	7.11669(–3)	1.84230(–1)	2.96616(–2)	3.66275(–1)	4.39767(–2)
0.9	4.44157(–2)	8.49586(–3)	2.15983(–1)	3.51587(–2)	4.24420(–1)	5.16363(–2)
1.0	5.50626(–2)	1.06484(–2)	2.62841(–1)	4.33653(–2)	5.03454(–1)	6.23423(–2)

Table 6
Poiseuille and thermal creep: comparison results for the case $\alpha=0.5$

	$-U_\mathrm{P}$			$Q_\mathrm{P}=U_\mathrm{T}$			$-Q_\mathrm{T}$		
$2a$	S	CES	[20]	S	CES	[20]	S	CES	[20]
0.10	4.5801	4.3156	4.3628	1.4012	1.5426	1.5632	7.9768	7.6317	7.7430
1.00	3.3928	3.2959	3.3270	4.9043(–1)	5.3760(–1)	5.285(–1)	2.4640	2.5170	2.5136
10.0	4.5837	4.5285	4.5490	8.7524(–2)	8.6266(–2)	8.42(–2)	3.5694(–1)	3.6251(–1)	3.616(–1)

Table 7
Poiseuille and thermal creep: comparison results for the case $\alpha=1$

	$-U_\mathrm{P}$			$Q_\mathrm{P}=U_\mathrm{T}$			$-Q_\mathrm{T}$		
$2a$	S	CES	[20]	S	CES	[20]	S	CES	[20]
0.10	2.0395	1.9259	1.9301	7.3268(–1)	7.9087(–1)	7.966(–1)	4.0546	3.8509	3.8669
1.00	1.5536	1.4863	1.5090	3.6546(–1)	4.0456(–1)	3.890(–1)	1.7537	1.8018	1.7846
10.0	2.7799	2.7220	2.7291	9.8147(–2)	9.3046(–2)	8.98(–2)	3.4063(–1)	3.4964(–1)	3.467(–1)

Table 8
Poiseuille flow: velocity and heat-flow profiles (CES model) for the case $2a=1$

	$\alpha=0.1$		$\alpha=0.5$		$\alpha=1.0$	
τ/a	$-u_\mathrm{P}(\tau)$	$q_\mathrm{P}(\tau)$	$-u_\mathrm{P}(\tau)$	$q_\mathrm{P}(\tau)$	$-u_\mathrm{P}(\tau)$	$q_\mathrm{P}(\tau)$
0.0	8.84866	3.78635(–1)	1.74191	2.99576(–1)	8.42636(–1)	2.40862(–1)
0.1	8.84637	3.78092(–1)	1.73946	2.98793(–1)	8.40049(–1)	2.39884(–1)
0.2	8.83948	3.76451(–1)	1.73209	2.96425(–1)	8.32242(–1)	2.36927(–1)
0.3	8.82784	3.73675(–1)	1.71965	2.92417(–1)	8.19060(–1)	2.31918(–1)
0.4	8.81122	3.69698(–1)	1.70188	2.86667(–1)	8.00220(–1)	2.24727(–1)
0.5	8.78923	3.64413(–1)	1.67835	2.79013(–1)	7.75264(–1)	2.15142(–1)
0.6	8.76122	3.57651(–1)	1.64836	2.69196(–1)	7.43460(–1)	2.02833(–1)
0.7	8.72612	3.49135(–1)	1.61078	2.56798(–1)	7.03591(–1)	1.87262(–1)
0.8	8.68192	3.38362(–1)	1.56347	2.41066(–1)	6.53440(–1)	1.67475(–1)
0.9	8.62395	3.24197(–1)	1.50156	2.20325(–1)	5.88008(–1)	1.41371(–1)
1.0	8.52657	3.00538(–1)	1.39899	1.85908(–1)	4.81439(–1)	9.84408(–2)

used $u_\mathrm{P}=1$ for the velocities of the plates. In Table 4 we compare the xy component of the pressure and the two flow rates U and Q as computed from the BGK model, the CES model and the linearized Boltzmann equation (LBE). The LBE results listed

