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Heat transfer and evaporation/condensation problems based on the linearized Boltzmann equation

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Abstract

A polynomial expansion procedure and an analytical discrete-ordinates method are used to solve four basic problems, all based on the linearized Boltzmann equation for rigid-sphere interactions, that describe heat transfer and/or evaporation–condensation between two parallel surfaces or for the case of a semi-infinite half space. Relevant to the case of two surfaces, the basic problem of heat transfer driven by a temperature difference at two confining walls described by a general Maxwell gas–surface interaction law (a mixture of specular and diffuse reflection) is solved for the case where different accommodation coefficients can be used for each of the two bounding surfaces. In addition, the classical problem of "reverse temperature gradient" in the theory of evaporation and condensation is also solved for the case of two parallel liquid–vapor interfaces kept at different temperatures. In regard to half-space applications, an evaporation/condensation problem based on a presumed known interface condition and a heat-conduction problem (with no net flow) driven by energy flow from a bounding surface with know properties are each solved with what is considered a high degree of accuracy.

Keywords: Boltzmann equation; Rarefied gas dynamics; Heat transfer; Evaporation–condensation

1. Introduction

In two recent works [1,2] a newly introduced polynomial expansion technique (relevant to the speed variable) and an analytical discrete-ordinates method [3] that has evolved from Chandrasekhar's work [4] in radiative transfer were used, in regard to a rigorous form of the linearized Boltzmann equation for rigid-sphere interactions, to solve the classical temperature-jump problem and a collection of basic flow problems (Couette flow, Poiseuille flow and thermal-creep flow) relevant to finite plane-parallel channels and to semi-infinite half spaces. While the first set of problems [1,2] was defined in terms of a general Maxwell boundary condition (a combination of specular and diffuse reflection and characterized by a single accommodation coefficient α), the mentioned solution procedure was extended in a following work [5] in order to make use of the Cercignani–Lampis [6] boundary condition that is characterized by two accommodation coefficients α_n and α_t. Continuing to investigate the basic problems in rarefied gas dynamics that can be described by the linearized Boltzmann equation for the case of rigid-sphere scattering, we now discuss our solutions of four heat-transfer and/or evaporation–condensation problems defined either for a gas maintained between two parallel surfaces or for the case of a semi-infinite half space bounded by a plane surface.

In regard to the four basic problems we consider here, we note that there already exist solutions, mostly based on highly numerical approaches (finite differences methods, numerical quadrature schemes, spline expansions and collocation techniques, for example) with accuracy sufficient for most physical applications. In this regard, there also has been considerable discussion about the mathematical formulation of these applications in the theory of rarefied gas dynamics. And so to be clear about this work, we can say that our goal is to provide essentially analytical solutions (to the considered problems) that define what we

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consider to be a high standard of accuracy. In addition to defining good numerical results, the new analytical solutions can be implemented at a computational cost much less, we believe, than the cost of evaluating basic quantities of interest with strictly numerical solutions. While most existing works deal with these problems one or two at a time, we consider these problems to have much in common (mathematically speaking), and so it seems especially efficient to consider all four of these problems in this single presentation.

To start this work, we consider the homogeneous and linearized Boltzmann equation written for rigid-sphere collisions as [7,8]

$$c\mu \frac{\partial}{\partial \tau} h(\tau, \boldsymbol{c}) = \varepsilon L\{h\}(\tau, \boldsymbol{c}), \tag{1}$$

where

$$L\{h\}(\tau, \boldsymbol{c}) = -\nu(c) h(\tau, \boldsymbol{c}) + \int_0^\infty \int_{-1}^1 \int_0^{2\pi} \mathrm{e}^{-c'^2} h(\tau, \boldsymbol{c}') K(\boldsymbol{c}' : \boldsymbol{c}) c'^2 \, \mathrm{d}\chi' \, \mathrm{d}\mu' \, \mathrm{d}c'. \tag{2}$$

Here the scattering kernel is written in the expanded form

$$K(\boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (2n+1)(2-\delta_{0,m}) P_n^m(\mu') P_n^m(\mu) k_n(c', c) \cos m(\chi' - \chi), \tag{3}$$

where the *normalized* Legendre functions are given (in terms of the Legendre polynomials) by

$$P_n^m(\mu) = \left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} \left(1-\mu^2\right)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_n(\mu), \quad n \geqslant m. \tag{4}$$

In addition,

$$\varepsilon = \sigma_0^2 n_0 \pi^{1/2} l, \tag{5}$$

where l is (at this point) an unspecified mean-free path, n_0 is the density and σ_0 is the scattering diameter of the gas particles. In this work, the spatial variable τ is measured in units of the mean-free path l, and $c(2kT_0/m)^{1/2}$ is the magnitude of the particle velocity. Also, k is the Boltzmann constant, m is the mass of a gas particle and T_0 is a reference temperature. The basic unknown $h(\tau, \boldsymbol{c})$ in Eq. (1) is a perturbation from an absolute Maxwellian distribution. Continuing, we note that the functions $k_n(c', c)$ in Eq. (3) are the components in an expansion of the scattering law (for rigid-sphere collisions) reported by Pekeris, Alterman, Finkelstein and Frankowski [9], and

$$\nu(c) = \frac{2c^2+1}{c} \int_0^c \mathrm{e}^{-x^2} \, \mathrm{d}x + \mathrm{e}^{-c^2} \tag{6}$$

is the collision frequency. And finally, we use spherical coordinates $(c, \arccos\mu, \chi)$ to define the (dimensionless) velocity vector $\boldsymbol{c}$.

2. The problems

2.1. A heat-transfer problem for the case of two parallel surfaces

For this problem, due to the presence of the walls located at $\tau = \mp a$, we must supplement Eq. (1) with appropriate boundary conditions. Noting that

$$h(\tau, \boldsymbol{c}) \Leftrightarrow h(\tau, c, \mu, \chi), \tag{7}$$

we follow Williams [10] and express the two required boundary conditions as

$$h(-a, c, \mu, \chi) - (1-\alpha_1) h(-a, c, -\mu, \chi) - \alpha_1 \mathcal{I}_-\{h\}(-a) = \alpha_1 \delta_1 (c^2 - 2) \tag{8a}$$

and

$$h(a, c, -\mu, \chi) - (1-\alpha_2) h(a, c, \mu, \chi) - \alpha_2 \mathcal{I}_+\{h\}(a) = \alpha_2 \delta_2 (c^2 - 2) \tag{8b}$$

for $\mu \in (0,1]$, $c \in [0,\infty)$ and all χ. Here

$$\mathcal{I}_{\mp}\{h\}(\tau) = \frac{2}{\pi}\int_0^\infty\int_0^1\int_0^{2\pi} \mathrm{e}^{-c^2} h(\tau, c, \mp\mu, \chi)\mu c^3 \,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{9}$$

and $\alpha_1 \in (0,1]$ and $\alpha_2 \in (0,1]$ are the accommodation coefficients associated with each of the two walls that are allowed to have different scattering properties. In addition, the two constants δ_1 and δ_2 are measures of the temperatures of the two walls. In a previous work [11] that reported an accurate solution of the heat-transfer problem in a plane channel as described by the BGK model [12], it was noted that Eq. (1) and the boundary conditions listed as Eqs. (8) do not define a unique solution $h(\tau, \boldsymbol{c})$. And so we will, once we have introduced more notation, supplement Eqs. (8) with another (normalization) condition.

