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AN EXACT SOLUTION IN THE THEORY OF LINE FORMATION

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SUMMARY

The singular eigenfunction expansion technique is used to solve rigorously the equation of radiative transfer describing the interlocking-doublet model in the theory of line scattering. The Planck function, represented as a linear function of the optical variable, is assumed to be independent of frequency over the range of interest, and the ratios η_j of the line-scattering coefficients to the continuous absorption coefficient are taken to be constants. The normal-mode expansion technique, in conjunction with the appropriate particular solution, is used to obtain a rigorous analytical solution for the radiation intensity valid anywhere in a semi-infinite medium subjected to zero incident radiation. Half-range completeness and orthogonality theorems applicable for the basis set used are employed to effect the desired solution with a minimum of effort, and, as a procedure alternative to older techniques, the Case method is used to construct the half-space S-matrix, useful when surface quantities are of principal interest.

1. INTRODUCTION

In the theory of line scattering the absorption process is normally characterized by a continuous absorption coefficient, and the scattering phenomenon is described by a distinctly different and idealized model of the scattering coefficient. In the simplest case (coherent scattering) it is assumed that no change in frequency occurs during the scattering process. In general, however, the scattering law for isotropic scattering would be represented by a function $f(\nu' \to \nu)$, such that $f(\nu' \to \nu)\, d\nu$ denotes the probability of a scattered photon of initial frequency ν' being scattered into a frequency interval $d\nu$ about ν.

An important model in the theory of line scattering is the concept of interlocking multiplets discussed by Woolley & Stibbs (1). Here the scattering is not described by a continuous function of frequency, but is assumed to be possible only between the various scattering lines.

One of the earlier rigorous solutions to line scattering problems was that given by Chandrasekhar (2) for the coherent scattering theory; the principles of invariance were used to construct the solution for the radiation intensity emitted from the surface of a semi-infinite half-space in which the Planck function was represented as a linear function of the optical variable. These invariance principles (3) were also used by Stibbs (4) to develop the exit distribution in a similar problem, but with a more general scattering law, and the same method was employed by Busbridge & Stibbs (5) for the interlocking multiplet model. In a more general study, the half-space scattering function $S(\nu', \nu, \mu', \mu)$ was developed by Busbridge (6) for the completely non-coherently scattering theory, and thus the solutions for the various surface quantities were made available.

In 1960 Case (**7**) introduced the method of singular eigenfunction expansions and applied the new technique to problems in neutron transport theory. Although the Case-method has found favour mostly in the realm of neutron transport theory, the technique has been utilized for several astrophysical applications; for example, the picket-fence model was discussed by Siewert & Zweifel (**8**), the transfer of polarized radiation in a free-electron atmosphere was investigated by Siewert & Fraley (**9**), and the method was used by Siewert & McCormick (**10**) in a study of line formation with anisotropic scattering.

The principal advantage of the normal-mode expansion technique is that solutions valid at any optical depth are obtained, as well as results for the various surface quantities of interest. An additional merit of the Case technique is that the usual idea of expansions in terms of complete and orthogonal basis sets is used to yield solutions with a minimum of manipulation.

In the present analysis the method of normal modes is used for the case of an interlocked doublet to generate a rigorous solution for the radiation intensity valid anywhere in a semi-infinite medium with a linearly varying Planck function. In Section 2 we give a brief formulation of the problem of interlocking multiplets, and in Section 3 we develop a general solution for the doublet. Section 4 is devoted to an alternative derivation of the scattering function, useful if only surface quantities are of interest.

2. BASIC FORMULATION

We consider initially the frequency-dependent equation of transfer (**3**)

$$\mu \frac{\partial}{\partial z} I_\nu(z, \mu) + \rho(z)(k_\nu + s_\nu) I_\nu(z, \mu) = \rho(z) \frac{(1-\epsilon_\nu)}{2} \int_0^\infty d\nu' \int_{-1}^{1} d\mu' s_{\nu'} f(\nu' \to \nu) I_{\nu'}(z, \mu') + \rho(z)(k_\nu + s_\nu \epsilon_\nu) B_\nu[T(z)], \quad (1)$$

where k_ν and s_ν are respectively the absorption and scattering coefficients, $B_\nu[T(z)]$ is the Planck blackbody function, $f(\nu' \to \nu)$ denotes the scattering law, and ϵ_ν is Eddington's collision constant (**4**). Also z is the position variable, $\rho(z)$ is the density of the medium, and μ is the direction cosine (as measured from the positive z-axis) of the propagating radiation. We now restrict our attention to a frequency band $\nu \in \Delta\nu$ within which it is assumed that k_ν, ϵ_ν, and $B_\nu[T(z)]$ are independent of frequency. We also introduce an optical variable

$$\tau = \int_0^z k\rho(z)\, dz, \quad (2)$$

define

$$\eta_\nu = s_\nu / k, \quad (3)$$

and thus rewrite equation (1) in the form

$$\mu \frac{\partial}{\partial \tau} I_\nu(\tau, \mu) + (1+\eta_\nu) I_\nu(\tau, \mu) = \frac{(1-\epsilon)}{2} \int_0^\infty d\nu' \int_{-1}^{1} d\mu' \eta_{\nu'} f(\nu' \to \nu) I_{\nu'}(\tau, \mu') + (1+\epsilon\eta_\nu) B[T(\tau)], \quad \nu \in \Delta\nu, \quad (4)$$

where a deleted subscript indicates that the function has been assumed frequency independent. Before representing the scattering law by the theory of interlocking multiplets, we assume that the Planck function for $\nu \in \Delta\nu$ is given by

$$B[T(\tau)] = a + b\tau, \quad (5)$$

where a and b are taken to be known constants.