Table 9
Thermal creep flow: velocity and heat-flow profiles (CES model) for the case $2a = 1$

	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 1.0$	
τ/a	$u_T(\tau)$	$-q_T(\tau)$	$u_T(\tau)$	$-q_T(\tau)$	$u_T(\tau)$	$-q_T(\tau)$
0.0	3.62774(−1)	1.74575	2.89736(−1)	1.33088	2.35639(−1)	1.01920
0.1	3.62638(−1)	1.74531	2.89204(−1)	1.32909	2.34795(−1)	1.01629
0.2	3.62229(−1)	1.74396	2.87598(−1)	1.32368	2.32240(−1)	1.00749
0.3	3.61535(−1)	1.74167	2.84878(−1)	1.31449	2.27914(−1)	9.92531(−1)
0.4	3.60539(−1)	1.73836	2.80974(−1)	1.30124	2.21700(−1)	9.70952(−1)
0.5	3.59215(−1)	1.73395	2.75774(−1)	1.28348	2.13417(−1)	9.41998(−1)
0.6	3.57516(−1)	1.72824	2.69099(−1)	1.26051	2.02776(−1)	9.04486(−1)
0.7	3.55370(−1)	1.72096	2.60660(−1)	1.23118	1.89310(−1)	8.56489(−1)
0.8	3.52645(−1)	1.71161	2.49936(−1)	1.19340	1.72188(−1)	7.94564(−1)
0.9	3.49047(−1)	1.69907	2.35775(−1)	1.14263	1.49586(−1)	7.11177(−1)
1.0	3.43035(−1)	1.67762	2.12264(−1)	1.05592	1.12398(−1)	5.69111(−1)

in Table 4 were deduced from Ref. [19]. In constructing Table 4 we have made some modifications to our notation in order to compare with Sone et al. [19]. While we continue to use $u_p = 1$, we have defined the channel half-thickness a by

$$a = \frac{2^{1/2}}{8\varepsilon_p k_*} \tag{143}$$

where we use k_* to denote the constant k used in Ref. [19] as a defining parameter. We note that we have found the results (denoted by the superscript $*$) of Sone et al. [19] are related to our physical quantities by the relations

$$P^*_{xy} = 2P_{xy}, \quad U^* = aU \quad \text{and} \quad Q^* = aQ. \tag{144a,b,c}$$

In Table 5 we list the velocity and heat-flow profiles deduced from the CES model for a particular channel width and three values of the accommodation coefficient. Tables 6 and 7 compare the CES results to similar results from the S model [2] and the more definitive results reported by Loyalka and Hickey [20] for the linearized Boltzmann equation. See also Ref. [21]. And finally in Tables 8 and 9 we list our results from an ADO computation of the velocity and heat-flow profiles.

9. Concluding remarks

In Refs. [3] and [4], where the CES model was used for some basic applications defined for semi-infinite media, it was suggested that the CES model can provide a good practical alternative to more computational intensive methods based on the linearized Boltzmann equation. Having complete this work on three basic problems in rarefied gas dynamics defined by flow in a finite channel, we also conclude that the CES model can we used to improve the results available from the BGK model without the computational expense required by definitive solutions of the linearized Boltzmann equation. While we consider our results for the component of the pressure tensor P_{xy}, the velocity profile $u(\tau)$ and the flow rate U to be especially good, the results for the heat flow profile $q(\tau)$ and the heat-flow rate Q must be considered not quite so good when compared to the results from the linearized Boltzmann equation. In fact, we have seen, in comparing the results from various model equations to results deduced from the linearized Boltzmann, that the heat flow profile and the heat flow rate are the quantities that are the most difficult to compute well from model approximations. And finally, in addition to the merits of the CES model, we have seen here another case where the ADO method [16] can be used to solve well model problems in the field of rarefied-gas dynamics.

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