Following the discussion from [10], we see that, while our problem is defined in terms of the unknown $h(\tau, \boldsymbol{c})$, we require only two elementary integrals of $h(\tau, \boldsymbol{c})$ in order to establish the density and temperature perturbations defined by

$$N(\tau) = \frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2} h(\tau, \boldsymbol{c}) c^2 \,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c \tag{10}$$

and

$$T(\tau) = \frac{2}{3\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2}(c^2 - 3/2) h(\tau, \boldsymbol{c}) c^2 \,\mathrm{d}\chi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{11}$$

or

$$N(\tau) = \frac{2}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\phi(\tau, c, \mu) c^2 \,\mathrm{d}\mu\,\mathrm{d}c \tag{12}$$

and

$$T(\tau) = \frac{4}{3\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}(c^2 - 3/2)\phi(\tau, c, \mu) c^2 \,\mathrm{d}\mu\,\mathrm{d}c, \tag{13}$$

where

$$\phi(\tau, c, \mu) = \frac{1}{2\pi}\int_0^{2\pi} h(\tau, \boldsymbol{c})\,\mathrm{d}\chi \tag{14}$$

is an azimuthal average. We can integrate Eqs. (1) and (8) over χ to find

$$c\mu\frac{\partial}{\partial\tau}\phi(\tau, c, \mu) + \varepsilon\nu(c)\phi(\tau, c, \mu) = \varepsilon\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c'^2}\phi(\tau, c', \mu')k(c', \mu' : c, \mu)c'^2\,\mathrm{d}\mu'\,\mathrm{d}c', \tag{15}$$

for $\tau \in (-a, a)$, $\mu \in [-1, 1]$ and $c \in [0, \infty)$, and

$$\phi(-a, c, \mu) - (1-\alpha_1)\phi(-a, c, -\mu) - 4\alpha_1 D_1 = \alpha_1\delta_1\big(c^2 - 2\big) \tag{16a}$$

and

$$\phi(a, c, -\mu) - (1-\alpha_2)\phi(a, c, \mu) - 4\alpha_2 D_2 = \alpha_2\delta_2\big(c^2 - 2\big), \tag{16b}$$

for $\mu \in (0, 1]$ and $c \in [0, \infty)$. Here

$$k(c', \mu' : c, \mu) = \int_0^{2\pi} K(\boldsymbol{c}' : \boldsymbol{c})\,\mathrm{d}\chi \tag{17}$$

or

$$k(c', \mu' : c, \mu) = \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)P_n(\mu')P_n(\mu)k_n(c', c). \tag{18}$$

In addition,

$$D_1 = \int_0^\infty \int_0^1 \mathrm{e}^{-c^2} \phi(-a, c, -\mu)\mu c^3 \,\mathrm{d}\mu \,\mathrm{d}c \tag{19a}$$

and

$$D_2 = \int_0^\infty \int_0^1 \mathrm{e}^{-c^2} \phi(a, c, \mu)\mu c^3 \,\mathrm{d}\mu \,\mathrm{d}c. \tag{19b}$$

We will see later in this work that Eqs. (16) are not sufficient to define a unique solution of Eq. (15), and so we will follow what was done in [11] and make use of the additional (normalizing) condition

$$\int_{-a}^{a} N(\tau)\,\mathrm{d}\tau = 0. \tag{20}$$

Since the equations required to define the (Pekeris) component functions $k_n(c', c)$ are available in other works, see, for example, [1,7–9], we do not list them here. However we do list the three identities

$$\nu(c) = \int_0^\infty \mathrm{e}^{-c'^2} k_0(c', c) c'^2 \,\mathrm{d}c', \tag{21a}$$

$$\nu(c)c = \int_0^\infty \mathrm{e}^{-c'^2} k_1(c', c) c'^3 \,\mathrm{d}c' \tag{21b}$$

and

$$\nu(c)c^2 = \int_0^\infty \mathrm{e}^{-c'^2} k_0(c', c) c'^4 \,\mathrm{d}c' \tag{21c}$$

that are available [13] from Eq. (1) and the conditions of conservation of mass, energy and momentum.

Now, multiplying Eq. (15) by $c^\beta \exp(-c^2)$, for $\beta = 2$ and 4, and integrating over all c and μ we find, after noting Eqs. (21a) and (21c) and using $k_n(c', c) = k_n(c, c')$, that

$$\frac{\mathrm{d}U}{\mathrm{d}\tau} = 0 \tag{22a}$$

and

$$\frac{\mathrm{d}Q}{\mathrm{d}\tau} = 0, \tag{22b}$$

where

$$U = \frac{2}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \phi(\tau, c, \mu)\mu c^3 \,\mathrm{d}\mu \,\mathrm{d}c \tag{23}$$

is a measure of the net flow and

$$Q = \frac{2}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \phi(\tau, c, \mu)\big(c^2 - 5/2\big)\mu c^3 \,\mathrm{d}\mu \,\mathrm{d}c \tag{24}$$

is the non-dimensional heat flux. Using the fact that U is a constant, we can use either of Eqs. (16) to conclude that $U = 0$. And so in regard to this problem we intend to compute the density and temperature perturbations as listed by Eqs. (12) and (13) and the non-dimensional heat flux as defined by Eq. (24). In order to compare with results available in other works, we also report values of the normalized heat flux defined as

$$q = \frac{Q}{Q_{fm}}, \tag{25}$$

where Q_{fm} is the non-dimensional heat flux for "free-molecular" conditions. The required result for Q_{fm} can be obtained by neglecting all terms in Eq. (15) that are proportional to ε, solving the resulting equation subject to Eqs. (16) and (20) and evaluating Eq. (24). In this way we find

$$Q_{fm} = \frac{\alpha_1\alpha_2(\delta_1 - \delta_2)}{(\alpha_1 + \alpha_2 - \alpha_1\alpha_2)\pi^{1/2}}. \tag{26}$$

2.2. The problem of a reverse temperature gradient

In a paper published in 1971, Pao [14] pointed out that the slope of the temperature profile in a saturated vapor between two parallel evaporating and condensing surfaces kept at different temperatures could, for special values of a certain parameter β, be in opposition to the imposed overall temperature gradient. We investigate this problem here. Following a paper by Thomas, Chang and Siewert [15], we consider that the vapor is confined between two parallel interfaces, one located at $\tau = -a$ and kept at temperature $T_0 - \Delta T$, while the other surface is located at $\tau = a$ and kept at temperature $T_0 + \Delta T$. The results of [14,15] were based on the BGK kinetic model, and so here we consider that within the vapor the perturbation function $h(\tau, \boldsymbol{c})$ can be described by Eq. (1), the linearized Boltzmann equation for rigid-sphere interactions. At each of the two inter-phase surfaces, we assume that the vapor molecules striking the surface are absorbed and re-emitted with a Maxwellian distribution of velocities characterized by the temperature at the respective surface, and so we continue to follow [14,15] and express the required boundary conditions as

$$h(-a, c, \mu, \chi) = -\Delta N - \big(c^2 - 3/2\big)\Delta T \tag{27a}$$

and

$$h(a, c, -\mu, \chi) = \Delta N + \big(c^2 - 3/2\big)\Delta T \tag{27b}$$

for $\mu \in (0, 1]$ and all c and χ. Here the density perturbation ΔN that corresponds to the temperature perturbation ΔT is taken [14,15] to be given by

$$\Delta N = \beta \Delta T, \tag{28}$$

where the constant β is considered to be known. Using Eq. (28), we rewrite Eqs. (27) as

$$h(-a, c, \mu, \chi) = -\big(\beta + c^2 - 3/2\big)\Delta T \tag{29a}$$

and

$$h(a, c, -\mu, \chi) = \big(\beta + c^2 - 3/2\big)\Delta T \tag{29b}$$

for $\mu \in (0, 1]$ and all c and χ. Here, as for the previously defined problem, we seek the density and temperature perturbations, so we integrate Eqs. (1) and (29) to obtain the balance equation

$$c\mu \frac{\partial}{\partial \tau}\phi(\tau, c, \mu) + \varepsilon\nu(c)\phi(\tau, c, \mu) = \varepsilon \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c'^2}\phi(\tau, c', \mu')k(c', \mu' : c, \mu)c'^2 \,\mathrm{d}\mu' \,\mathrm{d}c', \tag{30}$$

