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The linearized Boltzmann equation: Concise and accurate solutions to basic flow problems

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Abstract. A polynomial expansion procedure and an analytical discrete-ordinates method are used to solve a collection of basic flow problems based on a rigorous version of the linearized Boltzmann equation for rigid-sphere interactions. In particular, two half-space problems, Kramers and thermal creep, and three problems defined by flow in a plane-parallel channel, Poiseuille, thermal-creep and Couette flow, are solved (essentially) analytically in terms of a modern version of the discrete-ordinates method. The developed algorithms are implemented for general values of the accommodation coefficient to yield numerical results that can be considered a new standard of reference.

Keywords. Boltzmann equation, rarefied gas dynamics, Kramers, thermal creep, Poiseuille, Couette

1. Introduction

As one of several steps in evaluating the effectiveness of a new computational method it is considered important to study two basic half-space problems known as Kramers' problem and the thermal-creep problem and three problems, Poiseuille flow, thermal-creep flow and Couette flow, defined by flow in a finite plane-parallel channel. In addition to the important limiting cases (half-space problems), the three problems defined for finite channels allow us to study the flow of a rarefied gas that is driven by (i) movement parallel to the channel by one or both of the two confining walls, (ii) a pressure gradient or (iii) a temperature gradient in a direction parallel to the boundaries that enclose the gas. A new book by Cercignani [1] includes discussions of these basic flow problems and provides a guide to many works that have contributed to our understanding of all of these often-studied problems. In addition to Ref. [1], we have found a review article by Sharipov and Seleznev [2] to be of fundamental importance to our work here. As well as providing a comprehensive list of important references relative to basic problems in rarefied gas dynamics, Sharipov and Seleznev [2] have reported many good comparisons of numerical results derived from the linearized Boltzmann equation and various kinetic models and based on a diverse collection of computational

algorithms.

In a recent work [3] a newly introduced polynomial expansion technique (relevant to the speed variable) and an analytical discrete-ordinates method [4] that has evolved from Chandrasekhar's work [5] in radiative transfer were used to solve the classical temperature-jump problem based on a rigorous form of the linearized Boltzmann equation for rigid-sphere interactions. And so now we show that the methods of Ref. [3] can be used also to solve well the collection of flow problems considered in this work.

2. Mathematical formulation

To start this work, we follow Pekeris [6], who quotes Boltzmann [7], Hilbert [8] and Chapman and Cowling [9], and consider the linearized Boltzmann equation written in terms of $h(\tau, \boldsymbol{c})$, a perturbation to the velocity distribution function, for rigid-sphere collisions as

$$S(\boldsymbol{c}) + c\mu \frac{\partial}{\partial \tau} h(\tau, \boldsymbol{c}) = \varepsilon L\{h\}(\tau, \boldsymbol{c}) \tag{1}$$

where

$$L\{h\}(\tau, \boldsymbol{c}) = -\nu(c) h(\tau, \boldsymbol{c}) + \int_0^\infty \int_{-1}^1 \int_0^{2\pi} \mathrm{e}^{-c'^2} h(\tau, \boldsymbol{c}') K(\boldsymbol{c}', \boldsymbol{c}) c'^2 \, \mathrm{d}\chi' \, \mathrm{d}\mu' \, \mathrm{d}c'. \tag{2}$$

Here the scattering kernel is written in the expanded (Pekeris) form [6], *viz.*

$$K(\boldsymbol{c}', \boldsymbol{c}) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (2n+1)(2 - \delta_{0,m}) P_n^m(\mu') P_n^m(\mu) k_n(c', c) \cos m(\chi' - \chi) \tag{3}$$

where the *normalized* Legendre functions are given (in terms of the Legendre polynomials) by

$$P_n^m(\mu) = \left[\frac{(n-m)\,!}{(n+m)\,!} \right]^{1/2} (1 - \mu^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_n(\mu), \quad n \geq m. \tag{4}$$

In addition,

$$\varepsilon = \sigma_0^2 n_0 \pi^{1/2} l \tag{5}$$

where l is (at this point) an unspecified mean-free path, n_0 is the density and σ_0 is the scattering diameter of the gas particles. In this work, the spatial variable τ is measured in units of the mean-free path l and $c(2kT_0/m)^{1/2}$ is the magnitude of the particle velocity. Also, k is the Boltzmann constant, m is the mass of a gas particle and T_0 is a reference temperature. It should be noted that we have included in Eq. (1) an inhomogeneous driving term $S(\boldsymbol{c})$ that we will specify, along with an appropriate definition of the perturbation $h(\tau, \boldsymbol{c})$, for the three general types of flow (Couette, Poiseuille and thermal creep) we consider in this work. We note that the functions $k_n(c', c)$ in Eq. (3) are the components in an

expansion of the scattering law (for rigid-sphere collisions) reported by Pekeris [6], and

$$\nu(c) = \frac{2c^2+1}{c}\int_0^c \mathrm{e}^{-x^2}\,\mathrm{d}x + \mathrm{e}^{-c^2} \tag{6}$$

is the collision frequency. And finally, we use spherical coordinates $(\,c', \arccos\mu', \chi'\,)$ and $(\,c, \arccos\mu, \chi\,)$ to define the (dimensionless) velocity vectors $\boldsymbol{c}'$ and $\boldsymbol{c}$.

Since the component kernels $k_n(c', c)$ used in Eq. (3) are essential to our work here, we restate some basic results developed and reported by Pekeris [6], Pekeris and Alterman [10] and Pekeris, Alterman, Finkelstein and Frankowski [11]. First of all, Pekeris [6] gives (in our notation) the expression

$$k_n(c', c) = 2\int_0^\pi \left[(2/R)\mathrm{e}^{\omega^2} - R\right] P_n(\cos\theta)\sin\theta\,\mathrm{d}\theta \tag{7}$$

where

$$R = |\boldsymbol{c}' - \boldsymbol{c}| \quad \text{and} \quad \omega = (1/R)c'c\sin\theta \tag{8a,b}$$

and where θ is the angle between $\boldsymbol{c}'$ and $\boldsymbol{c}$. In Ref. [10] Pekeris and Alterman discussed the coefficients of viscosity and heat conduction and used the kernel functions $k_1(c', c)$ and $k_2(c', c)$ to define the Chapman-Enskog integral equations for viscosity and heat conduction. We write these equations here as

$$\nu(c)c^2 b(c) - \int_0^\infty \mathrm{e}^{-c'^2} b(c') k_2(c', c) c'^4\,\mathrm{d}c' = c^2 \tag{9}$$

and

$$\nu(c)ca(c) - \int_0^\infty \mathrm{e}^{-c'^2} a(c') k_1(c', c) c'^3\,\mathrm{d}c' = c(c^2 - 5/2) \tag{10a}$$

with

$$\int_0^\infty \mathrm{e}^{-c^2} a(c) c^4\,\mathrm{d}c = 0. \tag{10b}$$

Now, as discussed (for example) in Refs. [10,12], if we wish to use a mean-free path based on the viscosity, *i.e.*

$$l = l_\mathrm{p} = (\mu_*/p_0)(2kT_0/m)^{1/2} \tag{11}$$

where μ_* is the viscosity and $p_0 = n_0 kT_0$ is the pressure, then we should use in Eq. (1)

$$\varepsilon = \varepsilon_\mathrm{p} = \frac{16}{15\pi^{1/2}}\int_0^\infty \mathrm{e}^{-c^2} b(c) c^6\,\mathrm{d}c \tag{12}$$

where $b(c)$ is defined by Eq. (9). On the other hand, if we wish to use a mean-free path based on heat conduction,

$$l = l_\mathrm{t} = [4\lambda_*/(5n_0 k)][m/(2kT_0)]^{1/2}, \tag{13}$$

where λ_* is the heat-conduction coefficient, then in Eq. (1) we should use [10,13]

$$\varepsilon = \varepsilon_\mathrm{t} = \frac{16}{15\pi^{1/2}}\int_0^\infty \mathrm{e}^{-c^2} a(c) c^6\,\mathrm{d}c \tag{14}$$

where $a(c)$ is defined by Eqs. (10). While the component kernel functions $k_n(c', c)$, for $n = 1, 2$ and 3, are required for the Chapman-Enskog equations for viscosity and heat conduction and the so-called Burnett integral equations [12], we intend to use more of these component kernels in a truncated version of Eq. (3). We note here that in Ref. [11], Pekeris and co-workers have reported an ingenious set of expressions and recursion formulas that they used (along with a computer program) to evaluate analytically the expression listed here as Eq. (7). Pekeris *et al.* [11] also give explicit results for the cases up to and including $k_8(c', c)$, but we believe there is at least one misprint in those (extensive) results. As will be discussed in more detail later in this work, we have used the MAPLE software package to evaluate Eq. (7) analytically for essentially any value of n, but we have actually used the kernel functions (so far) only up to $n = 8$.

In regard to boundary conditions that supplement Eq. (1) we will be more explicit when considering the specific problems, but, in general, for half-space problems we consider Eq. (1) either with no driving term, but with a condition imposed on a diverging solution as τ tends of infinity (Kramers' problem), or with a specified driving term and a particular solution of the inhomogeneous equation. For both of these half-space problems we use a boundary condition at the wall ($\tau = 0$) of the form

$$h(0, c, \mu, \chi) - (1 - \alpha)h(0, c, -\mu, \chi) - \alpha\mathcal{I}\{h\}(0) = 0 \tag{15}$$

for $\mu \in (0, 1]$, $c \in [0, \infty)$ and all χ. Here, in general,

$$\mathcal{I}\{h\}(\tau) = \frac{2}{\pi}\int_0^\infty \int_0^1 \int_0^{2\pi} \mathrm{e}^{-c^2} h(\tau, c, -\mu, \chi) c^3 \mu \, \mathrm{d}\chi \, \mathrm{d}\mu \, \mathrm{d}c, \tag{16}$$

$\alpha \in (0, 1]$ is the accommodation coefficient and

$$h(\tau, \boldsymbol{c}) \Leftrightarrow h(\tau, c, \mu, \chi). \tag{17}$$

For the problems defined for flow in a channel, we consider that the walls are located at $\pm a$ and supplement Eq. (1) with the boundary condition

$$h(-a, c, \mu, \chi) - (1 - \alpha)h(-a, c, -\mu, \chi) - \alpha\mathcal{I}\{h\}(-a) = F(\boldsymbol{c}), \tag{18}$$

for $\mu \in (0, 1]$, all χ and all c, and a symmetry condition that relates the perturbation distribution function $h(\tau, c, \mu, \chi)$ to $h(-\tau, c, -\mu, \chi)$. Here $F(\boldsymbol{c})$ is taken to be specified.

3. Quantities of interest

In two previous works [14,15] the five flow problems we solve here were solved in terms of the CES model of the linearized Boltzmann equation, and so we now make use of those parts of Refs. [14,15] that are relevant here for the Boltzmann equation. In our notation $c\mu$ is the component of the (dimensionless) velocity

vector in the positive τ direction, and so if we let

$$c_\eta = c(1-\mu^2)^{1/2}\cos\chi \tag{19}$$

denote the component of velocity in the direction of the flow, then we can express the bulk velocity and heat-flow profiles we intend to compute as

$$u(\tau) = \frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2}h(\tau,\boldsymbol{c})c^3(1-\mu^2)^{1/2}\cos\chi\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c \tag{20}$$

and

$$q(\tau) = \frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2}h(\tau,\boldsymbol{c})(c^2-5/2)c^3(1-\mu^2)^{1/2}\cos\chi\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c. \tag{21}$$

For the problem of Couette flow we also intend to compute a component of the pressure tensor which we write as

$$P_{xy} = \frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2}h(\tau,\boldsymbol{c})c^4\mu(1-\mu^2)^{1/2}\cos\chi\,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c \tag{22}$$

where x and y are the spatial variables (measured in cm) that correspond respectively to η and τ.

