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# ON COMPUTING THE CHAPMAN-ENSKOG AND BURNETT FUNCTIONS

**L. B. Barichello[†], P. Rodrigues[‡] and C. E. Siewert[§]**
[†]Instituto de Matemática, Universidade Federal do Rio Grande do Sul
Av. Bento Gonçalves, 9500, 91509–900 Porto Alegre, RS, Brasil
lbaric@mat.ufrgs.br
[‡]Curso de Matemática, Universidade Regional Integrada do Alto Uruguai e das Missões
Av. Assis Brasil, 709, Cx. P. 184, 98400–000 Frederico Westphalen, RS, Brasil
patricia@fw.uri.br
[§]Mathematics Department, North Carolina State University
Raleigh, NC 27695–8205, USA
siewert@ncsu.edu

## ABSTRACT

An expansion and projection technique based on Legendre polynomials is used to solve, in an efficient and accurate way, the Chapman-Enskog equations for viscosity and heat transfer and to compute the Burnett functions required, for example, for Poiseuille-flow problems based on rigid-sphere collisions and the linearized Boltzmann equation.

## 1. INTRODUCTION

Within the context of the linearized Boltzmann equation for rigid-sphere collisions, the Chapman-Enskog functions for viscosity and heat transfer, as well as the Burnett functions, have been computed and the results reported [1–4]. However, since these functions are especially important in regard to analytical and computational solutions of the linearized Boltzmann equation, we report here an additional calculation that can be used as an alternative to the previously reported [1,4] computations that were based on the use of spline functions. While we consider the algorithm, based on Hermite cubic splines and a collocation procedure, that was used in Ref. [4] to be a good one, we wish here to report a computation that leads to the same numerical results, without any of the concerns that collocation methods can sometimes provoke.

The general class of problems we consider can be written as

$$\mathcal{L}_n\{f\}(c) = r(c), \quad c \in [0, \infty), \tag{1}$$

with $r(c)$ given, and with

$$\mathcal{L}_n\{f\}(c) = \nu(c) f(c) - \int_0^\infty \mathrm{e}^{-c'^2} f(c') k_n(c', c) c'^2 \, \mathrm{d}c'. \tag{2}$$

Here

$$\nu(c) = \frac{2c^2 + 1}{c} \int_0^c \mathrm{e}^{-x^2} \, \mathrm{d}x + \mathrm{e}^{-c^2} \tag{3}$$

is the "collision frequency" and the functions $k_n(c', c)$ are components in the Pekeris-Alterman "rigid-sphere" scattering kernel [2] written as

$$K(\boldsymbol{c}', \boldsymbol{c}) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (2n+1)(2-\delta_{0,m}) P_n^m(\mu') P_n^m(\mu) k_n(c', c) \cos m(\chi' - \chi). \tag{4}$$

The *normalized* Legendre functions

$$P_n^m(\mu) = \left[ \frac{(n-m)\,!}{(n+m)\,!} \right]^{1/2} (1-\mu^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_n(\mu), \quad n \geq m, \tag{5}$$

where $P_n(\mu)$ denotes the usual Legendre polynomial, are such that

$$\int_{-1}^{1} P_n^m(\mu) P_{n'}^m(\mu)\, \mathrm{d}\mu = \Big( \frac{2}{2n+1} \Big) \delta_{n,n'}. \tag{6}$$

In this work we use the component functions $k_n(c', c)$ only for $n = 1, 2,$ and 3. Since these basic functions were listed explicitly in Ref. [4], we do not re-list them here.

While the numerical approach we propose is general, the specific cases considered here are the Chapman-Enskog equation for viscosity,

$$\mathcal{L}_2\{B\}(c) = c^2, \tag{7}$$

the Chapman-Enskog equation for heat transfer,

$$\mathcal{L}_1\{A\}(c) = c(c^2 - 5/2), \tag{8a}$$

with the normalization condition

$$\int_0^{\infty} \mathrm{e}^{-c^2} A(c) c^3\, \mathrm{d}c = 0, \tag{8b}$$

and the two Burnett equations [1]

$$\mathcal{L}_1\{D\}(c) = 2cB(c) - 5c\varepsilon_{\mathrm{p}}, \tag{9a}$$

with the normalization [1]

$$\int_0^{\infty} \mathrm{e}^{-c^2} D(c) c^3\, \mathrm{d}c = 0, \tag{9b}$$

and

$$\mathcal{L}_3\{E\}(c) = 2cB(c), \tag{10}$$

where

$$\varepsilon_{\mathrm{p}} = \frac{16}{15} \pi^{-1/2} \int_0^{\infty} e^{-c^2} B(c) c^4\, \mathrm{d}c. \tag{11}$$