for $\tau \in (-a, a)$, $\mu \in [-1, 1]$ and all c, and the boundary conditions

$$\phi(-a, c, \mu) = -\big(\beta + c^2 - 3/2\big)\Delta T \tag{31a}$$

and

$$\phi(a, c, -\mu) = \big(\beta + c^2 - 3/2\big)\Delta T, \tag{31b}$$

for $\mu \in (0, 1]$ and all c. We note that the scattering kernel in Eq. (30) is still given by Eq. (18). Once $\phi(\tau, c, \mu)$ is available, we intend to compute, for a given value of β, the density and temperature profiles

$$N(\tau) = \frac{2}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2}\phi(\tau, c, \mu)c^2 \,\mathrm{d}\mu \,\mathrm{d}c \tag{32}$$

and

$$T(\tau) = \frac{4}{3\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2}\big(c^2 - 3/2\big)\phi(\tau, c, \mu)c^2 \,\mathrm{d}\mu \,\mathrm{d}c \tag{33}$$

as well as the flow rate U and the heat-flow rate Q as defined by Eqs. (23) and (24). We also seek the *critical* values β_T and β_Q defined (for a given vapor thickness $\delta = 2a$) so that for $\beta > \beta_T$ the temperature perturbation $T(\tau)$ will have at $\tau = 0$ a gradient in opposition to the overall temperature gradient and for $\beta > \beta_Q$ the heat flow Q will be in the direction opposite to the mass and energy flow. Finally, in order to have the solution developed here also establish the solution to a similar problem, discussed by Sone, Ohwada and Aoki [16] and defined by a generalized version of the boundary condition listed as Eqs. (29), we make available the "outward" flow

$$U_+ = \frac{2}{\pi^{1/2}} \int_0^\infty \int_0^1 \mathrm{e}^{-c^2} \phi(a, c, \mu) \mu c^3 \, \mathrm{d}\mu \, \mathrm{d}c. \tag{34}$$

2.3. Evaporation/condensation in a semi-infinite half space

Our discussion of evaporation/condensation in a semi-infinite half space is based on early papers by Pao [17] and Siewert and Thomas [18]. Both of these papers are based on the BGK kinetic model, while here we continue to define our analysis in terms of Eq. (1), the linearized Boltzmann equation for rigid-sphere interactions. If we consider that the function $h(\tau, \boldsymbol{c})$ in Eq. (1) defines a perturbation from an absolute Maxwellian distribution written in terms of the interface density n_0 and temperature T_0 then we seek here a solution of Eq. (1) that is bounded as τ tends to infinity and that satisfies at $\tau = 0$ the interface condition

$$h(0, c, \mu, \chi) = 0 \tag{35}$$

for $\mu \in (0, 1]$, $c \in [0, \infty)$ and all χ. For this problem we intend to compute the temperature and density profiles, and so again we can integrate Eq. (1) over χ and seek a bounded (as τ tends to infinity) solution of

$$c\mu \frac{\partial}{\partial \tau} \phi(\tau, c, \mu) + \varepsilon \nu(c) \phi(\tau, c, \mu) = \varepsilon \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c'^2} \phi(\tau, c', \mu') k(c', \mu' : c, \mu) c'^2 \, \mathrm{d}\mu' \, \mathrm{d}c', \tag{36}$$

for $\tau > 0$, $\mu \in [-1, 1]$ and all c, subject to the the boundary condition

$$\phi(0, c, \mu) = 0, \tag{37}$$

for $\mu \in (0, 1]$ and all c. We note that the distinction between condensation and evaporation is made here by the sign of the specified value of the flow

$$U = \frac{2}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \phi(\tau, c, \mu) \mu c^3 \, \mathrm{d}\mu \, \mathrm{d}c. \tag{38}$$

In this work we normalize our solution by imposing the condition $U = 1$, so that the computed density and temperature profiles

$$N(\tau) = \frac{2}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \phi(\tau, c, \mu) c^2 \, \mathrm{d}\mu \, \mathrm{d}c \tag{39}$$

and

$$T(\tau) = \frac{4}{3\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \left(c^2 - 3/2\right) \phi(\tau, c, \mu) c^2 \, \mathrm{d}\mu \, \mathrm{d}c \tag{40}$$

can (after multiplication by a given value of U) be used for both condensation and evaporation.

2.4. Heat transfer in a semi-infinite half space

The heat transfer problem solved in [17] and [18] for the case of a semi-infinite half space is defined so that the temperature and density perturbations are required to diverge as τ tend to infinity, but at the same time the conditions

$$\lim_{\tau \to \infty} \frac{\mathrm{d}}{\mathrm{d}\tau} T(\tau) = 1 \tag{41a}$$

and

$$\lim_{\tau \to \infty} \frac{\mathrm{d}}{\mathrm{d}\tau} N(\tau) = -1 \tag{41b}$$

are imposed. And since again $h(\tau, \boldsymbol{c})$ is taken to be a perturbation from a Maxwellian distribution with surface parameters n_0 and T_0, we follow [17] and [18] and use the boundary condition

$$h(0, c, \mu, \chi) = 0 \tag{42}$$

for $\mu \in (0, 1]$, $c \in [0, \infty)$ and all χ. As we seek the temperature and density profiles we can use

$$N(\tau) = \frac{2}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \phi(\tau, c, \mu) c^2 \, \mathrm{d}\mu \, \mathrm{d}c \tag{43}$$

and

$$T(\tau) = \frac{4}{3\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \big(c^2 - 3/2\big) \phi(\tau, c, \mu) c^2 \, \mathrm{d}\mu \, \mathrm{d}c \tag{44}$$

along with

$$c\mu \frac{\partial}{\partial \tau} \phi(\tau, c, \mu) + \varepsilon \nu(c) \phi(\tau, c, \mu) = \varepsilon \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c'^2} \phi(\tau, c', \mu') k(c', \mu' : c, \mu) c'^2 \, \mathrm{d}\mu' \, \mathrm{d}c', \tag{45}$$

for $\tau > 0$, $\mu \in [-1, 1]$ and all c, and the boundary condition

$$\phi(0, c, \mu) = 0, \tag{46}$$

for $\mu \in (0, 1]$ and all c. For this problem there is no flow, and so we also impose the condition

$$\int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \phi(\tau, c, \mu) \mu c^3 \, \mathrm{d}\mu \, \mathrm{d}c = 0. \tag{47}$$

We note that this heat-transfer problem is very similar to the classical temperature-jump problem as defined, for example, by Welander [19]. In fact, while the density profile here differs by an additive constant from the density profile defined [1] by the temperature-jump problem (for the case of diffuse reflection), the temperature perturbation is same as for the temperature-jump problem.