It is clear from Eqs. (20–22) that the information we seek is expressed in terms of certain moments of $h(\tau,\boldsymbol{c})$, and so we can make a convenient simplification in our formulation. Considering the form of the scattering kernel given by Eq. (3), we introduce

$$\psi(\tau,c,\mu) = \frac{1}{\pi}(1-\mu^2)^{-1/2}\int_0^{2\pi} h(\tau,\boldsymbol{c})\cos\chi\,\mathrm{d}\chi \tag{23}$$

and rewrite Eqs. (20–22) as

$$u(\tau) = \frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\psi(\tau,c,\mu)c^3(1-\mu^2)\,\mathrm{d}\mu\,\mathrm{d}c, \tag{24}$$

$$q(\tau) = \frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\psi(\tau,c,\mu)(c^2-5/2)c^3(1-\mu^2)\,\mathrm{d}\mu\,\mathrm{d}c \tag{25}$$

and

$$P_{xy} = \frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\psi(\tau,c,\mu)c^4\mu(1-\mu^2)\,\mathrm{d}\mu\,\mathrm{d}c. \tag{26}$$

We can multiply Eq. (1) by $\cos\chi$ and integrate to find, after noting Eq. (3), that $\psi(\tau,c,\mu)$ must satisfy the balance equation

$$S^*(c,\mu) + c\mu\frac{\partial}{\partial\tau}\psi(\tau,c,\mu) = \varepsilon L^*\{\psi\}(\tau,c,\mu) \tag{27}$$

where

$$S^*(c,\mu) = \frac{1}{\pi}(1-\mu^2)^{-1/2}\int_0^{2\pi} S(\boldsymbol{c})\cos\chi\,\mathrm{d}\chi \tag{28}$$

and

$$L^*\{\psi\}(\tau,c,\mu) = -\nu(c)\psi(\tau,c,\mu) + \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c'^2}\psi(\tau,c'\mu')k(c',\mu':c,\mu)c'^2\,\mathrm{d}\mu'\,\mathrm{d}c'. \tag{29}$$

Here

$$k(c',\mu':c,\mu) = (1-\mu'^2)\sum_{n=1}^{\infty}\Pi_n(\mu')\Pi_n(\mu)k_n(c',c) \tag{30}$$

with

$$\Pi_n(\mu) = \Big[\frac{2n+1}{2n(n+1)}\Big]^{1/2}\frac{\mathrm{d}}{\mathrm{d}\mu}P_n(\mu), \quad n \geq 1. \tag{31}$$

Note that

$$\int_{-1}^1 (1-\mu^2)\Pi_n(\mu)\Pi_{n'}(\mu)\,\mathrm{d}\mu = \delta_{n,n'}. \tag{32}$$

For the two considered half-space problems, Eq. (15) yields

$$\psi(0,c,\mu) - (1-\alpha)\psi(0,c,-\mu) = 0 \tag{33}$$

for $\mu \in (0,1]$ and all c. For the problems defined for finite channels, we note that in addition to a symmetry relation (to be defined) between $\psi(\tau,c,\mu)$ and $\psi(-\tau,c,-\mu)$, we find from Eq. (18) that $\psi(\tau,c,\mu)$ must satisfy the boundary condition

$$\psi(-a,c,\mu) - (1-\alpha)\psi(-a,c,-\mu) = F^*(c,\mu) \tag{34}$$

for $\mu \in (0,1]$ and all c. Here

$$F^*(c,\mu) = \frac{1}{\pi}(1-\mu^2)^{-1/2}\int_0^{2\pi} F(\boldsymbol{c})\cos\chi\,\mathrm{d}\chi. \tag{35}$$

Having given a general introduction to the class of flow problems we intend to solve here, we can continue with more specific formulations.

4. The problems

While the problems we solve here have much in common, there are naturally some differences, and so we now can be more specific in our mathematical statements of the problems.

4.1 Kramers' problem

For Kramers' problem we consider that $h(\tau,\boldsymbol{c})$ represents a perturbation from an absolute Maxwellian distribution, and so we express the velocity distribution function as

$$f(\tau,\boldsymbol{c}) = f_0(c)[1+h(\tau,\boldsymbol{c})] \tag{36}$$

where

$$f_0(c) = n_0\big[m/(2\pi kT_0)\big]^{3/2}\mathrm{e}^{-c^2}. \tag{37}$$

Our formulation of this problem does not have a driving term in Eq. (1), and so we seek a solution of

$$c\mu\frac{\partial}{\partial\tau}\psi(\tau,c,\mu) = \varepsilon L^*\{\psi\}(\tau,c,\mu) \tag{38}$$

that satisfies the boundary condition

$$\psi(0,c,\mu) - (1-\alpha)\psi(0,c,-\mu) = 0, \tag{39}$$

for $\mu \in (0,1]$ and all c. Since Eqs. (38) and (39) are both homogeneous, there can be no flow unless we require $\psi(\tau,c,\mu)$ to diverge as τ tends to infinity. In Ref. [16] some solutions to Eq. (1) that are linear in τ were listed. We can use one of those solutions in order to write

$$\psi(\tau,c,\mu) = \mathcal{K}\{\psi_*(\tau,c,\mu) + \frac{2}{\varepsilon}[c\varepsilon\tau - \mu B(c)]\} \tag{40}$$

where $B(c) = c^2 b(c)$ is defined as a solution of Eq. (9), $\mathcal{K}$ is a normalizing constant and $\psi_*(\tau,c,\mu)$ is a bounded (as τ tends to infinity) solution of Eq. (38). Substituting Eq. (40) into Eq. (39), we find the boundary condition to be satisfied by $\psi_*(\tau,c,\mu)$, *viz.*

$$\psi_*(0,c,\mu) - (1-\alpha)\psi_*(0,c,-\mu) = \frac{2}{\varepsilon}(2-\alpha)\mu B(c), \tag{41}$$

for $\mu \in (0,1]$ and all c. Once we have found $\psi_*(\tau,c,\mu)$ we can compute the desired bulk velocity and heat-flow profiles from

$$u(\tau) = \mathcal{K}\big\{\tau + \frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\psi_*(\tau,c,\mu)c^3(1-\mu^2)\,\mathrm{d}\mu\,\mathrm{d}c\big\} \tag{42}$$

and

$$q(\tau) = \mathcal{K}\big\{\frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\psi_*(\tau,c,\mu)c^3(c^2-5/2)(1-\mu^2)\,\mathrm{d}\mu\,\mathrm{d}c\big\}. \tag{43}$$

4.2 The half-space thermal-creep problem

For the case of thermal creep, the flow is caused by a constant temperature gradient in a direction parallel to the wall, and so it is helpful to linearize about a local Maxwellian rather than the absolute Maxwellian as was done in Eqs. (36) and (37). We follow Williams [17] and express the velocity distribution function as

$$f(\tau,\eta,\boldsymbol{c}) = f_0(c)\{1 + [(c^2-3/2)K_\eta + R_\eta]\eta + h(\tau,\boldsymbol{c})\} \tag{44}$$

where $f_0(c)$ is given by Eq. (37) and we have expressed the imposed temperature and density variations as

$$T(\eta) = T_0(1 + K_\eta\eta) \tag{45}$$

and

$$n(\eta) = n_0(1 + R_\eta \eta). \tag{46}$$

We continue to use T_0 and n_0 as convenient reference values of the temperature and density, η is used to define (in terms of the mean-free path l) the direction of flow and K_η and R_η are the constant gradients (in dimensionless units) of the temperature and density. For the problem of thermal creep we take $K_\eta = -R_\eta$, introduce $k_{\rm T} = K_\eta$ and consider the balance equation

$$S^*(c,\mu) + c\mu \frac{\partial}{\partial \tau}\psi(\tau, c, \mu) = \varepsilon L^*\{\psi\}(\tau, c, \mu) \tag{47}$$

where the operator L^* is defined by Eq. (29) and where

$$S^*(c,\mu) = c(c^2 - 5/2)k_{\rm T}. \tag{48}$$

For the case of a mixture of specular and diffuse reflection at the wall, the solution of Eq. (47) that we seek should satisfy the boundary condition

$$\psi(0, c, \mu) - (1 - \alpha)\psi(0, c, -\mu) = 0, \tag{49}$$

for $\mu \in (0, 1]$ and all c. Continuing, we now impose the normalization $k_{\rm T} = 1$ and write

$$\psi(\tau, c, \mu) = \psi_*(\tau, c, \mu) + \psi_{\rm ps}(\tau, c, \mu) \tag{50}$$

where

$$\psi_{\rm ps}(\tau, c, \mu) = -A(c)/\varepsilon \tag{51}$$

is a particular solution of Eq. (47), where $A(c) = ca(c)$ is a solution of Eqs. (10) and where $\psi_*(\tau, c, \mu)$ is a bounded (as τ tends to infinity) solution of the homogeneous version of Eq. (47). Substituting Eq. (50) into Eq. (49), we find the boundary condition

$$\psi_*(0, c, \mu) - (1 - \alpha)\psi_*(0, c, -\mu) = \alpha A(c)/\varepsilon, \tag{52}$$

for $\mu \in (0, 1]$ and $c \in [0, \infty)$. The velocity profile and the heat-flow profile can, after we note Eq. (10b), now be expressed as

$$u(\tau) = \frac{1}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 {\rm e}^{-c^2} \psi_*(\tau, c, \mu) c^3 (1 - \mu^2)\, {\rm d}\mu\, {\rm d}c \tag{53}$$

and

$$q(\tau) = -\frac{5\varepsilon_{\rm t}}{4\varepsilon} + \frac{1}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 {\rm e}^{-c^2} \psi_*(\tau, c, \mu) c^3 (c^2 - 5/2)(1 - \mu^2)\, {\rm d}\mu\, {\rm d}c \tag{54}$$

where $\varepsilon_{\rm t}$ is available from Eq.(14).

4.3 Couette flow

For the Couette-flow problem we continue to follow Ref. [17] and consider that $h(\tau, \boldsymbol{c})$ represents the perturbation from an absolute Maxwellian, and so we again express the distribution function as

$$f(\tau, \boldsymbol{c}) = f_0(c)[1 + h(\tau, \boldsymbol{c})] \tag{55}$$

where

$$f_0(c) = n_0\big[m/(2\pi k T_0)\big]^{3/2} \mathrm{e}^{-c^2}. \tag{56}$$

For this problem there is no driving term in Eq. (1), and so

$$S^*(c, \mu) = 0. \tag{57}$$

In addition, we consider that the two plates (walls) are given velocities $\pm u_\mathrm{p}$, and so the known term in the boundary condition listed as Eq. (35) takes the form [17]

$$F^*(c, \mu) = 2\alpha c u_\mathrm{p}. \tag{58}$$

As a result of the wall velocities, we can make use of the (anti) symmetry condition

$$\psi(-\tau, c, -\mu) = -\psi(\tau, c, \mu) \tag{59}$$

for all τ, c and μ. For the problem of Couette flow we intend to compute, in addition to the quantities listed by Eqs. (24–26), the half-channel mass and heat-flow rates defined as

$$U = \frac{1}{2a^2} \int_0^a u(\tau)\, \mathrm{d}\tau \tag{60}$$

and

$$Q = \frac{1}{2a^2} \int_0^a q(\tau)\, \mathrm{d}\tau. \tag{61}$$

4.4. Poiseuille and thermal-creep flow in a channel

The problems of Poiseuille flow and thermal-creep flow in a plane channel have much in common, and so we find it convenient to consider the two problems formulated together. Here the flow is caused by a constant pressure gradient (Poiseuille flow) and a constant temperature gradient (thermal-creep flow) in a direction parallel to the walls, and so it is helpful to linearize about a local Maxwellian rather than the absolute Maxwellian as was done in Eqs. (36) and (37). We express the distribution function as [17]

$$f(\tau, \eta, \boldsymbol{c}) = f_0(c)\{1 + [(c^2 - 3/2)K_\eta + R_\eta]\eta + h(\tau, \boldsymbol{c})\} \tag{62}$$

where $f_0(c)$ is given by Eq. (37) and we have expressed the imposed temperature and density variations as