## 2. THE DEVELOPMENT

We start by expressing the desired (approximate) solution of Eq. (1) as

$$f(c) = \sum_{k=0}^{K} a_k P_k(2\mathrm{e}^{-c} - 1) \tag{12}$$

where the constants $\{a_k\}$ are to be determined. To find the coefficients $\{a_k\}$ required in Eq. (12), we substitute Eq. (12) into Eq. (1), multiply the resulting equation by

$$\boldsymbol{W}(c) = \boldsymbol{P}^{\mathrm{T}}(c) c^2 \mathrm{e}^{-c^2}, \tag{13}$$

where the superscript T denotes the transpose operation and

$$\boldsymbol{P}(c) = \left[P_0(2\mathrm{e}^{-c} - 1), P_1(2\mathrm{e}^{-c} - 1), ..., P_K(2\mathrm{e}^{-c} - 1)\right], \tag{14}$$

and integrate over all $c$ to obtain a system of linear algebraic equations for the desired coefficients $\{a_k\}$. Since the kernel functions are such that $k_n(c', c) = k_n(c, c')$, we are able to use symmetry features of matrices involved in the procedure to reduce the numerical work required to evaluate these matrices. However, as mentioned in Refs. [6] and [7], some care must be exercised in evaluating the kernel functions $k_n(c', c)$.

After we solve the linear system that defines the coefficients $\{a_k\}$, our first solution is given by Eq. (12); however an alternative "post-processed" result can be obtained by using Eq. (12) in Eq. (1) to find

$$f(c) = [r(c) + \sum_{k=0}^{K} a_k F_k(c)]/\nu(c) \tag{15}$$

where

$$F_k(c) = \int_0^\infty \mathrm{e}^{-c'^2} P_k(2\mathrm{e}^{-c'} - 1) k_n(c', c) c'^2 \, \mathrm{d}c'. \tag{16}$$

## 3. NUMERICAL RESULTS AND CONCLUSIONS

In evaluating our numerical solutions for the two Chapman-Enskog functions $A(c)$ and $B(c)$ and the two Burnett functions $D(c)$ and $E(c)$, we consider, as have others [1-4], that the domain of significant interest is the subset $c \in [0, 5]$ of the complete domain $[0, \infty)$. As the use of these functions typically involves a multiplication by $\exp(-c^2)$ the chosen domain of interest seems suitable for various applications that require these functions. The numerical results we obtained here are essentially in perfect agreement with previous work [4]. However, some general comments on the implementation of the developed algorithms can be given. So, in our work here we have two basic approximation parameters: $K+1$ is the number of basis functions used in Eq. (12) and $M$ is the number of Gauss points used to evaluate the input matrices which define the linear system for the constants $\{a_k\}$. While we have not attempted a definitive study to find the minimum values of $K$ and $M$ to obtain a given degree of accuracy, we did find that we could recover the seven-digit results listed in Ref. [4] with $K = 100$ and $M = 200$. More importantly we found, for $c \in [0, 5]$, that we had to use the post-processed result listed as Eq. (15) only for, say, $c < 0.2$ and $c > 4.2$. And so, since we found the desired results mostly from the simple result listed as Eq. (12), our final algorithm was especially efficient.

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## REFERENCES

[1] S. K. Loyalka and K. A. Hickey. “Plane Poiseuille flow: near continuum results for a rigid sphere gas.” *Physica A*, **160**, 395–408 (1989).

[2] C. L. Pekeris and Z. Alterman. “Solution of the Boltzmann-Hilbert integral equation II. The coefficients of viscosity and heat conduction.” *Proc. Natl. Acad. Sci.*, **43**, 998–1007 (1957).

[3] T. Ohwada and Y. Sone. “Analysis of thermal stress slip flow and negative thermophoresis using the Boltzmann equation for hard-sphere molecules.” *Eur. J. Mech. B/Fluids*, **11**, 389–414 (1992).

[4] C. E. Siewert. “On computing the Chapman-Enskog functions for viscosity and heat transfer and the Burnett functions.” *JQSRT*, **74**, 789–796 (2002).

[5] L. B. Barichello and C. E. Siewert. “Some comments on modeling the linearized Boltzmann equation.” *JQSRT*, **77**, 43–59 (2003).

[6] C. E. Siewert. “The linearized Boltzmann equation: a concise and accurate solution of the temperature-jump problem.” *JQSRT*, **77**, 417–432 (2003).

[7] C. E. Siewert. “The linearized Boltzmann equation: concise and accurate solutions to basic flow problems.” *Z. Angew. Math. Phys.*, **54**, 273–303 (2003).

18 ICTT, Rio de Janeiro, RJ, Brazil.