3. A polynomial representation and the ADO method

As noted in the previous section of this work, all four of the considered problems are based on the balance equation

$$c\mu \frac{\partial}{\partial \tau} \phi(\tau, c, \mu) + \varepsilon \nu(c) \phi(\tau, c, \mu) = \varepsilon \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c'^2} \phi(\tau, c', \mu') k(c', \mu' : c, \mu) c'^2 \, \mathrm{d}\mu' \, \mathrm{d}c', \tag{48}$$

where the kernel function is given by Eq. (18), and various boundary or other conditions that are different for each of the problems. And so now we follow our previous work with the temperature-jump problem [1] and approximate the solution of Eq. (48) by making use of the polynomial representation

$$\phi(\tau, c, \mu) = \sum_{k=0}^{K} P_k\big(2\,\mathrm{e}^{-c} - 1\big) g_k(\tau, \mu), \tag{49}$$

where the Legendre polynomials are denoted by $P_k(x)$ and where the functions $g_k(\tau, \mu)$ are to be determined. We now truncate the kernel function listed as Eq. (18) after $L + 1$ terms, substitute Eq. (49) into the resulting form of Eq. (48), multiply that equation by

$$W_i(c) = c^2 \, \mathrm{e}^{-c^2} P_i\big(2\,\mathrm{e}^{-c} - 1\big), \tag{50}$$

for $i = 0, 1, \ldots, K$, and integrate over all c to obtain a coupled system of "multigroup" equations which we write as

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{A} \boldsymbol{G}(\tau, \mu) + \varepsilon \boldsymbol{S} \boldsymbol{G}(\tau, \mu) = \varepsilon \sum_{l=0}^{L} \boldsymbol{B}_l P_l(\mu) \int_{-1}^{1} P_l(\mu') \boldsymbol{G}(\tau, \mu') \, \mathrm{d}\mu'. \tag{51}$$

Here the $K+1$ vector-valued function $\boldsymbol{G}(\tau,\mu)$ has components $g_k(\tau,\mu)$ and the $(K+1)\times(K+1)$ constants are given by

$$\boldsymbol{A}=\int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{P}^{\mathrm{T}}(c)\boldsymbol{P}(c)c^3\,\mathrm{d}c, \tag{52}$$

$$\boldsymbol{S}=\int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{P}^{\mathrm{T}}(c)\boldsymbol{P}(c)\nu(c)c^2\,\mathrm{d}c \tag{53}$$

and

$$\boldsymbol{B}_l=\frac{2l+1}{2}\int_0^\infty\int_0^\infty \mathrm{e}^{-c'^2}\,\mathrm{e}^{-c^2}k_l(c',c)\boldsymbol{P}^{\mathrm{T}}(c')\boldsymbol{P}(c)c'^2c^2\,\mathrm{d}c'\,\mathrm{d}c, \tag{54}$$

where the superscript T is used to denote the transpose operation, and where

$$\boldsymbol{P}(c)=\big[P_0\big(2\,\mathrm{e}^{-c}-1\big),P_1\big(2\,\mathrm{e}^{-c}-1\big),\dots,P_K\big(2\,\mathrm{e}^{-c}-1\big)\big]. \tag{55}$$

We note, since $k_l(c',c)=k_l(c,c')$, that the matrices $\boldsymbol{B}_l$ are symmetric. We note also that a computation of the matrices listed as Eq. (54) will require some care to do well; however as discussed in [1], an evaluation of all the input matrices $\boldsymbol{A}$, $\boldsymbol{S}$ and $\boldsymbol{B}_l$ can be done once only and stored for later use.

We find it convenient to multiply Eq. (51) by $\boldsymbol{A}^{-1}$ and then to consider

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{G}(\tau,\mu)+\varepsilon\boldsymbol{\Sigma}\boldsymbol{G}(\tau,\mu)=\varepsilon\sum_{l=0}^{L}\boldsymbol{C}_lP_l(\mu)\int_{-1}^{1}P_l(\mu')\boldsymbol{G}(\tau,\mu')\,\mathrm{d}\mu', \tag{56}$$

where

$$\boldsymbol{\Sigma}=\boldsymbol{A}^{-1}\boldsymbol{S} \tag{57a}$$

and

$$\boldsymbol{C}_l=\boldsymbol{A}^{-1}\boldsymbol{B}_l. \tag{57b}$$

At this point we introduce our [3] analytical discrete-ordinates method (ADO method) and use a "half-range" quadrature scheme to approximate Eq. (56) by writing

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{G}(\tau,\mu)+\varepsilon\boldsymbol{\Sigma}\boldsymbol{G}(\tau,\mu)=\varepsilon\sum_{l=0}^{L}P_l(\mu)\boldsymbol{C}_l\sum_{n=1}^{N}w_n\boldsymbol{G}_{l,n}(\tau), \tag{58}$$

where to compact our notation we have introduced

$$\boldsymbol{G}_{l,n}(\tau)=P_l(\mu_n)\big[\boldsymbol{G}(\tau,\mu_n)+(-1)^l\boldsymbol{G}(\tau,-\mu_n)\big]. \tag{59}$$

Here the N quadrature points $\{\mu_n\}$ and the N weights $\{w_n\}$ are defined for use on the integration interval $[0,1]$. Eq. (58) clearly has separable exponential solutions, so we use ν as a separation constant and substitute

$$\boldsymbol{G}(\tau,\mu)=\boldsymbol{\Phi}(\nu,\mu)\,\mathrm{e}^{-\varepsilon\tau/\nu} \tag{60}$$

into that equation to find

$$\big[\boldsymbol{\Sigma}-(\mu/\nu)\boldsymbol{I}\big]\boldsymbol{\Phi}(\nu,\mu)=\sum_{l=0}^{L}P_l(\mu)\boldsymbol{C}_l\sum_{n=1}^{N}w_n\boldsymbol{\Phi}_{l,n}(\nu), \tag{61}$$

where $\boldsymbol{I}$ is the identity matrix and

$$\boldsymbol{\Phi}_{l,n}(\nu)=P_l(\mu_n)\big[\boldsymbol{\Phi}(\nu,\mu_n)+(-1)^l\boldsymbol{\Phi}(\nu,-\mu_n)\big]. \tag{62}$$

If we now evaluate Eq. (61) at $\mu=\pm\mu_i$, for $i=1,2,\dots,N$, then we can obtain an eigenvalue problem we can solve numerically to establish a collection of elementary solutions of a discrete version of Eq. (61). Omitting the details of this approach, all of which were reported explicitly in [1], we express our first result as

$$\boldsymbol{G}(\tau,\pm\mu_i)=\sum_{j=1}^{J}\big[A_j\boldsymbol{\Phi}(\nu_j,\pm\mu_i)\,\mathrm{e}^{-\varepsilon\tau/\nu_j}+B_j\boldsymbol{\Phi}(\nu_j,\mp\mu_i)\,\mathrm{e}^{\varepsilon\tau/\nu_j}\big], \tag{63}$$

for $i = 1, 2, \dots, N$. Here the arbitrary constants $\{A_j\}$ and $\{B_j\}$ are to be determined by the boundary and other conditions of a given problem, the separation constants $\{\nu_j\}$ and the elementary solutions $\boldsymbol{\Phi}(\nu_j, \pm\mu_i)$ are, at this point, considered known (from the numerical solution of the mentioned eigenvalue problem) and $J = N(K+1)$.

We found in [1] that four of the separation constants tended to infinity as the order N of our quadrature scheme increased, and so we neglect these separation constants in Eq. (63) and use instead four known exact solutions of Eq. (48). We also redefine the arbitrary constants in Eq. (63) and write our general (approximate) solution to Eq. (48) in the form

$$\phi(\tau, c, \pm\mu_i) = \phi_a(\tau, c, \pm\mu_i) + \boldsymbol{P}(c) \sum_{j=3}^{J} \big[A_j \boldsymbol{\Phi}(\nu_j, \pm\mu_i)\, \mathrm{e}^{-\varepsilon(a+\tau)/\nu_j} + B_j \boldsymbol{\Phi}(\nu_j, \mp\mu_i)\, \mathrm{e}^{-\varepsilon(a-\tau)/\nu_j}\big], \tag{64}$$

where

$$\phi_a(\tau, c, \pm\mu_i) = \pm A_1 c\mu_i + A_2\big(c^2 - 5/2\big) + B_1 + B_2\big[\big(c^2 - 5/2\big)\varepsilon\tau \mp \mu_i A(c)\big], \tag{65}$$

and where $A(c)$ is the Chapman–Enskog function related to thermal conductivity [20,21]. Having established Eqs. (64) and (65), we are ready to use these results to solve the four problems considered in this work.