$$T(\eta) = T_0(1 + K_\eta \eta) \tag{63a}$$

and

$$n(\eta) = n_0(1 + R_\eta \eta). \tag{63b}$$

We continue to use T_0 and n_0 as convenient reference values of the temperature and density, η is used to define (in terms of the mean-free path l) the direction of flow and K_η and R_η are the constant gradients (in dimensionless units) of the temperature and density. Now, as noted by Williams [17], the problem of Poiseuille flow has $K_\eta = 0$ and R_η arbitrary, while thermal-creep flow is defined by $K_\eta = -R_\eta$, with R_η arbitrary. To distinguish the two problems we use subscript labels P and T, and so we consider that the defining equation for $h(\tau, \boldsymbol{c})$ is the inhomogeneous form given by Eq. (1) with

$$S(\boldsymbol{c}) = c(1 - \mu^2)^{1/2} \cos\chi[(c^2 - 5/2)k_\mathrm{T} + k_\mathrm{P}], \tag{64}$$

where $k_\mathrm{T} = K_\eta$ and $k_\mathrm{P} = R_\eta + K_\eta$. In this work Poiseuille flow is defined by $k_\mathrm{P} = 1$ and $k_\mathrm{T} = 0$, while $k_\mathrm{T} = 1$ and $k_\mathrm{P} = 0$ defines the case of thermal-creep flow. Making use again of the definition introduced in Eq. (23), we find that here we must solve

$$S^*(c, \mu) + c\mu \frac{\partial}{\partial \tau}\psi(\tau, c, \mu) = \varepsilon L^*\{\psi\}(\tau, c, \mu) \tag{65}$$

where the operator L^* is defined by Eq. (29) and where

$$S^*(c, \mu) = c[(c^2 - 5/2)k_\mathrm{T} + k_\mathrm{P}]. \tag{66}$$

For the Poiseuille and thermal-creep problems the wall velocity u_p is zero, and so we seek a solution of Eq. (65) that satisfies the symmetry condition

$$\psi(-\tau, c, -\mu) = \psi(\tau, c, \mu), \tag{67}$$

for all τ, c and μ, and the boundary condition

$$\psi(-a, c, \mu) - (1 - \alpha)\psi(-a, c, -\mu) = 0, \tag{68}$$

for $\mu \in (0, 1]$ and all c. For the two flow problems defined in this section, we compute the velocity and heat-flow profiles

$$u(\tau) = \frac{1}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \psi(\tau, c, \mu) c^3 (1 - \mu^2)\, \mathrm{d}\mu\, \mathrm{d}c \tag{69}$$

and

$$q(\tau) = \frac{1}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \psi(\tau, c, \mu)(c^2 - 5/2) c^3 (1 - \mu^2)\, \mathrm{d}\mu\, \mathrm{d}c. \tag{70}$$

We also seek the full-channel mass and heat-flow rates defined as

$$U = \frac{1}{2a^2} \int_{-a}^a u(\tau)\, \mathrm{d}\tau \tag{71}$$

and

$$Q = \frac{1}{2a^2} \int_{-a}^a q(\tau)\, \mathrm{d}\tau. \tag{72}$$

Now, since we wish to base our ADO solution on the homogeneous form of Eq. (65), we write

$$\psi(\tau, c, \mu) = \psi_*(\tau, c, \mu) + \psi_{\rm ps}(\tau, c, \mu) \tag{73}$$

where $\psi_{\rm ps}(\tau, c, \mu)$ is a particular solution (that has the correct symmetry) of Eq. (65) and $\psi_*(\tau, c, \mu)$ is a solution of the homogeneous version of Eq. (65) that has the symmetry property

$$\psi_*(-\tau, c, -\mu) = \psi_*(\tau, c, \mu), \tag{74}$$

for all τ, c and μ, and that satisfies the boundary condition

$$\psi_*(-a, c, \mu) - (1-\alpha)\psi_*(-a, c, -\mu) = \mathcal{R}(c, \mu), \tag{75}$$

for $\mu \in (0, 1]$ and all c. Considering that we have found the required particular solution, we can write the known term in Eq. (75) as

$$\mathcal{R}(c, \mu) = (1-\alpha)\psi_{\rm ps}(-a, c, -\mu) - \psi_{\rm ps}(-a, c, \mu). \tag{76}$$

In regard to particular solutions, we note that for the thermal-creep problem the particular solution given by Eq. (51) can also be used here. On the other hand, we quote from Refs. [12,15,16] and express a particular solution for Poiseuille flow as

$$\psi_{\rm ps}(\tau, c, \mu) = \{c(\varepsilon\tau)^2 - c(\varepsilon a)^2 - 2B(c)\varepsilon\tau\mu + D(c)/5 + E(c)(5\mu^2-1)/5\}/(\varepsilon\varepsilon_{\rm p}). \tag{77}$$

Here $A(c)$ and $B(c)$ are the solutions of the Chapman-Enskog equations for heat conduction and viscosity and $D(c)$ and $E(c)$ are solutions of the so-called [12,18] Burnett equations. More specifically $D(c)$ is a solution of

$$\nu(c)D(c) - \int_0^\infty {\rm e}^{-c'^2} D(c')k_1(c', c)c'^2\,{\rm d}c' = 2cB(c) - 5c\varepsilon_{\rm p} \tag{78a}$$

subject to the normalization condition

$$\int_0^\infty {\rm e}^{-c^2} D(c)c^3\,{\rm d}c = 0, \tag{78b}$$

and $E(c)$ is a solution of

$$\nu(c)E(c) - \int_0^\infty {\rm e}^{-c'^2} E(c')k_3(c', c)c'^2\,{\rm d}c' = 2cB(c). \tag{79}$$

Here, as with Eqs. (9) and (10), $k_1(c', c)$ and $k_3(c', c)$ are component kernel functions from the Pekeris theory [6,10,11].

Taking note of Eq. (73) and adding the subscripts P and T, respectively, we find we can express the desired bulk velocity and heat-flow profiles in terms of the final unknown $\psi_*(\tau, c, \mu)$; first for Poiseuille flow we can write

$$u_{\rm P}(\tau) = \frac{\varepsilon}{2\varepsilon_{\rm p}}(\tau^2 - a^2) + \frac{1}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 {\rm e}^{-c^2} \psi_*(\tau, c, \mu)c^3(1-\mu^2)\,{\rm d}\mu\,{\rm d}c \tag{80}$$

and

$$q_{\rm P}(\tau) = \frac{4d_5}{15\varepsilon\varepsilon_{\rm p}} + \frac{1}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 {\rm e}^{-c^2} \psi_*(\tau, c, \mu)(c^2 - 5/2)c^3(1-\mu^2)\,{\rm d}\mu\,{\rm d}c. \quad (81)$$

Here

$$d_5 = \frac{1}{\pi^{1/2}} \int_0^\infty {\rm e}^{-c^2} D(c) c^5\,{\rm d}c. \quad (82)$$

The equivalent results for thermal-creep flow in a plane channel are

$$u_{\rm T}(\tau) = \frac{1}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 {\rm e}^{-c^2} \psi_*(\tau, c, \mu)c^3(1-\mu^2)\,{\rm d}\mu\,{\rm d}c \quad (83)$$

and

$$q_{\rm T}(\tau) = -\frac{5\varepsilon_{\rm t}}{4\varepsilon} + \frac{1}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 {\rm e}^{-c^2} \psi_*(\tau, c, \mu)(c^2 - 5/2)c^3(1-\mu^2)\,{\rm d}\mu\,{\rm d}c. \quad (84)$$

Once the velocity and heat-flow profiles are established we can compute the flow rates from

$$U_{\rm P} = \frac{1}{2a^2} \int_{-a}^a u_{\rm P}(\tau)\,{\rm d}\tau \quad \text{and} \quad Q_{\rm P} = \frac{1}{2a^2} \int_{-a}^a q_{\rm P}(\tau)\,{\rm d}\tau, \quad (85{\rm a,b})$$

for Poiseuille flow, and

$$U_{\rm T} = \frac{1}{2a^2} \int_{-a}^a u_{\rm T}(\tau)\,{\rm d}\tau \quad \text{and} \quad Q_{\rm T} = \frac{1}{2a^2} \int_{-a}^a q_{\rm T}(\tau)\,{\rm d}\tau \quad (86{\rm a,b})$$

for thermal-creep flow in a channel.

5. A polynomial representation

In Ref. [3] a polynomial expansion technique and an ADO (analytical discrete-ordinates) method [4] were used to solve a version of the temperature-jump problem that was based on the linearized Boltzmann equation. Here, in regard to flow problems, we must solve a balance equation that results from a projection of the original Boltzmann equation that differs from that used for the temperature-jump problem; however much of what was done in Ref. [3] can be used in this work. And so we seek a solution of

$$[c\mu\frac{\partial}{\partial\tau} + \varepsilon\nu(c)]\psi_*(\tau, c, \mu) = \varepsilon \int_0^\infty \int_{-1}^1 {\rm e}^{-c'^2} \psi_*(\tau, c', \mu')k(c', \mu' : c, \mu)c'^2\,{\rm d}\mu'\,{\rm d}c' \quad (87)$$

that satisfies the boundary condition

$$\psi_*(\tau_*, c, \mu) - (1-\alpha)\psi_*(\tau_*, c, -\mu) = \mathcal{R}(c, \mu) \quad (88)$$

for $\mu \in (0, 1]$ and all c. Here $\mathcal{R}(c, \mu)$ is considered a known quantity. To compact our presentation we use $\tau_* = 0$ for half-space problems, whereas of flow in finite

channels we take $\tau_* = -a$. In addition to a symmetry condition such as given in Eqs. (59) and (67) for channel flow, we will require for half-space problems that the solution $\psi_*(\tau, c, \mu)$ be bounded as τ tends to infinity. The scattering kernel $k(c', \mu' : c, \mu)$ is given by Eq. (30), but we now make our first approximation: we truncate Eq. (30) and write

$$k(c', \mu' : c, \mu) = (1 - \mu'^2) \sum_{l=1}^{L} \Pi_l(\mu')\Pi_l(\mu) k_l(c', c) \tag{89}$$

where the polynomials $\Pi_l(\mu)$ are defined by Eq. (31) and the component functions $k_l(c', c)$ are those of Pekeris [6]. At this point we approximate the required solution by a representation in terms of Legendre polynomials, *viz.*

$$\psi_*(\tau, c, \mu) = \sum_{k=0}^{K} P_k(2\mathrm{e}^{-c} - 1) g_k(\tau, \mu) \tag{90}$$

where the functions $g_k(\tau, \mu)$ are to be determined. We now substitute Eq. (90) into Eq. (87), multiply the resulting equation by

$$W_i(c) = c^2 \mathrm{e}^{-c^2} P_i(2\mathrm{e}^{-c} - 1), \quad i = 0, 1, 2, ..., K, \tag{91}$$

and integrate over all c to obtain the coupled system

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{A}\boldsymbol{G}(\tau, \mu) + \varepsilon \boldsymbol{S}\boldsymbol{G}(\tau, \mu) = \varepsilon \sum_{l=1}^{L} \boldsymbol{B}_l \Pi_l(\mu) \int_{-1}^{1} (1 - \mu'^2) \Pi_l(\mu') \boldsymbol{G}(\tau, \mu') \, \mathrm{d}\mu'. \tag{92}$$

Here the $K+1$ vector-valued function $\boldsymbol{G}(\tau, \mu)$ has components $g_k(\tau, \mu)$ and the $(K+1) \times (K+1)$ constants are given by

$$\boldsymbol{A} = \int_0^\infty \mathrm{e}^{-c^2} \boldsymbol{P}^{\mathrm{T}}(c) \boldsymbol{P}(c) c^3 \, \mathrm{d}c, \tag{93}$$

$$\boldsymbol{S} = \int_0^\infty \mathrm{e}^{-c^2} \boldsymbol{P}^{\mathrm{T}}(c) \boldsymbol{P}(c) \nu(c) c^2 \, \mathrm{d}c \tag{94}$$

and

$$\boldsymbol{B}_l = \int_0^\infty \int_0^\infty \mathrm{e}^{-c'^2} \mathrm{e}^{-c^2} k_l(c', c) \boldsymbol{P}^{\mathrm{T}}(c') \boldsymbol{P}(c) c'^2 c^2 \, \mathrm{d}c' \, \mathrm{d}c \tag{95}$$

where the superscript T is used to denote the transpose operation, and where

$$\boldsymbol{P}(c) = \left[P_0(2\mathrm{e}^{-c} - 1), P_1(2\mathrm{e}^{-c} - 1), \cdots, P_K(2\mathrm{e}^{-c} - 1)\right]. \tag{96}$$

We note, since $k_l(c', c) = k_l(c, c')$, that the matrices $\boldsymbol{B}_l$ are symmetric. We note also that a computation of the matrices listed as Eq. (95) will require some care to do well; however, an evaluation of all the input matrices $\boldsymbol{A}$, $\boldsymbol{S}$ and $\boldsymbol{B}_l$ can be done once only and stored for later use. Some details of these initially required computations will be given in a subsequent section of this work.