4. The solutions

4.1. A heat-transfer problem for the case of two parallel surfaces

In regard to this problem, we intend to use the boundary conditions listed as Eqs. (16), with $\delta_1 = 1$ and $\delta_2 = -1$, and the condition listed as Eq. (20) to determine the arbitrary constants in the solution defined by Eqs. (64) and (65). In order to construct a linear system from which to determine these constants, we multiply Eqs. (16) by

$$W_i(c) = c^2\, \mathrm{e}^{-c^2} P_i\big(2\,\mathrm{e}^{-c} - 1\big), \tag{66}$$

for $i = 0, 1, \dots, K$, integrate over all c and then evaluate the resulting equations at the N quadrature points $\{\mu_k\}$. This procedure leads to a "square" linear system from which to determine the required constants; however, we note that a constant is a solution of Eqs. (48) that also satisfies the homogeneous version of Eqs. (16), and so the constant B_1 in Eq. (65) cannot be determined from the linear system. It is for this reason that we use, after finding all of the other arbitrary constants, the condition listed as Eq. (20) to determine B_1. It is clear that without the use of Eq. (20) this heat-transfer problem does not have a unique solution. And so to start we omit the constant B_1 in Eq. (65) and focus our attention on the other constants. This procedure leads to a "non-square" linear system, and so we elected to obtain a square system by adding one of the equations to another and then omitting one of these two equations. In this way we were able to use a standard routine to solve the resulting square system of linear algebraic equations. Once the constants in Eqs. (64) and (65) were determined so as to establish $\phi(\tau, c, \mu)$, we can evaluate the quantities of interest.

In regard to our basic results, we should make note of an important aspect (not mentioned before) of our solution that results from the fact that the expansion given by Eq. (49) and the ADO method are approximations. While the quantities U and Q as defined by exact theory must be constants, this is not necessarily so after the Legendre expansion and the use of the discrete-ordinates method. However, we have seen that as the order of the expansion and the order of the ADO method increase we can expect U and Q to approach constant values. And so, in computing the heat flux Q we have ignored the exponential terms and the coefficient A_1 in Eqs. (64) and (65) to obtain

$$Q = -\frac{5}{4}\varepsilon_t B_2, \tag{67}$$

where

$$\varepsilon_t = \frac{16}{15\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2} A(c) c^5\, \mathrm{d}c \tag{68}$$

or

$$\varepsilon_t = 0.679630049\dots. \tag{69}$$

We also find

$$N(\tau) = -A_2 - B_2\varepsilon\tau + B_1 + \sum_{j=3}^{J} \big[A_j\, \mathrm{e}^{-\varepsilon(a+\tau)/\nu_j} + B_j\, \mathrm{e}^{-\varepsilon(a-\tau)/\nu_j}\big] N_j \tag{70}$$

and

$$T(\tau) = A_2 + B_2\varepsilon\tau + \sum_{j=3}^{J}\big[A_j\,\mathrm{e}^{-\varepsilon(a+\tau)/\nu_j} + B_j\,\mathrm{e}^{-\varepsilon(a-\tau)/\nu_j}\big]T_j \tag{71}$$

where

$$N_j = 2\pi^{-1/2}\boldsymbol{P}_0\boldsymbol{N}(\nu_j), \tag{72}$$

$$T_j = (4/3)\pi^{-1/2}\big[\boldsymbol{P}_2 - (3/2)\boldsymbol{P}_0\big]\boldsymbol{N}(\nu_j), \tag{73}$$

$$\boldsymbol{N}(\nu_j) = \sum_{n=1}^{N} w_n\big[\boldsymbol{\Phi}(\nu_j, \mu_n) + \boldsymbol{\Phi}(\nu_j, -\mu_n)\big] \tag{74}$$

and

$$\boldsymbol{P}_n = \int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{P}(c)c^{n+2}\,\mathrm{d}c. \tag{75}$$

In the next section of this work we discuss some computational aspects of the solution developed here, and we report some relevant numerical results.

4.2. *The problem of a reverse temperature gradient*

Looking back to Eqs. (31), we see that for this problem we must define the constants in Eqs. (64) and (65) so that the solution will satisfy the conditions

$$\phi(-a, c, \mu) = -\big(\beta + c^2 - 3/2\big)\Delta T \tag{76a}$$

and

$$\phi(a, c, -\mu) = \big(\beta + c^2 - 3/2\big)\Delta T, \tag{76b}$$

for $\mu \in (0, 1]$ and all c. We find it convenient to express the desired solution as

$$\phi(\tau, c, \mu) = \big[(\beta - 3/2)\phi_1(\tau, c, \mu) + \phi_2(\tau, c, \mu)\big]\Delta T, \tag{77}$$

where $\phi_1(\tau, c, \mu)$ and $\phi_2(\tau, c, \mu)$ can each be expressed in the form given by Eqs. (64) and (65). However, the first of these two special solutions must satisfy the boundary conditions

$$\phi_1(-a, c, \mu) = -1 \tag{78a}$$

and

$$\phi_1(a, c, -\mu) = 1, \tag{78b}$$

for $\mu \in (0, 1]$ and all c, while the second solution must satisfy the boundary conditions

$$\phi_2(-a, c, \mu) = -c^2 \tag{79a}$$

and

$$\phi_2(a, c, -\mu) = c^2, \tag{79b}$$

for $\mu \in (0, 1]$ and all c. Entering Eqs. (64) and (65) into Eqs. (78) and (79), "projecting" against $W_i(c)$ and evaluating the resulting equations at the quadrature points $\{\mu_k\}$, we obtain square linear systems that can be solved to establish the arbitrary constants required to complete the two special solutions $\phi_1(\tau, c, \mu)$ and $\phi_2(\tau, c, \mu)$. Once these special solutions are available, the required results can, for any value of β, be obtained from

$$N(\tau) = \big[(\beta - 3/2)N_1(\tau) + N_2(\tau)\big]\Delta T, \tag{80a}$$

$$T(\tau) = \big[(\beta - 3/2)T_1(\tau) + T_2(\tau)\big]\Delta T, \tag{80b}$$

$$U = \big[(\beta - 3/2)U_1 + U_2\big]\Delta T, \tag{80c}$$

$$Q = \big[(\beta - 3/2)Q_1 + Q_2\big]\Delta T \tag{80d}$$

and

$$U_+ = \big[(\beta - 3/2)U_{1+} + U_{2+}\big]\Delta T. \tag{80e}$$

If we use labels k =1 and 2 with Eqs. (64) and (65) to identify the two special problems, we can express the quantities required in Eqs. (80) as

$$Q_k = -\frac{5}{4}\varepsilon_t B_2^k, \tag{81a}$$

$$N_k(\tau) = -B_2^k\varepsilon\tau + \sum_{j=3}^{J} A_j^k N_j\big[\mathrm{e}^{-\varepsilon(a+\tau)/\nu_j} - \mathrm{e}^{-\varepsilon(a-\tau)/\nu_j}\big], \tag{81b}$$

$$T_k(\tau) = B_2^k\varepsilon\tau + \sum_{j=3}^{J} A_j^k T_j\big[\mathrm{e}^{-\varepsilon(a+\tau)/\nu_j} - \mathrm{e}^{-\varepsilon(a-\tau)/\nu_j}\big] \tag{81c}$$

and

$$U_k = A_1^k/2. \tag{81d}$$

In addition to the general results given by Eqs. (80) we can use the special solutions to obtain the critical values β_T and β_Q. For a given value of the vapor thickness $\delta = 2a$, we can differentiate (with respect to τ) Eq. (80b), set the resulting equation (evaluated at $\tau = 0$) equal to zero and solve for $\beta = \beta_T$. In this way we find

$$\beta_T = 3/2 - T_2'(0)/T_1'(0). \tag{82}$$

In a similar way we can obtain from Eq. (80d) the critical value $\beta = \beta_Q$ for which the heat flow at $\tau = 0$ changes sign. We find