Now in regard to the boundary condition subject to which we must solve Eq. (92), we use Eq. (90) in Eq. (88) and then multiply the resulting equation by $W_i(c)$ and integrate over all c to obtain

$$\boldsymbol{F}[\boldsymbol{G}(\tau_*,\mu)-(1-\alpha)\boldsymbol{G}(\tau_*,-\mu)]=\boldsymbol{T}(\mu) \tag{97}$$

for $\mu\in(0,1]$. Here the additional input constants are

$$\boldsymbol{F}=\int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{P}^{\mathrm{T}}(c)\boldsymbol{P}(c)c^2\,\mathrm{d}c \tag{98}$$

and

$$\boldsymbol{T}(\mu)=\int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{P}^{\mathrm{T}}(c)\mathcal{R}(c,\mu)c^2\,\mathrm{d}c. \tag{99}$$

And so, we now must solve Eq. (92) subject to the boundary condition given as Eq. (97); however, in order to make use of a previously reported [19] ADO solution of a multigroup neutron transport problem, we multiply Eq. (92) by $\boldsymbol{A}^{-1}$ and Eq. (97) by $\boldsymbol{F}^{-1}$ to obtain the final forms we solve, *viz.*

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{G}(\tau,\mu)+\varepsilon\boldsymbol{\Sigma}\boldsymbol{G}(\tau,\mu)=\varepsilon\sum_{l=1}^{L}\boldsymbol{C}_l\Pi_l(\mu)\int_{-1}^{1}(1-\mu'^2)\Pi_l(\mu')\boldsymbol{G}(\tau,\mu')\,\mathrm{d}\mu' \tag{100}$$

and

$$\boldsymbol{G}(\tau_*,\mu)-(1-\alpha)\boldsymbol{G}(\tau_*,-\mu)=\boldsymbol{Q}(\mu) \tag{101}$$

for $\mu\in(0,1]$. Here

$$\boldsymbol{\Sigma}=\boldsymbol{A}^{-1}\boldsymbol{S},\quad \boldsymbol{C}_l=\boldsymbol{A}^{-1}\boldsymbol{B}_l\quad\text{and}\quad \boldsymbol{Q}(\mu)=\boldsymbol{F}^{-1}\boldsymbol{T}(\mu). \tag{102a,b,c}$$

And so now we continue by developing our analytical discrete-ordinates solution of the transport problem defined by Eqs. (100) and (101).

6. An analytical discrete-ordinates solution

Since our discrete-ordinates solution of Eqs. (100) and (101) follows closely work previously reported [3,19], we can be brief here. We begin by using a "half-range" quadrature scheme to approximate the integral term in Eq. (100), and so we write

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{G}(\tau,\mu)+\varepsilon\boldsymbol{\Sigma}\boldsymbol{G}(\tau,\mu)=\varepsilon\sum_{l=1}^{L}\Pi_l(\mu)\boldsymbol{C}_l\sum_{n=1}^{N}(1-\mu_n^2)w_n\boldsymbol{G}_{l,n}(\tau) \tag{103}$$

where to compact our notation we have introduced

$$\boldsymbol{G}_{l,n}(\tau)=\Pi_l(\mu_n)[\boldsymbol{G}(\tau,\mu_n)+(-1)^l\boldsymbol{G}(\tau,-\mu_n)]\,. \tag{104}$$

Here the N quadrature points $\{\mu_n\}$ and the N weights $\{w_n\}$ are defined for use on the integration interval $[0,1]$. Equation (103) has separable exponential solutions, so we use ν as a separation constant and substitute

$$\boldsymbol{G}(\tau,\mu)=\boldsymbol{\Phi}(\nu,\mu)\mathrm{e}^{-\varepsilon\tau/\nu} \tag{105}$$

into that equation to find

$$\big[\boldsymbol{\Sigma} - (\mu/\nu)\boldsymbol{I}\big]\boldsymbol{\Phi}(\nu,\mu) = \sum_{l=1}^{L} \Pi_l(\mu)\boldsymbol{C}_l \sum_{n=1}^{N} w_n(1-\mu_n^2)\boldsymbol{\Phi}_{l,n}(\nu) \tag{106}$$

where $\boldsymbol{I}$ is the identity matrix and

$$\boldsymbol{\Phi}_{l,n}(\nu) = \Pi_l(\mu_n)[\boldsymbol{\Phi}(\nu,\mu_n) + (-1)^l\boldsymbol{\Phi}(\nu,-\mu_n)]. \tag{107}$$

If we now evaluate Eq. (106) at $\mu = \pm\mu_i$, for $i = 1, 2, ..., N$, then we can obtain

$$\big[\boldsymbol{D} - (1/\nu)\boldsymbol{M}\big]\boldsymbol{\Phi}_+(\nu) = \sum_{l=1}^{L} \boldsymbol{\Pi}_l\boldsymbol{C}_l\boldsymbol{G}_l(\nu) \tag{108}$$

and

$$\big[\boldsymbol{D} + (1/\nu)\boldsymbol{M}\big]\boldsymbol{\Phi}_-(\nu) = \sum_{l=1}^{L} (-1)^l\boldsymbol{\Pi}_l\boldsymbol{C}_l\boldsymbol{G}_l(\nu) \tag{109}$$

where

$$\boldsymbol{\Phi}_+(\nu) = \big[\boldsymbol{\Phi}^{\mathrm{T}}(\nu,\mu_1),\, \boldsymbol{\Phi}^{\mathrm{T}}(\nu,\mu_2),\, \cdots,\, \boldsymbol{\Phi}^{\mathrm{T}}(\nu,\mu_N)\big]^{\mathrm{T}} \tag{110a}$$

and

$$\boldsymbol{\Phi}_-(\nu) = \big[\boldsymbol{\Phi}^{\mathrm{T}}(\nu,-\mu_1),\, \boldsymbol{\Phi}^{\mathrm{T}}(\nu,-\mu_2),\, \cdots,\, \boldsymbol{\Phi}^{\mathrm{T}}(\nu,-\mu_N)\big]^{\mathrm{T}}. \tag{110b}$$

In addition, we have used the $J \times J$ matrices

$$\boldsymbol{M} = \mathrm{diag}\big\{\mu_1\boldsymbol{I}, \mu_2\boldsymbol{I}, ..., \mu_N\boldsymbol{I}\big\} \tag{111a}$$

and

$$\boldsymbol{D} = \mathrm{diag}\big\{\boldsymbol{\Sigma}, \boldsymbol{\Sigma}, ..., \boldsymbol{\Sigma}\big\} \tag{111b}$$

along with

$$\boldsymbol{G}_l(\nu) = \boldsymbol{\Pi}_l^{\mathrm{T}}\boldsymbol{W}[\boldsymbol{\Phi}_+(\nu) + (-1)^l\boldsymbol{\Phi}_-(\nu)]. \tag{112}$$

Note that we have introduced the composite dimension $J = N(K+1)$ and that here the matrix

$$\boldsymbol{W} = \mathrm{diag}\big\{w_1(1-\mu_1^2)\boldsymbol{I}, w_2(1-\mu_2^2)\boldsymbol{I}, ..., w_N(1-\mu_N^2)\boldsymbol{I}\big\} \tag{113a}$$

is also $J \times J$, while the matrices

$$\boldsymbol{\Pi}_l = \big[\Pi_l(\mu_1)\boldsymbol{I},\, \Pi_l(\mu_2)\boldsymbol{I}, \cdots, \Pi_l(\mu_N)\boldsymbol{I}\big]^{\mathrm{T}} \tag{113b}$$

are $J \times (K+1)$. We now let

$$\boldsymbol{U} = \boldsymbol{\Phi}_+(\nu) + \boldsymbol{\Phi}_-(\nu) \quad \text{and} \quad \boldsymbol{V} = \boldsymbol{\Phi}_+(\nu) - \boldsymbol{\Phi}_-(\nu) \tag{114a,b}$$

so that we can take the sum and the difference of Eqs. (108) and (109) to obtain

$$\boldsymbol{E}\boldsymbol{X} = \frac{1}{\nu}\boldsymbol{Y} \quad \text{and} \quad \boldsymbol{H}\boldsymbol{Y} = \frac{1}{\nu}\boldsymbol{X} \tag{115a,b}$$

where

$$\boldsymbol{E} = \big\{\boldsymbol{D} - \sum_{l=1}^{L} \boldsymbol{\Pi}_l \boldsymbol{C}_l [1 + (-1)^l] \boldsymbol{\Pi}_l^{\mathrm{T}} \boldsymbol{W}\big\} \boldsymbol{M}^{-1} , \tag{116a}$$

$$\boldsymbol{H} = \big\{\boldsymbol{D} - \sum_{l=1}^{L} \boldsymbol{\Pi}_l \boldsymbol{C}_l [1 - (-1)^l] \boldsymbol{\Pi}_l^{\mathrm{T}} \boldsymbol{W}\big\} \boldsymbol{M}^{-1} , \tag{116b}$$

$$\boldsymbol{X} = \boldsymbol{M}\boldsymbol{U} \quad \text{and} \quad \boldsymbol{Y} = \boldsymbol{M}\boldsymbol{V} . \tag{117a,b}$$

We can eliminate between Eqs. (117) to obtain the eigenvalue problems

$$\big(\boldsymbol{H}\boldsymbol{E}\big)\boldsymbol{X} = \lambda \boldsymbol{X} \quad \text{and} \quad \big(\boldsymbol{E}\boldsymbol{H}\big)\boldsymbol{Y} = \lambda \boldsymbol{Y} \tag{118a,b}$$

where $\lambda = 1/\nu^2$. We note that the required separation constants $\{\nu_j\}$ are readily available once we find the eigenvalues $\{\lambda_j\}$ defined by either of Eqs. (118). We choose to express our results in terms of the eigenvalues and eigenvectors defined by Eq. (118a). Continuing, we let λ_j and $\boldsymbol{X}(\lambda_j)$, for $j = 1, 2, \ldots, J$, denote the collection of eigenvalues and eigenvectors of Eq. (118a). The separation constants we require clearly occur in plus-minus pairs, and so letting ν_j , for the $j = 1, 2, \ldots, J$, denote the reciprocal of the positive square root of λ_j , we can use Eqs. (114), (115) and (117) to obtain

$$\boldsymbol{\Phi}_+(\nu_j) = (1/2)\boldsymbol{M}^{-1}\big(\boldsymbol{I} + \nu_j \boldsymbol{E}\big)\boldsymbol{X}(\lambda_j) \tag{119a}$$

and

$$\boldsymbol{\Phi}_-(\nu_j) = (1/2)\boldsymbol{M}^{-1}\big(\boldsymbol{I} - \nu_j \boldsymbol{E}\big)\boldsymbol{X}(\lambda_j) \tag{119b}$$

for $j = 1, 2, \ldots, J$. We note that $\boldsymbol{I}$ in Eqs. (119) is the $J \times J$ identity matrix and that

$$\boldsymbol{\Phi}_+(-\nu_j) = \boldsymbol{\Phi}_-(\nu_j) , \tag{120}$$

and so at this point we have available all we require for defining our discrete-ordinates solution to Eq. (100). We therefore write

$$\boldsymbol{G}(\tau, \pm\mu_i) = \sum_{j=1}^{J} \big[A_j \boldsymbol{\Phi}(\nu_j, \pm\mu_i) \mathrm{e}^{-\varepsilon(\tau-\tau_*)/\nu_j} + B_j \boldsymbol{\Phi}(\nu_j, \mp\mu_i) \mathrm{e}^{\varepsilon(\tau+\tau_*)/\nu_j}\big], \tag{121}$$

for $i = 1, 2, ..., N$. Here the arbitrary constants $\{A_j\}$ and $\{B_j\}$ are to be determined by a boundary condition and (when appropriate) a symmetry condition for a given problem, and to be clear we note that the quantities $\boldsymbol{\Phi}(\nu_j, \mu_i)$ and $\boldsymbol{\Phi}(\nu_j, -\mu_i)$ are to be taken from the components of $\boldsymbol{\Phi}_\pm(\nu_j)$ that are available from Eqs. (119).

Now, as was anticipated, we have observed in our numerical work that one of the separation constants, say ν_1 , tends to infinity as the order J of the eigenvalue system is increased. And so our procedure is to ignore this separation constant in Eq. (121) and to use instead two of the exact solutions available from, say,

Ref. [16]. This means that we now write our discrete-ordinates version of Eq. (90) as

$$\psi_*(\tau, c, \pm\mu_i) = A_1 c + B_1[c\varepsilon\tau \mp \mu_i B(c)] + \boldsymbol{P}(c)\boldsymbol{G}_*(\tau, \pm\mu_i) \tag{122}$$

where $\boldsymbol{P}(c)$ is given by Eq. (96) and where

$$\boldsymbol{G}_*(\tau, \pm\mu_i) = \sum_{j=2}^{J} \left[A_j \boldsymbol{\Phi}(\nu_j, \pm\mu_i) \mathrm{e}^{-\varepsilon(\tau-\tau_*)/\nu_j} + B_j \boldsymbol{\Phi}(\nu_j, \mp\mu_i) \mathrm{e}^{\varepsilon(\tau+\tau_*)/\nu_j}\right]. \tag{123}$$

To complete our solution we must determine the constants $\{A_j\}$ and $\{B_j\}$ from a boundary condition and, when appropriate, a symmetry condition relevant to a specific problem.

7. Solutions to the problems

Having developed our polynomial expansion technique and the analytical discrete-ordinates method, we are now ready to solve the problems explicitly defined in Section 4 of this work.

7.1. Kramers' problem

Since we have a half-space problem, we must, in Eqs. (122) and (123), use $\tau_* = 0$ and set the coefficients $\{B_j\}$ equal to zero. And so noting Eqs. (41) and (88), we can write

$$\mathcal{R}(c, \mu) = (2/\varepsilon)(2 - \alpha)\mu B(c), \tag{124}$$

and therefore we can determine the constants $\{A_j\}$ from a system of linear algebraic equations we find from a form of Eq. (122) that has been modified to take into account the fact that one of the exact solutions used in Eq. (122) remains in our solution here. We write this linear system as

$$A_1 \boldsymbol{V}_1 + \sum_{j=2}^{J} A_j \boldsymbol{V}(\nu_j) = \boldsymbol{R} \tag{125}$$

where

$$\boldsymbol{V}_1 = \alpha\left[\boldsymbol{E}_1^{\mathrm{T}}, \boldsymbol{E}_1^{\mathrm{T}}, ..., \boldsymbol{E}_1^{\mathrm{T}}\right]^{\mathrm{T}} \tag{126}$$

and, for $j = 2, 4, ...J$,

$$\boldsymbol{V}(\nu_j) = \left[\boldsymbol{\Gamma}_1^{\mathrm{T}}(\nu_j), \boldsymbol{\Gamma}_2^{\mathrm{T}}(\nu_j), ..., \boldsymbol{\Gamma}_N^{\mathrm{T}}(\nu_j)\right]^{\mathrm{T}}. \tag{127}$$

Here

$$\boldsymbol{\Gamma}_i(\nu_j) = \boldsymbol{\Phi}(\nu_j, \mu_i) - (1 - \alpha)\boldsymbol{\Phi}(\nu_j, -\mu_i) \tag{128}$$

and

$$\boldsymbol{R} = \left[\boldsymbol{Q}^{\mathrm{T}}(\mu_1), \boldsymbol{Q}^{\mathrm{T}}(\mu_2), ..., \boldsymbol{Q}^{\mathrm{T}}(\mu_N)\right]^{\mathrm{T}} \tag{129}$$

where $\boldsymbol{Q}(\mu)$ is given by Eqs. (99), (102c) and (124). In addition

$$\boldsymbol{E}_1 = \boldsymbol{F}^{-1}\boldsymbol{P}_1^{\mathrm{T}} \tag{130}$$

where, in general,

$$\boldsymbol{P}_n = \int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{P}(c)c^{n+2}\,\mathrm{d}c. \tag{131}$$

Once we have solved the linear system given as Eq. (125) we can use Eq. (122) in Eqs. (42) and (43) to obtain the bulk velocity and heat-flow profiles for Kramers' problem; we find, after adding the subscript P and introducing the normalization $\mathcal{K} = 1$,

$$u_{\mathrm{P}}(\tau) = \tau + \zeta_{\mathrm{P}} + \sum_{j=2}^{J} A_j N_j \mathrm{e}^{-\varepsilon\tau/\nu_j} \tag{132}$$

and

$$q_{\mathrm{P}}(\tau) = \sum_{j=2}^{J} A_j M_j \mathrm{e}^{-\varepsilon\tau/\nu_j} \tag{133}$$

where

$$N_j = \pi^{-1/2}\boldsymbol{P}_1\boldsymbol{N}(\nu_j) \tag{134a}$$

and

$$M_j = \pi^{-1/2}[\boldsymbol{P}_3 - (5/2)\boldsymbol{P}_1]\boldsymbol{N}(\nu_j). \tag{134b}$$

Here

$$\boldsymbol{N}(\nu_j) = \sum_{n=1}^{N} w_n(1-\mu_n^2)[\boldsymbol{\Phi}(\nu_j,\mu_n) + \boldsymbol{\Phi}(\nu_j,-\mu_n)]. \tag{135}$$

To complete our discussion of Kramers' problem, we note that

$$\zeta_{\mathrm{P}} = A_1/2 \tag{136}$$

is the viscous-slip coefficient.

7.2. The half-space thermal-creep problem

As for Kramers' problem we must, in Eqs. (122) and (123), use $\tau_* = 0$ and set the coefficients $\{B_j\}$ equal to zero. And so, noting that for this problem (with the normalization $k_{\mathrm{T}} = 1$)

$$\mathcal{R}(c,\mu) = (\alpha/\varepsilon)A(c), \tag{137}$$

we can determine the constants $\{A_j\}$ by solving a linear system that uses Eq. (137) instead of Eq. (124) to define the right-hand side of Eq. (125). Once we have solved the linear system we can use Eq. (122) in Eqs. (53) and (54) to obtain the bulk

velocity and heat-flow profiles for the half-space thermal-creep problem; we find, after adding the subscript T,

$$u_{\rm T}(\tau) = \zeta_{\rm T} + \sum_{j=2}^{J} A_j N_j {\rm e}^{-\varepsilon\tau/\nu_j} \tag{138}$$

and

$$q_{\rm T}(\tau) = -\frac{5\varepsilon_{\rm t}}{4\varepsilon} + \sum_{j=2}^{J} A_j M_j {\rm e}^{-\varepsilon\tau/\nu_j} \tag{139}$$

where we continue to make use of Eqs. (134) and (135). Here

$$\zeta_{\rm T} = A_1/2 \tag{140}$$

is the thermal-slip coefficient.

7.3. Couette flow

For this problem the flow is not driven by a inhomogeneous driving term in Eq. (1), and so we can construct our solution directly from Eqs. (122) and (123). Noting the (anti) symmetry condition listed as Eq. (59) and the fact the channel is defined by $\tau \in [-a, a]$, we see that, in regard to Eqs. (122) and (123), we can take $\tau_* = -a$, $A_1 = 0$, and $A_j = -B_j$ for $j = 2, 3, ..., J$. And so we express the solution we seek as

$$\psi(\tau, c, \pm\mu_i) = B_1[c\varepsilon\tau \mp \mu_i B(c)] + \boldsymbol{P}(c)\boldsymbol{G}_*(\tau, \pm\mu_i) \tag{141}$$

where now

$$\boldsymbol{G}_*(\tau, \pm\mu_i) = \sum_{j=2}^{J} B_j \big[\boldsymbol{\Phi}(\nu_j, \mp\mu_i){\rm e}^{-\varepsilon(a-\tau)/\nu_j} - \boldsymbol{\Phi}(\nu_j, \pm\mu_i){\rm e}^{-\varepsilon(a+\tau)/\nu_j}\big]. \tag{142}$$

We have seen that $\psi(\tau, c, \mu)$ should satisfy the boundary condition

$$\psi(-a, c, \mu) - (1-\alpha)\psi(-a, c, -\mu) = 2\alpha c u_{\rm p} \tag{143}$$

for $\mu \in (0, 1]$ and all c, and so we can substitute Eqs. (141) and (142) into Eq. (143) to find, after a projection against the functions listed in Eq. (91),

$$B_1 \boldsymbol{V}_1 + \sum_{j=2}^{J} B_j \boldsymbol{V}(\nu_j) = \boldsymbol{R} \tag{144}$$

where now, instead of Eq. (126), we

$$\boldsymbol{V}_1 = -\varepsilon a\alpha \big[\boldsymbol{E}_1^{\rm T}, \boldsymbol{E}_1^{\rm T}, ..., \boldsymbol{E}_1^{\rm T}\big]^{\rm T} - (2-\alpha)\big[\mu_1 \boldsymbol{B}^{\rm T}, \mu_2 \boldsymbol{B}^{\rm T}, ..., \mu_N \boldsymbol{B}^{\rm T}\big]^{\rm T}, \tag{145}$$

and, instead of Eq. (128), we have

$$\boldsymbol{\Gamma}_i(\nu_j) = [1 - \alpha + {\rm e}^{-2\varepsilon a/\nu_j}]\boldsymbol{\Phi}(\nu_j, -\mu_i) - [1 + (1-\alpha){\rm e}^{-2\varepsilon a/\nu_j}]\boldsymbol{\Phi}(\nu_j, \mu_i). \tag{146}$$

In addition we also have replaced Eq. (129) with

$$\boldsymbol{R} = 2\alpha u_{\rm p}\left[\boldsymbol{E}_1^{\rm T}, \boldsymbol{E}_1^{\rm T}, ..., \boldsymbol{E}_1^{\rm T}\right]^{\rm T}, \tag{147}$$

and finally

$$\boldsymbol{B} = \boldsymbol{F}^{-1}\int_0^\infty {\rm e}^{-c^2}\boldsymbol{P}^{\rm T}(c)B(c)c^2\,{\rm d}c. \tag{148}$$

Once we have solved the linear system listed as Eq. (144) to obtain the constants $\{B_j\}$, we can evaluate the desired quantities from Eqs. (24–26) and Eqs. (60) and (61). In this way we find

$$u(\tau) = \frac{\varepsilon\tau}{2}B_1 + \sum_{j=2}^{J} B_j N_j[{\rm e}^{-\varepsilon(a-\tau)/\nu_j} - {\rm e}^{-\varepsilon(a+\tau)/\nu_j}], \tag{149}$$

$$q(\tau) = \sum_{j=2}^{J} B_j M_j[{\rm e}^{-\varepsilon(a-\tau)/\nu_j} - {\rm e}^{-\varepsilon(a+\tau)/\nu_j}], \tag{150}$$

$$P_{xy} = -(1/4)\varepsilon_{\rm p}B_1, \tag{151}$$

$$U = \frac{\varepsilon}{8}B_1 + \frac{1}{2\varepsilon a^2}\sum_{j=2}^{J} B_j\nu_j N_j(1-{\rm e}^{-\varepsilon a/\nu_j})^2 \tag{152}$$

and

$$Q = \frac{1}{2\varepsilon a^2}\sum_{j=2}^{J} B_j\nu_j M_j(1-{\rm e}^{-\varepsilon a/\nu_j})^2 \tag{153}$$

where we continue to make use of Eqs. (134) and (135).