$$\beta_Q = 3/2 - Q_2(0)/Q_1(0). \tag{83}$$

While we intend to compute U_1, U_2, U_{1+} and U_{2+} independently, we can use Eqs. (78b) and (79b) to conclude that

$$U_{1+} = U_1 + 1/\big(2\pi^{1/2}\big) \tag{84a}$$

and

$$U_{2+} = U_2 + 1/\pi^{1/2} \tag{84b}$$

which can be used as low-level checks on our computations. In addition, we note that Sharipov [22] has communicated the expression

$$U_2 - \frac{5}{2}U_1 = Q_1 \tag{85}$$

which also can be used as low-level check on our computations. Eq. (85) follows from the reciprocity arguments of Onsager [23,24] and Sharipov [25,26] and can be obtained here in the following way: we first write the defining balance equations for $\phi_1(\tau, c, \mu)$ and $\phi_2(\tau, c, -\mu)$ as

$$c\mu\frac{\partial}{\partial\tau}\phi_1(\tau, c, \mu) = \mathcal{L}_1\{\phi_1\}(\tau, c, \mu) \tag{86a}$$

and

$$-c\mu\frac{\partial}{\partial\tau}\phi_2(\tau, c, -\mu) = \mathcal{L}_1\{\phi_2\}(\tau, c, -\mu), \tag{86b}$$

where

$$\mathcal{L}_1\{\phi\}(\tau, c, \mu) = -\varepsilon\nu(c)\phi(\tau, c, \mu) + \varepsilon\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c'^2}\phi(\tau, c', \mu')k(c', \mu' : c, \mu)c'^2\,\mathrm{d}\mu'\,\mathrm{d}c', \tag{87}$$

and where $k(c', \mu' : c, \mu)$ is given by Eq. (18). We can now multiply Eqs. (86a) and (86b), respectively, by

$$\mathrm{e}^{-c^2}\phi_2(\tau, c, -\mu)c^2$$

and

$$\mathrm{e}^{-c^2}\phi_1(\tau, c, \mu)c^2,$$

integrate over all c and μ, and subtract the resulting two equations, one from the other, to obtain an equation we can integrate over τ from $-a$ to a to find

$$\int_0^\infty\int_0^1 \mathrm{e}^{-c^2}\big[\phi_1(a, c, \mu)c^2 - \phi_2(a, c, \mu)\big]c^3\mu\,\mathrm{d}\mu\,\mathrm{d}c = 0. \tag{88}$$

In obtaining Eq. (88), we have we made use of Eqs. (78) and (79) and the fact that

$$\phi_\alpha(\tau, c, \mu) = -\phi_\alpha(-\tau, c, -\mu) \tag{89}$$

for $\alpha = 1$ and 2. Finally, we can use Eqs. (84) in Eq. (88), along with the basic definitions given by Eqs. (24) and (34), to find the desired result, viz.

$$U_2 - \frac{5}{2}U_1 = Q_1. \tag{90}$$

4.3. Evaporation/condensation in a semi-infinite half space

For this half-space problem we conclude from Eqs. (64) and (65) that we can set $a = 0$ and express the desired solution as

$$\phi(\tau, c, \pm\mu_i) = \phi_a(\tau, c, \pm\mu_i) + \boldsymbol{P}(c)\sum_{j=3}^{J} A_j\boldsymbol{\Phi}(\nu_j, \pm\mu_i)\,\mathrm{e}^{-\varepsilon\tau/\nu_j}, \tag{91}$$

where, recalling that we have normalized this problem by taking $U = 1$, we now have

$$\phi_a(\tau, c, \pm\mu_i) = \pm 2c\mu_i + A_2\big(c^2 - 5/2\big) + B_1. \tag{92}$$

Entering Eqs. (91) and (92) into Eq. (37), "projecting" against $W_i(c)$ and evaluating the resulting equation at the quadrature points $\{\mu_k\}$, we obtain a square linear systems that can be solved to establish the arbitrary constants required to complete the solution $\phi(\tau, c, \mu)$. In this way we find the final results for the density and temperature profiles can be written as

$$N(\tau) = -A_2 + B_1 + \sum_{j=3}^{J} A_j N_j\,\mathrm{e}^{-\varepsilon\tau/\nu_j} \tag{93a}$$

and

$$T(\tau) = A_2 + \sum_{j=3}^{J} A_j T_j\,\mathrm{e}^{-\varepsilon\tau/\nu_j}, \tag{93b}$$

where N_j and T_j are given by Eqs. (72) and (73). To be clear, we note again that Eqs. (93) are the density and temperature perturbations for the (evaporation) case $U = 1$. To have results for another value of U requires only that we multiply Eqs. (93) by a given value of U.

4.4. Heat transfer in a semi-infinite half space

Here the solution $\phi(\tau, c, \mu)$ is required to diverge as τ tends to infinity, but at the same time we must also satisfy Eqs. (41). And so we express our solution from Eqs. (64) and (65) as

$$\phi(\tau, c, \pm\mu_i) = \phi_a(\tau, c, \pm\mu_i) + \boldsymbol{P}(c)\sum_{j=3}^{J} A_j\boldsymbol{\Phi}(\nu_j, \pm\mu_i)\,\mathrm{e}^{-\varepsilon\tau/\nu_j}, \tag{94}$$

where we now have

$$\phi_a(\tau, c, \pm\mu_i) = A_2\big(c^2 - 5/2\big) + B_1 + (1/\varepsilon)\big[\big(c^2 - 5/2\big)\varepsilon\tau \mp \mu_i A(c)\big]. \tag{95}$$

Now, we substitute Eqs. (94) and (95) into Eq. (46), "project" against $W_i(c)$ and evaluate the resulting equation at the quadrature points $\{\mu_k\}$ to obtain a square linear systems that can be solved to establish the arbitrary constants required to complete the solution $\phi(\tau, c, \mu)$. In this way we find the final results for the density and temperature profiles can be written as

$$N(\tau) = -A_2 + B_1 - \tau + \sum_{j=3}^{J} A_j N_j \, \mathrm{e}^{-\varepsilon\tau/\nu_j} \tag{96a}$$

and

$$T(\tau) = A_2 + \tau + \sum_{j=3}^{J} A_j T_j \, \mathrm{e}^{-\varepsilon\tau/\nu_j}, \tag{96b}$$

where (still) N_j and T_j are given by Eqs. (72) and (73).

5. Numerical results

Because of the importance given to the problem of heat transfer in a plane-parallel medium, there exits a large body of work devoted to the subject. While we can mention the early works by Wang Chang and Uhlenbeck [27], Gross and Ziering [28], Willis [29], Frankowski, Alterman and Pekeris [30], we refer to Cercignani's recent book [31] for a vast list of related works. As far as our specific interest here, we have made use of two works by Bassanini, Cercignani and Pagani [32,33], as well as papers by Thomas, Chang, and Siewert [34], Valougeorgis and Thomas [35], Ohwada, Aoki and Sone [36] and Siewert [11]. In regard to the mentioned works [11,27–36], only the paper by Ohwada Aoki and Sone [36] bases the analysis on the linearized Boltzmann equation, and so this work is the one most relevant to us here. The papers by Bassanini, Cercignani and Pagani [32,33] are based on the use of variational and numerical methods and the classical BGK model. Thomas, Chang and Siewert [34] reported the first definitive results for the BGK model and Valougeorgis and Thomas [35] used the so-called $\mathrm{F_N}$ method and the BGK model to solve the heat transfer problem for flow between two walls that are characterized by two independent accommodation constants. Our work here thus can be seen as incorporating the generality of the Valougeorgis and Thomas paper [35] with the more definitive model (the linearized Boltzmann equation) used by Ohwada, Aoki and Sone [36].