7.4. Poiseuille and thermal-creep flow in a channel

For these two problems the flow is driven by an inhomogeneous driving term in Eq. (1), and so, as already discussed, we require a particular solution as well as $\psi_*(\tau, c, \mu)$, which we can construct from Eqs. (122) and (123). Noting the symmetry condition listed as Eq. (67) and the fact the channel is defined by $\tau \in [-a, a]$, we see that, in regard to Eqs. (122) and (123), we can take $\tau_* = -a$, $B_1 = 0$, and $B_j = A_j$ for $j = 2, 3, ..., J$. And so we write

$$\psi_*(\tau, c, \pm\mu_i) = A_1 c + \boldsymbol{P}(c)\boldsymbol{G}_*(\tau, \pm\mu_i) \tag{154}$$

where now

$$\boldsymbol{G}_*(\tau, \pm\mu_i) = \sum_{j=2}^{J} A_j\left[\boldsymbol{\Phi}(\nu_j, \pm\mu_i){\rm e}^{-\varepsilon(a+\tau)/\nu_j} + \boldsymbol{\Phi}(\nu_j, \mp\mu_i){\rm e}^{-\varepsilon(a-\tau)/\nu_j}\right]. \tag{155}$$

From Eqs. (75) and (76) we see that $\psi_*(\tau, c, \mu)$ should satisfy the boundary condition listed as Eq. (88) where

$$\mathcal{R}(c,\mu) = -[2a(2-\alpha)/\varepsilon_{\rm p}]\mu B(c) - \alpha[D(c) + E(c)(5\mu^2 - 1)]/(5\varepsilon\varepsilon_{\rm p}) \tag{156a}$$

for Poiseuille flow ($k_{\rm P} = 1$ and $k_{\rm T} = 0$) and

$$\mathcal{R}(c,\mu) = (\alpha/\varepsilon)A(c) \tag{156b}$$

for thermal-creep flow ($k_{\rm P} = 0$ and $k_{\rm T} = 1$). To find the constants $\{A_j\}$, we substitute Eqs. (154) and (155) into Eq. (88) to find, after a projection against the functions listed in Eq. (91),

$$A_1 \boldsymbol{V}_1 + \sum_{j=2}^{J} A_j \boldsymbol{V}(\nu_j) = \boldsymbol{R} \tag{157}$$

where (again)

$$\boldsymbol{V}_1 = \alpha\big[\boldsymbol{E}_1^{\rm T}, \boldsymbol{E}_1^{\rm T}, ..., \boldsymbol{E}_1^{\rm T}\big]^{\rm T} \tag{158}$$

and, for $j = 2, 4, ...J$,

$$\boldsymbol{V}(\nu_j) = \big[\boldsymbol{\Gamma}_1^{\rm T}(\nu_j), \boldsymbol{\Gamma}_2^{\rm T}(\nu_j), ..., \boldsymbol{\Gamma}_N^{\rm T}(\nu_j)\big]^{\rm T}. \tag{159}$$

Now, instead of Eq. (128) or Eq. (146), we find

$$\boldsymbol{\Gamma}_i(\nu_j) = [1 - (1-\alpha){\rm e}^{-2\varepsilon a/\nu_j}]\boldsymbol{\Phi}(\nu_j, \mu_i) - [1 - \alpha - {\rm e}^{-2\varepsilon a/\nu_j}]\boldsymbol{\Phi}(\nu_j, -\mu_i). \tag{160}$$

Finally,

$$\boldsymbol{R} = \big[\boldsymbol{Q}^{\rm T}(\mu_1), \boldsymbol{Q}^{\rm T}(\mu_2), ..., \boldsymbol{Q}^{\rm T}(\mu_N)\big]^{\rm T} \tag{161}$$

where $\boldsymbol{Q}(\mu)$ is defined, for the two problems, by using each of Eqs. (156) in Eqs. (99) and (102c).

Once we have solved the linear system given as Eq. (157) we can use Eq. (154) in Eqs. (80), (81)and (85) to obtain the desired results for the case of Poiseuille flow, *viz.*

$$u_{\rm P}(\tau) = \frac{\varepsilon}{2\varepsilon_{\rm p}}(\tau^2 - a^2) + \frac{1}{2}A_1 + \sum_{j=2}^{J} A_j N_j[{\rm e}^{-\varepsilon(a+\tau)/\nu_j} + {\rm e}^{-\varepsilon(a-\tau)/\nu_j}], \tag{162}$$

$$q_{\rm P}(\tau) = \frac{4d_5}{15\varepsilon\varepsilon_{\rm p}} + \sum_{j=2}^{J} A_j M_j[{\rm e}^{-\varepsilon(a+\tau)/\nu_j} + {\rm e}^{-\varepsilon(a-\tau)/\nu_j}], \tag{163}$$

$$U_{\rm P}(\tau) = -\frac{a\varepsilon}{3\varepsilon_{\rm p}} + \frac{1}{2a}A_1 + \frac{1}{\varepsilon a^2}\sum_{j=2}^{J} A_j \nu_j N_j(1 - {\rm e}^{-2\varepsilon a/\nu_j}) \tag{164}$$

and

$$Q_{\rm P}(\tau) = \frac{4d_5}{15a\varepsilon\varepsilon_{\rm p}} + \frac{1}{\varepsilon a^2}\sum_{j=2}^{J} A_j \nu_j M_j(1 - {\rm e}^{-2\varepsilon a/\nu_j}) \tag{165}$$

where d_5, N_j and M_j are as given by Eqs. (82), (134) and (135). The equivalent results for thermal-creep flow in a channel are

$$u_{\rm T}(\tau) = \frac{1}{2}A_1 + \sum_{j=2}^{J} A_j N_j [{\rm e}^{-\varepsilon(a+\tau)/\nu_j} + {\rm e}^{-\varepsilon(a-\tau)/\nu_j}], \tag{166}$$

$$q_{\rm T}(\tau) = -\frac{5\varepsilon_{\rm t}}{4\varepsilon} + \sum_{j=2}^{J} A_j M_j [{\rm e}^{-\varepsilon(a+\tau)/\nu_j} + {\rm e}^{-\varepsilon(a-\tau)/\nu_j}], \tag{167}$$

$$U_{\rm T}(\tau) = \frac{1}{2a}A_1 + \frac{1}{\varepsilon a^2}\sum_{j=2}^{J} A_j \nu_j N_j (1 - {\rm e}^{-2\varepsilon a/\nu_j}) \tag{168}$$

and

$$Q_{\rm T}(\tau) = -\frac{5\varepsilon_{\rm t}}{4a\varepsilon} + \frac{1}{\varepsilon a^2}\sum_{j=2}^{J} A_j \nu_j M_j (1 - {\rm e}^{-2\varepsilon a/\nu_j}). \tag{169}$$

As our solutions for the five considered flow problems, all based on the linearized Boltzmann equation for rigid-sphere interactions, are established, we are ready to evaluate them for some data cases of interest.

8. Numerical results

Our solutions of the five flow problems, Kramers, half-space thermal creep, Couette, Poiseuille and thermal-creep flow in a finite channel, as based on the linearized Boltzmann equation for rigid-sphere interactions, involve various analytical and numerical approximations which must be mentioned. First of all, the infinite series in Eq. (30) has been truncated terms so as to yield the representation given by Eq. (89). To date, we have used at most $L = 8$ in the expansion. The Maple software package was used to obtain analytical expressions for the component kernels; however, because we observed some loss of accuracy when using these analytical expressions, we have included in our computation some asymptotic expansions for small values of c', when $c' < c$. Our next approximation is illustrated by Eq. (90) where $K + 1$ terms in a polynomial expansion was used to model the speed dependence (the c variable) of our solution. Then after a projection against the functions listed in Eq. (91), we obtained the coupled system of transport equations and boundary conditions listed as Eqs. (100) and (101). At this point we have introduced our analytical discrete-ordinates method by approximating the integral term in Eq. (100) by an N-point half-range quadrature scheme, *viz.* a standard Gauss-Legendre scheme mapped onto the interval [0,1]. Finally we have evaluated all the input parameters, for example Eqs. (93–95), by mapping an M-point Gauss-Legendre quadrature scheme onto the positive real axis. And so in this work we have four approximation parameters: $\{L, M, K, N\}$. In regard to linear algebra, we have used the driver program RG from the EISPACK collection

Table 1. Viscous-slip ($\varepsilon = \varepsilon_{\rm p}$) and thermal-slip ($\varepsilon = \varepsilon_{\rm t}$) coefficients

	$\zeta_{\rm P}$		$\zeta_{\rm T}$	
α	CES	LBE	CES	LBE
0.1	1.704462(1)	1.70478(1)	2.671726(–1)	2.65765(–1)
0.2	8.169615	8.17248	2.770231(–1)	2.74450(–1)
0.3	5.203049	5.20563	2.864184(–1)	2.82900(–1)
0.4	3.713778	3.71609	2.953902(–1)	2.91124(–1)
0.5	2.815562	2.81761	3.039673(–1)	2.99133(–1)
0.6	2.212984	2.21478	3.121761(–1)	3.06938(–1)
0.7	1.779429	1.78098	3.200405(–1)	3.14547(–1)
0.8	1.451586	1.45292	3.275826(–1)	3.21968(–1)
0.9	1.194279	1.19540	3.348226(–1)	3.29210(–1)
1.0	9.864009(–1)	9.87328(–1)	3.417790(–1)	3.36280(–1)

Table 2. The velocity profile $u_{\rm P}(\tau)$ for Kramers' problem ($\varepsilon = \varepsilon_{\rm p}$)

τ	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
0.0	1.6472(1)	4.7032	2.3851	1.4150	8.9295(–1)	7.1553(–1)
0.1	1.6771(1)	4.9753	2.6322	1.6386	1.0943	9.0630(–1)
0.2	1.6956(1)	5.1494	2.7960	1.7925	1.2389	1.0463
0.3	1.7111(1)	5.2982	2.9380	1.9282	1.3684	1.1729
0.4	1.7252(1)	5.4336	3.0686	2.0541	1.4899	1.2922
0.5	1.7383(1)	5.5606	3.1920	2.1740	1.6064	1.4071
0.6	1.7507(1)	5.6821	3.3105	2.2898	1.7195	1.5189
0.7	1.7627(1)	5.7994	3.4255	2.4025	1.8301	1.6284
0.8	1.7743(1)	5.9137	3.5379	2.5130	1.9388	1.7363
0.9	1.7857(1)	6.0256	3.6482	2.6218	2.0461	1.8429
1.0	1.7968(1)	6.1356	3.7568	2.7292	2.1523	1.9484
2.0	1.9023(1)	7.1839	4.7987	3.7649	3.1820	2.9752

[20] to solve our eigenvalue/eigenvector problem and packages from the LINPACK collection [21] to solve our linear systems. We note, in regard to Eq. (118a) and the fact that $\nu = 1/\lambda^{1/2}$, that we have not encountered any complex separation constants ($\nu_j, j = 2, 4, ..., J$) for the solution parameters used to date.

We list in Tables 1–5 our results for some typical half-space cases using

$$\varepsilon = \varepsilon_{\rm p} = 0.449027806... \tag{170a}$$

for Kramers' problem, and

$$\varepsilon = \varepsilon_{\rm t} = 0.679630049... \tag{170b}$$

for the half-space thermal-creep problem, to define mean-free paths based, respectively, on viscosity and on thermal conductivity. In Tables 6–9 we report our

Table 3. The heat-flow profile $q_{\rm P}(\tau)$ for Kramers' problem ($\varepsilon = \varepsilon_{\rm p}$)

τ	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
0.0	2.3959(–1)	2.0559(–1)	1.7417(–1)	1.4509(–1)	1.1813(–1)	1.0538(–1)
0.1	1.9023(–1)	1.6449(–1)	1.4042(–1)	1.1786(–1)	9.6680(–2)	8.6568(–2)
0.2	1.6365(–1)	1.4191(–1)	1.2148(–1)	1.0225(–1)	8.4101(–2)	7.5405(–2)
0.3	1.4360(–1)	1.2475(–1)	1.0699(–1)	9.0207(–2)	7.4326(–2)	6.6698(–2)
0.4	1.2735(–1)	1.1079(–1)	9.5136(–2)	8.0318(–2)	6.6261(–2)	5.9497(–2)
0.5	1.1373(–1)	9.9035(–2)	8.5131(–2)	7.1943(–2)	5.9409(–2)	5.3370(–2)
0.6	1.0206(–1)	8.8948(–2)	7.6523(–2)	6.4720(–2)	5.3487(–2)	4.8068(–2)
0.7	9.1923(–2)	8.0173(–2)	6.9021(–2)	5.8415(–2)	4.8307(–2)	4.3427(–2)
0.8	8.3039(–2)	7.2468(–2)	6.2424(–2)	5.2862(–2)	4.3739(–2)	3.9331(–2)
0.9	7.5193(–2)	6.5655(–2)	5.6584(–2)	4.7939(–2)	3.9685(–2)	3.5694(–2)
1.0	6.8224(–2)	5.9596(–2)	5.1385(–2)	4.3553(–2)	3.6069(–2)	3.2448(–2)
2.0	2.7542(–2)	2.4123(–2)	2.0852(–2)	1.7718(–2)	1.4709(–2)	1.3248(–2)

Table 4. The velocity profile $u_{\rm T}(\tau)$ for the half-space thermal-creep problem ($\varepsilon = \varepsilon_{\rm t}$)

τ	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
0.0	2.3877(–1)	2.0477(–1)	1.7337(–1)	1.4432(–1)	1.1741(–1)	1.0469(–1)
0.1	2.4634(–1)	2.2627(–1)	2.0732(–1)	1.8938(–1)	1.7238(–1)	1.6421(–1)
0.2	2.5011(–1)	2.3715(–1)	2.2478(–1)	2.1296(–1)	2.0163(–1)	1.9614(–1)
0.3	2.5279(–1)	2.4491(–1)	2.3731(–1)	2.2996(–1)	2.2283(–1)	2.1935(–1)
0.4	2.5484(–1)	2.5089(–1)	2.4699(–1)	2.4313(–1)	2.3931(–1)	2.3742(–1)
0.5	2.5648(–1)	2.5567(–1)	2.5474(–1)	2.5369(–1)	2.5256(–1)	2.5196(–1)
0.6	2.5782(–1)	2.5957(–1)	2.6108(–1)	2.6236(–1)	2.6344(–1)	2.6391(–1)
0.7	2.5892(–1)	2.6282(–1)	2.6636(–1)	2.6958(–1)	2.7250(–1)	2.7387(–1)
0.8	2.5985(–1)	2.6554(–1)	2.7079(–1)	2.7564(–1)	2.8013(–1)	2.8225(–1)
0.9	2.6064(–1)	2.6784(–1)	2.7454(–1)	2.8078(–1)	2.8660(–1)	2.8937(–1)
1.0	2.6131(–1)	2.6981(–1)	2.7774(–1)	2.8516(–1)	2.9212(–1)	2.9544(–1)
2.0	2.6456(–1)	2.7935(–1)	2.9333(–1)	3.0657(–1)	3.1912(–1)	3.2517(–1)

results, all based on using $\varepsilon = \varepsilon_{\rm p}$, for Couette flow, and Tables 10–14 are devoted to Poiseuille flow and thermal-creep flow in a plane channel. Note that here we use $\varepsilon = \varepsilon_{\rm p}$ for both Poiseuille flow and thermal-creep flow. While we believe our results to be correct to within one unit in last digit given, we have no proof of the accuracy. However, we have found our solution to be stable with regard to changes in the approximation parameters $\{L, M, K, N\}$. Although we have made no special effort to find the minimum values of $\{L, M, K, N\}$ to achieve a given accuracy, we have typically used the values $\{8, 200, 40, 30\}$. In addition to the fact that our solution appears to be stable with respect to the approximation parameters, we have looked at the special cases of (i) the BGK model and (ii) the CES model. Results for these two models were obtained simply by replacing

Table 5. The heat-flow profile $-q_{\mathrm{T}}(\tau)$ for the half-space thermal-creep problem ($\varepsilon = \varepsilon_{\mathrm{t}}$)

τ	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
0.0	1.1662	1.0071	8.5846(–1)	7.1946(–1)	5.8921(–1)	5.2713(–1)
0.1	1.1930	1.0838	9.8034(–1)	8.8227(–1)	7.8910(–1)	7.4422(–1)
0.2	1.2056	1.1202	1.0390	9.6162(–1)	8.8780(–1)	8.5212(–1)
0.3	1.2141	1.1450	1.0791	1.0162	9.5599(–1)	9.2683(–1)
0.4	1.2205	1.1635	1.1090	1.0569	1.0070	9.8276(–1)
0.5	1.2254	1.1778	1.1322	1.0886	1.0467	1.0264
0.6	1.2293	1.1891	1.1507	1.1139	1.0784	1.0612
0.7	1.2324	1.1984	1.1657	1.1344	1.1042	1.0896
0.8	1.2350	1.2059	1.1781	1.1513	1.1255	1.1129
0.9	1.2371	1.2122	1.1883	1.1653	1.1432	1.1324
1.0	1.2389	1.2175	1.1969	1.1771	1.1580	1.1487
2.0	1.2472	1.2418	1.2365	1.2315	1.2266	1.2242

Table 6. Couette flow with $u_{\mathrm{p}} = 1$: a component P_{xy} of the reduced pressure tensor ($\varepsilon = \varepsilon_{\mathrm{p}}$)

	$\alpha = 0.1$		$\alpha = 1.0$	
$2a$	CES	LBE	CES	LBE
1.0(–7)	2.96942(–2)	2.96942(–2)	5.64190(–1)	5.64190(–1)
1.0(–3)	2.96926(–2)	2.96927(–2)	5.63636(–1)	5.63647(–1)
1.0(–1)	2.95505(–2)	2.95533(–2)	5.20156(–1)	5.20868(–1)
1.0	2.85847(–2)	2.85927(–2)	3.39977(–1)	3.40502(–1)
1.0(1)	2.26813(–2)	2.26781(–2)	8.35227(–2)	8.35098(–2)
1.0(3)	9.67035(–4)	9.67029(–4)	9.98031(–4)	9.98029(–4)
1.0(7)	9.99997(–8)	9.99997(–8)	1.00000(–7)	1.00000(–7)

the true component kernels (and the collision frequency for the case of the BGK model) by relevant model kernel functions ($L = 1$ for BGK and $L = 2$ for CES). In this way we were able to confirm to many figures of accuracy previously reported [14,15,22,23] results.

Continuing, we note that Wakabayashi, Ohwada and Golse [24] have reported the viscous and thermal-slip coefficients as deduced from a strictly numerical solution of the linearized Boltzmann equation for rigid-sphere collisions. As is not unusual in the field of rarefied gas dynamics, some care must be taken in comparing results from different works; Ref. [24], for example, uses a mean-free path based on

$$\varepsilon = 2^{1/2}/4 \tag{171}$$

while we use either $\varepsilon = \varepsilon_{\mathrm{p}}$ or $\varepsilon = \varepsilon_{\mathrm{t}}$. However, after a change of mean-free

Table 7. Couette flow ($\varepsilon = \varepsilon_{\rm p}$) with $u_{\rm p} = 1$: the flow rate $-U$

	$\alpha = 0.1$		$\alpha = 1.0$	
$2a$	CES	LBE	CES	LBE
0.10	5.4108(–2)	5.3191(–2)	7.4199(–1)	7.2929(–1)
1.00	2.3125(–2)	2.3115(–2)	2.2678(–1)	2.2737(–1)
10.0	1.1556(–2)	1.1584(–2)	4.2142(–2)	4.2192(–2)

Table 8. Couette flow ($\varepsilon = \varepsilon_{\rm p}$) with $u_{\rm p} = 1$: the heat-flow rate Q

	$\alpha = 0.1$		$\alpha = 1.0$	
$2a$	CES	LBE	CES	LBE
0.10	1.1294(–2)	9.3667(–3)	1.4479(–1)	1.1892(–1)
1.00	4.3490(–3)	3.2993(–3)	2.6986(–2)	2.2451(–2)
10.0	1.7913(–4)	1.7731(–4)	2.8598(–4)	3.0697(–4)

paths, we find that we have good, though not perfect, agreement with Ref. [24]. For example, Wakabayashi, Ohwada and Golse report from 4 to 6 figures of data in their tabulations of the slip coefficients, and we agree to four significant figures. We also agree well with the velocity and heat-flow profiles available in Ref. [24]. We also have seen that we agree well with the results reported by Sone, Takata and Ohwada [25] for the problem of Couette flow. Again, since a different mean-free path and (slightly) different definitions are used by by Sone *et al.*, some care

Table 9. Couette flow ($\varepsilon = \varepsilon_{\rm p}$) with $u_{\rm p} = 1$: velocity and heat-flow profiles for the case $2a = 1$

	$\alpha = 0.1$		$\alpha = 1.0$	
τ/a	$-u(\tau)$	$q(\tau)$	$-u(\tau)$	$q(\tau)$
0.0	0.0	0.0	0.0	0.0
0.1	4.2247(–3)	5.7604(–4)	4.3188(–2)	4.0656(–3)
0.2	8.4806(–3)	1.1590(–3)	8.6559(–2)	8.1681(–3)
0.3	1.2802(–2)	1.7563(–3)	1.3031(–1)	1.2347(–2)
0.4	1.7229(–2)	2.3767(–3)	1.7469(–1)	1.6648(–2)
0.5	2.1814(–2)	3.0311(–3)	2.1998(–1)	2.1125(–2)
0.6	2.6632(–2)	3.7347(–3)	2.6662(–1)	2.5855(–2)
0.7	3.1799(–2)	4.5105(–3)	3.1525(–1)	3.0946(–2)
0.8	3.7525(–2)	5.3996(–3)	3.6704(–1)	3.6587(–2)
0.9	4.4293(–2)	6.4950(–3)	4.2461(–1)	4.3192(–2)
1.0	5.4771(–2)	8.3205(–3)	5.0206(–1)	5.2963(–2)

Table 10. Poiseuille and thermal creep ($\varepsilon = \varepsilon_{\rm p}$): comparison results for the case $\alpha = 0.1$

	$-U_{\rm P}$		$Q_{\rm P} = U_{\rm T}$		$-Q_{\rm T}$	
$2a$	CES	LBE	CES	LBE	CES	LBE
0.10	1.9984(1)	2.0243(1)	4.1416	4.1701	2.0437(1)	2.0650(1)
1.00	1.7522(1)	1.7564(1)	7.1489(–1)	7.1258(–1)	3.4555	3.4557
10.0	1.8737(1)	1.8743(1)	7.9621(–2)	7.9140(–2)	3.7492(–1)	3.7488(–1)

Table 11. Poiseuille and thermal creep ($\varepsilon = \varepsilon_{\rm p}$): comparison results for the case $\alpha = 0.5$

	$-U_{\rm P}$		$Q_{\rm P} = U_{\rm T}$		$-Q_{\rm T}$	
$2a$	CES	LBE	CES	LBE	CES	LBE
0.10	4.3156	4.3868	1.5426	1.5680	7.6317	7.7797
1.00	3.2959	3.3264	5.3760(–1)	5.2876(–1)	2.5170	2.5138
10.0	4.5285	4.5346	8.6266(–2)	8.4299(–2)	3.6251(–1)	3.6167(–1)

must be exercised in comparing the results of Ref.[25] with the results given in this work.

In regard to Poiseuille flow and thermal-creep flow in a plane channel, we have found two references to be especially helpful: the first [26] by Ohwada, Sone and Aoki considers only the case of diffuse reflection ($\alpha = 1$), while Loyalka and Hickey [27] allow in their analysis general values of the accommodation coefficient. We find it easier to compare our numerical results with those of Ref. [27] since Loyalka and Hickey use the same definitions and units as used in this work; however, we find numerical agreement with Ref. [27] that at best is four significant figures, but for some results we have only two figures of agreement. Our comparisons with Ref. [26] seem better. While we have not evaluated all of the results reported by Ohwada, Sone and Aoki [26], we typically find three to four figures of agreement with their tabulations and qualitative agreement with the results reported [26] in graphical form.