In regard to our numerical work, we have typically used the approximation parameters $K = 30$ and $N = 30$ in obtaining the results we list in Tables 1–4. However, to establish some confidence in our numerical results we found convergence (to the digits listed in the tables) by increasing (and decreasing) these two approximation parameters. In Tables 1 and 2 we list the density and temperature perturbations for two selected cases. In Table 3 we report our values of the normalized heat flux. In all of our computations, we have made use of the mean-free path based on thermal conductivity, i.e., we have used

$$\varepsilon = \varepsilon_t. \tag{97}$$

It can be seen from Tables 1 and 2 that the BGK model yields reasonable results (essentially two significant figures of accuracy) when compared to the linearized Boltzmann equation for hard-sphere interactions.

Table 1
Temperature and density profiles for the case $a = 1$, $\alpha_1 = 0.7$ and $\alpha_2 = 0.3$

	$T(-a+2\eta a)$		$N(-a+2\eta a)$	
η	BGK	LBE	BGK	LBE
0.0	6.7458(−1)	6.7606(−1)	−2.6459(−1)	−2.5368(−1)
0.1	5.9637(−1)	5.9100(−1)	−1.9310(−1)	−1.8265(−1)
0.2	5.3998(−1)	5.3593(−1)	−1.4119(−1)	−1.3342(−1)
0.3	4.8860(−1)	4.8692(−1)	−9.3472(−2)	−8.8300(−2)
0.4	4.3929(−1)	4.4033(−1)	−4.7442(−2)	−4.4787(−2)
0.5	3.9063(−1)	3.9450(−1)	−1.9405(−3)	−1.7756(−3)
0.6	3.4155(−1)	3.4824(−1)	4.3875(−2)	4.1504(−2)
0.7	2.9093(−1)	3.0025(−1)	9.0919(−2)	8.5893(−2)
0.8	2.3700(−1)	2.4843(−1)	1.4064(−1)	1.3278(−1)
0.9	1.7585(−1)	1.8785(−1)	1.9638(−1)	1.8556(−1)
1.0	8.4067(−2)	8.4272(−2)	2.7976(−1)	2.6933(−1)

Table 2
Temperature and density profiles for the case $a = 2.5$, $\alpha_1 = 1.0$ and $\alpha_2 = 0.5$

	$T(-a+2\eta a)$		$N(-a+2\eta a)$	
η	BGK	LBE	BGK	LBE
0.0	8.2673(−1)	8.2333(−1)	−5.7874(−1)	−5.6834(−1)
0.1	6.7574(−1)	6.6684(−1)	−4.3714(−1)	−4.2825(−1)
0.2	5.5806(−1)	5.5223(−1)	−3.2388(−1)	−3.1770(−1)
0.3	4.4682(−1)	4.4421(−1)	−2.1553(−1)	−2.1140(−1)
0.4	3.3803(−1)	3.3853(−1)	−1.0898(−1)	−1.0661(−1)
0.5	2.2994(−1)	2.3353(−1)	−2.9448(−3)	−2.2547(−3)
0.6	1.2127(−1)	1.2816(−1)	1.0350(−1)	1.0237(−1)
0.7	1.0579(−2)	2.1165(−2)	2.1140(−1)	2.0807(−1)
0.8	−1.0458(−1)	−8.9846(−2)	3.2258(−1)	3.1637(−1)
0.9	−2.3031(−1)	−2.1157(−1)	4.4183(−1)	4.3158(−1)
1.0	−4.1030(−1)	−4.0267(−1)	6.0750(−1)	5.9529(−1)

Table 3
The normalized heat flux q

α_1	α_2	$a = 0.1$	$a = 0.5$	$a = 1.0$	$a = 1.5$	$a = 2.0$	$a = 2.5$
0.7	0.1	9.85339(−1)	9.44283(−1)	9.04244(−1)	8.69276(−1)	8.37448(−1)	8.08046(−1)
0.7	0.3	9.61123(−1)	8.61658(−1)	7.75502(−1)	7.08001(−1)	6.52110(−1)	6.04641(−1)
0.7	0.5	9.42048(−1)	8.03655(−1)	6.93253(−1)	6.12618(−1)	5.49559(−1)	4.98494(−1)
0.7	0.7	9.26730(−1)	7.60977(−1)	6.36428(−1)	5.49802(−1)	4.84614(−1)	4.33435(−1)
0.7	0.9	9.14234(−1)	7.28469(−1)	5.95003(−1)	5.05448(−1)	4.39902(−1)	3.89560(−1)
0.7	1.0	9.08832(−1)	7.15013(−1)	5.78282(−1)	4.87862(−1)	4.22420(−1)	3.72597(−1)
0.9	0.1	9.85019(−1)	9.43315(−1)	9.02501(−1)	8.66748(−1)	8.34195(−1)	8.04152(−1)
0.9	0.3	9.58139(−1)	8.52558(−1)	7.61852(−1)	6.91376(−1)	6.33546(−1)	5.84853(−1)
0.9	0.5	9.34745(−1)	7.83172(−1)	6.65323(−1)	5.81139(−1)	5.16540(−1)	4.65054(−1)
0.9	0.7	9.14234(−1)	7.28469(−1)	5.95003(−1)	5.05448(−1)	4.39902(−1)	3.89560(−1)
0.9	0.9	8.96135(−1)	6.84283(−1)	5.41521(−1)	4.50276(−1)	3.85821(−1)	3.37630(−1)
0.9	1.0	8.87870(−1)	6.65245(−1)	5.19319(−1)	4.27947(−1)	3.64341(−1)	3.17305(−1)
1.0	1.0	8.78053(−1)	6.43426(−1)	4.94554(−1)	4.03495(−1)	3.41131(−1)	2.95558(−1)

Table 4
Comparison results for the heat flow Q for the case $\alpha_1 = \alpha_2 = 1$

	$k = 0.1$	$k = 0.5$	$k = 1.0$	$k = 2.0$	$k = 5.0$	$k = 8.0$	$k = 10.0$
[36]	3.246(−1)	7.170(−1)	8.577(−1)	9.609(−1)	1.046	1.073	1.083
This work	3.247(−1)	7.171(−1)	8.576(−1)	9.609(−1)	1.046	1.073	1.083

In order to check our results against those of Ohwada, Aoki and Sone [36], we compare in Table 4 our values of the non-dimensional heat flux to the results reported in [36]. In order to make this comparison, we have used the expression

$$a = \frac{2^{1/2}}{8\varepsilon k}, \tag{98}$$

where k is the scaling factor used in [36], to define the half thickness of the gas. While we believe our results for q and for Q are, in general, correct to six figures of accuracy, we see that we have essentially perfect agreement with the four significant figures reported by Ohwada, Aoki and Sone [36].