Table 12. Poiseuille and thermal creep ($\varepsilon = \varepsilon_{\rm p}$): comparison results for the case $\alpha = 1$

	$-U_{\rm P}$		$Q_{\rm P} = U_{\rm T}$		$-Q_{\rm T}$	
$2a$	CES	LBE	CES	LBE	CES	LBE
0.10	1.9259	1.9499	7.9087(–1)	7.9928(–1)	3.8509	3.9037
1.00	1.4863	1.5067	4.0456(–1)	3.8908(–1)	1.8018	1.7830
10.0	2.7220	2.7296	9.3046(–2)	8.9950(–2)	3.4964(–1)	3.4674(–1)

Table 13. Poiseuille flow ($\varepsilon = \varepsilon_{\rm p}$): velocity and heat-flow profiles for the case $2a = 1$

	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 1.0$	
τ/a	$-u_{\rm P}(\tau)$	$q_{\rm P}(\tau)$	$-u_{\rm P}(\tau)$	$q_{\rm P}(\tau)$	$-u_{\rm P}(\tau)$	$q_{\rm P}(\tau)$
0.0	8.8693	3.7272(–1)	1.7574	2.8922(–1)	8.5378(–1)	2.2669(–1)
0.1	8.8671	3.7231(–1)	1.7549	2.8860(–1)	8.5117(–1)	2.2590(–1)
0.2	8.8602	3.7106(–1)	1.7475	2.8673(–1)	8.4327(–1)	2.2348(–1)
0.3	8.8486	3.6895(–1)	1.7350	2.8355(–1)	8.2994(–1)	2.1938(–1)
0.4	8.8320	3.6592(–1)	1.7172	2.7899(–1)	8.1090(–1)	2.1348(–1)
0.5	8.8101	3.6188(–1)	1.6936	2.7288(–1)	7.8568(–1)	2.0559(–1)
0.6	8.7822	3.5667(–1)	1.6635	2.6501(–1)	7.5357(–1)	1.9539(–1)
0.7	8.7473	3.5006(–1)	1.6258	2.5499(–1)	7.1335(–1)	1.8239(–1)
0.8	8.7035	3.4160(–1)	1.5785	2.4212(–1)	6.6281(–1)	1.6567(–1)
0.9	8.6461	3.3022(–1)	1.5167	2.2482(–1)	5.9696(–1)	1.4322(–1)
1.0	8.5499	3.1004(–1)	1.4143	1.9460(–1)	4.8979(–1)	1.0463(–1)

Table 14. Thermal-creep flow ($\varepsilon = \varepsilon_{\rm p}$): velocity and heat-flow profiles for the case $2a = 1$

	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 1.0$	
τ/a	$u_{\rm T}(\tau)$	$-q_{\rm T}(\tau)$	$u_{\rm T}(\tau)$	$-q_{\rm T}(\tau)$	$u_{\rm T}(\tau)$	$-q_{\rm T}(\tau)$
0.0	3.6061(–1)	1.7429	2.8169(–1)	1.3193	2.2268(–1)	9.9636(–1)
0.1	3.6050(–1)	1.7425	2.8126(–1)	1.3178	2.2199(–1)	9.9383(–1)
0.2	3.6018(–1)	1.7414	2.7996(–1)	1.3132	2.1987(–1)	9.8616(–1)
0.3	3.5963(–1)	1.7395	2.7775(–1)	1.3054	2.1629(–1)	9.7312(–1)
0.4	3.5883(–1)	1.7368	2.7457(–1)	1.2942	2.1113(–1)	9.5425(–1)
0.5	3.5777(–1)	1.7332	2.7032(–1)	1.2790	2.0422(–1)	9.2884(–1)
0.6	3.5640(–1)	1.7284	2.6484(–1)	1.2593	1.9530(–1)	8.9575(–1)
0.7	3.5466(–1)	1.7223	2.5785(–1)	1.2340	1.8391(–1)	8.5313(–1)
0.8	3.5242(–1)	1.7144	2.4886(–1)	1.2011	1.6927(–1)	7.9763(–1)
0.9	3.4941(–1)	1.7036	2.3677(–1)	1.1561	1.4960(–1)	7.2182(–1)
1.0	3.4411(–1)	1.6844	2.1573(–1)	1.0763	1.1581(–1)	5.8837(–1)

While we have some confidence in our numerical results, there can be some doubt: the use of a maximum value of $L = 8$ could be a source of error. We can emphasize (again) that computing the component kernels $k_n(c', c)$ accurately is no simple task especially as n increases. This too could be a source of error in our final results. On the other hand the approximations used in Eq. (90), the projections defined by Eq. (91) and the analytical discrete-ordinates are all considered exceptionally good procedures for the considered problems.

To conclude this section we give some idea about the computation time required for our FORTRAN implementations of our solutions. It should be noted that no

special effort was made to "tune" the codes for speed and that no optimization options were used. While some problems are numerically more difficult than others, we note that our implementation (that uses our subroutines for the Chapman-Enskog and Burnett functions) of the solution of the Poiseuille-flow problem for the case $2a = 1$ required 26 seconds of CPU time on a 1.2 GHz mobile Pentium III, without using any stored data and based on the approximating parameters $\{8, 200, 20, 10\}$, to find the velocity and heat-flow profiles and the flow rates for 10 values of the accommodation coefficient. Using stored input data, we found, for this case, that the code required less than 2 seconds on the same machine. We found from this timing example results that agree to within one or two units in the last place of the five figures reported in our tables.

9. Concluding comments

We have used a new polynomial expansion technique and the Pekeris [6] expanded form of the scattering kernel basic to the linearized Boltzmann equation for rigid-sphere collisions to define a system of coupled transport problems that has been solved efficiently and accurately with a modern version [4] of the discrete-ordinates method usually associated with Chandrasekhar [5] and the field of radiative transfer. While there exist other basic works [24-27] that report numerical results for the five flow problems solved is this work, we are of the opinion that our (nearly) analytical solutions are more efficient in regard to accuracy and computer-time requirements than the older, strictly numerical solutions. In addition, we note that only a modest effort should be required to extend the work reported here so as to include the Cercignani-Lampis boundary condition recently implemented [23,28–30] for some model equations. Finally, we are of the opinion that to solve the considered flow problems and to have a FORTRAN implementation that yields four or five significant figures of accuracy in less than 2 seconds seems a good contribution to the general field of rarefied gas dynamics.

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References

[1] C. Cercignani, *Rarefied Gas Dynamics*, Cambridge University Press, Cambridge 2000.
[2] F. Sharipov and V. Seleznev, Data on internal rarefied gas flows, *J. Phys. Chem. Ref. Data* **27** (1998), 657.
[3] C. E. Siewert, The linearized Boltzmann equation: A concise and accurate solution of the temperature-jump problem, *J. Quant. Spectros. Radiat. Transfer*, (in press).

[4] L. B. Barichello and C. E. Siewert, A discrete-ordinates solution for a non-grey model with complete frequency redistribution, *J. Quant. Spectros. Radiat. Transfer* **62** (1999), 665.

[5] S. Chandrasekhar, *Radiative Transfer*, Oxford University Press, London 1950.

[6] C. L. Pekeris, Solution of the Boltzmann-Hilbert integral equation. *Proc. Natl. Acad. Sci.* **41** (1955), 661.

[7] L. Boltzmann, *Collected Works*, **1** (1872), 361.

[8] D. Hilbert, *Grundzüge einer Allgemeinen Theorie der Linearen Integralgleichungen*, Chelsea Publishing Co., New York 1953.

[9] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-uniform Gases*, Cambridge University Press, Cambridge 1952.

[10] C. L. Pekeris and Z. Alterman, Solution of the Boltzmann-Hilbert integral equation II. the coefficients of viscosity and heat conduction, *Proc. Natl. Acad. Sci.* **43** (1957), 998.

[11] C. L. Pekeris, Z. Alterman, L. Finkelstein and K. Frankowski, Propagation of sound in a gas of rigid spheres, *Phys. Fluids* **5** (1962), 1608.

[12] S. K. Loyalka and K. A. Hickey, Plane Poiseuille flow: near continuum results for a rigid sphere gas, *Physica A* **160** (1989), 395.

[13] S. K. Loyalka and J. H. Ferziger, Model dependence of the temperature slip coefficient, *Phys. Fluids* **11** (1968), 1668.

[14] C. E. Siewert, Two half-space problems based on a synthetic-kernel model of the linearized Boltzmann equation, *J. Quant. Spectros. Radiat. Transfer* **75** (2002), 21.

[15] C. E. Siewert, Poiseuille, thermal creep and Couette flow: Results based on the CES model of the linearized Boltzmann equation, *Euro. J. Mech. B/Fluids* **21** (2002), 579.

[16] L. B. Barichello and C. E. Siewert, Some comments on modeling the linearized Boltzmann equation, *J. Quant. Spectros. Radiat. Transfer* **77** (2003), 43.

[17] M. M. R. Williams, A review of the rarefied gas dynamics theory associated with some classical problems in flow and heat transfer, *Z. Angew. Math. Phys.* **52** (2001), 500.

[18] C. E. Siewert, On computing the Chapman-Enskog functions for viscosity and heat transfer and the Burnett functions, *J. Quant. Spectros. Radiat. Transfer* **74** (2002), 789.

[19] C. E. Siewert, A discrete-ordinates solution for multigroup transport theory with upscattering, *J. Quant. Spectros. Radiat. Transfer* **64** (2000), 255.

[20] B. T. Smith, J. M. Boyle, J. J. Dongarra, B. S. Garbow, Y. Ikebe, V. C. Klema and C. B. Moler, *Matrix Eigensystem Routines – EISPACK Guide*, Springer-Verlag, Berlin 1976.

[21] J. J. Dongarra, J. R. Bunch, C. B. Moler and G. W. Stewart, *LINPACK User's Guide*, SIAM, Philadelphia 1979.

[22] L. B. Barichello, M. Camargo, P. Rodrigues and C. E. Siewert, Unified solutions to classical flow problems based on the BGK model, *Z. Angew. Math. Phys.* **52** (2001), 517.

[23] C. E. Siewert and Felix Sharipov, Model equations in rarefied gas dynamics: viscous-slip and thermal-slip coefficients, *Phys. Fluids* **14** (2002), 4123.

[24] M. Wakabayashi, T. Ohwada, and F. Golse, Numerical analysis of the shear and thermal creep flows of a rarefied gas over the plane wall of a Maxwell-type boundary on the basis of the linearized Boltzmann equation for hard-sphere molecules, *Eur. J. Mech. B/Fluids* **15** (1996), 175.

[25] Y. Sone, S. Takata and T. Ohwada, Numerical analysis of the plane Couette flow of a rarefied gas on the basis of the linearized Boltzmann equation for hard-sphere molecules, *Eur. J. Mech. B/Fluids* **9** (1990), 273.

[26] T. Ohwada, Y. Sone and K. Aoki, Numerical analysis of the Poiseuille and thermal transpiration flows between two parallel plates on the basis of the Boltzmann equation for hard-sphere molecules, *Phys. Fluids A* **1** (1989), 2042.

[27] S. K. Loyalka and K. A. Hickey, Kinetic theory of thermal transpiration and the mechanocaloric effect: Planar flow of a rigid sphere gas with arbitrary accommodation at the surface, *J. Vac. Sci. Technol. A* **9** (1991), 158.

[28] F. Sharipov, Application of the Cercignani–Lampis scattering kernel to calculations of rarefied gas flows. I. Plane flow between two parallel plates, *Eur. J. Mech. B/Fluids* **21** (2002), 113.
[29] C. E. Siewert, Generalized boundary conditions for the S-model kinetic equations basic to flow in a plane channel, *J. Quant. Spectros. Radiat. Transfer* **72** (2002), 75.
[30] F. Sharipov, Application of the Cercignani–Lampis scattering kernel to calculations of rarefied gas flows. II. Slip and jump coefficients, submitted for publication.

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