In regard to the problem of reverse temperature gradient, we have made use of early works by Pao [14] and by Thomas, Chang and Siewert [15] that are based on the BGK kinetic model. While the paper of Pao [14] gave the first asymptotic ($\delta \to \infty$) result $\beta_T(\infty) = 3.5$, the first definitive (BGK) result $\beta_T(\infty) = 3.7723$ was reported in [15]. In more recent times, Sone, Ohwada and Aoki [16] used the linearized Boltzmann equation (for rigid-sphere interactions) to find $\beta_T(\infty) = 3.6992$; see also a paper by Aoki and Masukawa [37] that makes use of a non-linear BGK model equation. Here, using the linearized Boltzmann equation for rigid spheres, our Legendre expansion and the ADO method, we obtained the result $\beta_T(\infty) = 3.6996$. Our general results for this problem are given in Tables 5–8. In Tables 5, 6 and 7 we give our results for the two special problems, labeled with the indices $k = 1$ and 2, so that desired results for any specified value of the parameter β is available as a linear combination

Table 5
The problem of a reverse temperature gradient

a	$-U_1$	$-U_2$	Q_1	$-Q_2$
0.1	5.3688(−1)	1.1036	2.3857(−1)	5.0037(−1)
0.5	5.0653(−1)	1.0991	1.6728(−1)	3.6307(−1)
1.0	4.9467(−1)	1.1108	1.2586(−1)	2.7541(−1)
1.5	4.8885(−1)	1.1207	1.0147(−1)	2.2268(−1)
2.0	4.8522(−1)	1.1279	8.5124(−2)	1.8705(−1)
2.5	4.8267(−1)	1.1333	7.3351(−2)	1.6127(−1)
5.0	4.7628(−1)	1.1473	4.3408(−2)	9.5481(−2)
9.0	4.7263(−1)	1.1553	2.6263(−2)	5.7770(−2)
13.0	4.7105(−1)	1.1588	1.8827(−2)	4.1413(−2)

Table 6
The problem of a reverse temperature gradient

a	β_T	β_Q	$-U_{1+}$	$-U_{2+}$
0.1	5.1637	3.5974	2.5478(−1)	5.3943(−1)
0.5	4.5030	3.6705	2.2444(−1)	5.3487(−1)
1.0	4.0607	3.6882	2.1257(−1)	5.4662(−1)
1.5	3.8468	3.6946	2.0676(−1)	5.5647(−1)
2.0	3.7443	3.6974	2.0312(−1)	5.6374(−1)
2.5	3.6979	3.6986	2.0058(−1)	5.6914(−1)
5.0	3.6838	3.6996	1.9419(−1)	5.8310(−1)
9.0	3.6984	3.6996	1.9054(−1)	5.9113(−1)
13.0	3.6996	3.6996	1.8896(−1)	5.9461(−1)
15.0	3.6996	3.6996	1.8846(−1)	5.9570(−1)

Table 7
The problem of a reverse temperature gradient for the case $a = 1$

τ	$-T_1(\tau)$	$N_1(\tau)$	$T_2(\tau)$	$N_2(\tau)$
0.0	0.0	0.0	0.0	0.0
0.1	1.0697(−2)	1.6599(−2)	2.7426(−2)	−1.2010(−2)
0.2	2.1398(−2)	3.3490(−2)	5.5063(−2)	−2.3504(−2)
0.3	3.2105(−2)	5.1000(−2)	8.3140(−2)	−3.3908(−2)
0.4	4.2826(−2)	6.9529(−2)	1.1193(−1)	−4.2509(−2)
0.5	5.3569(−2)	8.9622(−2)	1.4180(−1)	−4.8346(−2)
0.6	6.4352(−2)	1.1208(−1)	1.7324(−1)	−4.9982(−2)
0.7	7.5214(−2)	1.3824(−1)	2.0706(−1)	−4.5053(−2)
0.8	8.6239(−2)	1.7061(−1)	2.4468(−1)	−2.9062(−2)
0.9	9.7687(−2)	2.1534(−1)	2.8942(−1)	8.9304(−3)
1.0	1.1178(−1)	3.1031(−1)	3.5985(−1)	1.3620(−1)

Table 8
Comparison results for the critical value β_T

	$k = 0.01$	$k = 0.1$	$k = 0.6$	$k = 1.0$	$k = 6.0$	$k = 10.0$	$k = 20.0$
[16]	3.6992	3.692	4.590	4.863	5.237	5.227	5.185
This work	3.6996	3.693	4.590	4.863	5.237	5.226	5.177

of these results: see Eqs. (80). In Table 6 we also list our results for β_T and β_Q, as defined by Eqs. (82) and (83), for selected values of the half thickness a. In regard to the numerical work, we have, as for the problem of heat transfer, typically used the approximation parameters $K = 30$ and $N = 30$ in obtaining the results we list in Tables 5–8. Again, while we have no definitive proof of the accuracy we achieved here, we believe our results are correct to all digits given. To compare our work here with the best results available, we list in Table 8 our values and those of [16] for the critical value β_T as a function of the parameter k used in Eq. (98) to define the vapor half thickness.

Table 9
Half-space temperature and density profiles

	Evaporation/condensation		Heat transfer	
τ	$-T(\tau)$	$-N(\tau)$	$T(\tau)$	$-N(\tau)$
0.0	4.0355(−1)	1.3073	8.6474(−1)	4.1042(−1)
0.1	4.2151(−1)	1.4586	1.0902	6.0070(−1)
0.2	4.3036(−1)	1.5190	1.2452	7.3850(−1)
0.3	4.3636(−1)	1.5570	1.3819	8.6335(−1)
0.4	4.4074(−1)	1.5835	1.5091	9.8153(−1)
0.5	4.4406(−1)	1.6031	1.6301	1.0956
0.6	4.4663(−1)	1.6180	1.7469	1.2068
0.7	4.4865(−1)	1.6296	1.8607	1.3160
0.8	4.5024(−1)	1.6389	1.9722	1.4237
0.9	4.5152(−1)	1.6463	2.0818	1.5302
1.0	4.5254(−1)	1.6524	2.1899	1.6357
2.0	4.5611(−1)	1.6779	3.2303	2.6642
5.0	4.5576(−1)	1.6853	6.2476	5.6781
7.0	4.5563(−1)	1.6855	8.2484	7.6789

Table 10
Comparison results: evaporation/condensation problem

	$-T(0)$	$-N(0)$	$-T(\infty)$	$-N(\infty)$
[38]	4.0368(−1)	1.3074	4.5566(−1)	1.6856
[39]	4.036 (−1)	1.307	4.556 (−1)	1.686
This work	4.0355(−1)	1.3073	4.5559(−1)	1.6855

The formulation of the evaporation/condensation problem we considered in this work is based on two early papers by Pao [17] and Siewert and Thomas [18] both of which are based on the BGK kinetic model. However, Sone, Ohwada and Aoki [38] and Loyalka [39] have, more recently, solved this same problem as based on the linearized Boltzmann equation for rigid-sphere interactions. While the four papers mentioned [17,18,38,39] are the ones most related to our work here, a new paper by Sone [40] gives an extensive discussion of general evaporation and condensation problems and is a good source of other reference material. Our numerical results for the temperature and density perturbations are given in Table 9, and in Table 10 we compare our most basic results to those of Sone, Ohwada and Aoki [38] and to those of Loyalka [39]. We believe our results to be correct to all digits given.

Finally, the heat-transfer problem solved in this work was also discussed in terms of the BGK kinetic model in [17] and [18]. In Table 9 we report our results for the temperature and density profiles. As mentioned previously in this work, the temperature perturbation is the same as for the temperature-jump problem [1] for the case of diffuse reflection (accommodation coefficient equal to unity), while the density perturbation differs only by an additive constant from the density profile for that same temperature-jump problem.

To conclude this section, we note that we base the confidence we have in the reported numerical results essentially on two observations: firstly, our results are stable with respect to changes in the number of Legendre terms used in Eq. (49) and with respect to changes in the order N of the ADO method used, and secondly, our codes yielded known BGK results when the Pekeris components in the scattering kernel and the collision frequency were replaced by the appropriate BGK versions of these quantities. Of course, this is not a proof of the accuracy achieved since we kept only the first nine terms in Eq. (18), and the "multigroup" constants listed as Eqs. (52)–(54) where evaluated by numerical integration.

6. Concluding comments

We have used a new polynomial expansion technique and the Pekeris [7] expanded form of the scattering kernel basic to the linearized Boltzmann equation for rigid-sphere collisions to define a system of coupled transport problems that has been solved efficiently and accurately with a modern version [3] of the discrete-ordinates method usually associated with Chandrasekhar [4] and the field of radiative transfer. While there exist other basic works that report numerical results for the four problems solved here, we are of the opinion that our computational methods are especially efficient in regard to accuracy and computer-time requirements when compared to existing solutions.

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