The principle of interlocking multiplets is based on the proposition that scattering can take place only between the various scattering lines lying within the frequency band $\Delta\nu$. Woolley & Stibbs (1) give a phenomenological derivation of the equations of transfer for the case of three interlocked scattering lines. Their arguments may be generalized immediately to give the system of N equations for N interlocking scattering lines:

$$\mu \frac{\partial}{\partial \tau} I_i(\tau, \mu) + (1+\eta_i) I_i(\tau, \mu) = \frac{(1-\epsilon)}{2} \alpha_i \sum_{j=1}^{N} \eta_j \int_{-1}^{1} I_j(\tau, \mu')\, d\mu' + (1+\epsilon\eta_i)(a+b\tau), \quad i = 1, 2, 3, \ldots, N, \tag{6}$$

where

$$\alpha_i = \frac{\eta_i}{\sum_{j=1}^{N} \eta_j}, \tag{7a}$$

and thus

$$\sum_{i=1}^{N} \alpha_i = 1. \tag{7b}$$

Here (5) the discrete subscripts are used to denote that the various quantities correspond to the central line scattering frequencies ν_i, $i = 1, 2, 3, \ldots, N$.

Since we wish to make use of previous work (8) regarding the solution to equation (6), we introduce a new optical variable $x = (1+\eta_N)\tau$, where without loss of generality we take the parameters η_j to be ordered such that

$$\eta_1 > \eta_2 > \eta_3 > \ldots > \eta_N,$$

and define

$$\sigma_i = \frac{1+\eta_i}{1+\eta_N}, \quad i = 1, 2, 3 \ldots N, \tag{8a}$$

$$\lambda_i = \frac{1+\epsilon\eta_i}{1+\eta_i}, \quad i = 1, 2, 3 \ldots N, \tag{8b}$$

$$\alpha = a, \quad \beta = \frac{b}{1+\eta_N} \tag{8c}$$

and

$$c_{ij} = \frac{\frac{1}{2}(1-\epsilon)\alpha_i\eta_j}{1+\eta_N}, \quad i, j = 1, 2, 3 \ldots N, \tag{8d}$$

so that the equation of transfer may be written in the matrix form

$$\mu \frac{\partial}{\partial x} \mathbf{I}(x, \mu) + \mathbf{\Sigma I}(x, \mu) = \mathbf{C} \int_{-1}^{1} \mathbf{I}(x, \mu')\, d\mu' + \mathbf{\Sigma}\, \mathbf{\Lambda}(\alpha + \beta x). \tag{9}$$

Here the elements of the $\mathbf{\Lambda}$-vector are λ_i, the elements of the $\mathbf{C}$-matrix are c_{ij}, $\mathbf{\Sigma}$ is a diagonal matrix with $(\mathbf{\Sigma})_{ii} = \sigma_i$, and $\mathbf{I}(x, \mu)$ is a column vector containing $I_i(x, \mu)$, $i = 1, 2, 3 \ldots N$ as elements. Finally it follows from equation (8a) that the parameters σ_i are ordered such that

$$\sigma_1 > \sigma_2 > \sigma_3 > \ldots > \sigma_N = 1.$$

3. GENERAL ANALYSIS

In this section we develop a rigorous solution to equation (6) for the interlocking doublet ($N = 2$); however, since the normal modes and necessary expansion theorems for general N have been established (11), the analysis given here can be extended to the general case. We wish to use the normal modes and methods of solution introduced by Siewert & Zweifel (**8**) and thus state the considered problem in a convenient notation: we seek a solution to

$$\mu \frac{\partial}{\partial x}\mathbf{I}(x,\mu)+\mathbf{\Sigma}\,\mathbf{I}(x,\mu) = \mathbf{C}\int_{-1}^{1}\mathbf{I}(x,\mu')\,d\mu'+\mathbf{\Sigma}\,\mathbf{\Lambda}(\alpha+\beta x) \tag{10}$$

subject to the two boundary conditions

(a) $\mathbf{I}(0,\mu) = 0, \quad \mu \in (0, 1)$,
(b) $\lim_{x\to\infty}\mathbf{I}(x,\mu)\to\mathbf{I}_p(x,\mu)$.

Here $\mathbf{I}_p(x,\mu)$ is a particular solution to equation (10) and is found immediately by assuming a linear function of x:

$$\mathbf{I}_p(x,\mu) = [(\alpha+\beta x)\mathbf{E}-\beta\mu\,\mathbf{\Sigma}^{-1}]\begin{bmatrix}1\\1\end{bmatrix}, \tag{11}$$

which agrees with the result given by Busbridge & Stibbs (**5**). The matrix $\mathbf{E}$ here denotes the identity element.

Since the normal modes for the homogeneous version of equation (10) have been determined (**8, 11**), we write our desired solution as a linear sum of those solutions which vanish at infinity, and include as well the particular solution $\mathbf{I}_p(x,\mu)$:

$$\mathbf{I}(x,\mu) = \mathbf{I}_p(x,\mu)+A(\eta_0)\mathbf{\Phi}(\eta_0,\mu)\,e^{-x/\eta_0}+\int_0^{1/\sigma}[A_1(\eta)\mathbf{\Phi}_1(\eta,\mu) + A_2(\eta)\mathbf{\Phi}_2(\eta,\mu)]\,e^{-x/\eta}\,d\eta+\int_{1/\sigma}^{1}A_3(\eta)\mathbf{\Phi}_3(\eta,\mu)\,e^{-x/\eta}\,d\eta. \tag{12}$$

Here $A(\eta_0)$, $A_1(\eta)$, $A_2(\eta)$ and $A_3(\eta)$ are the unknown expansion coefficients which must be determined such that boundary condition (a) is satisfied; we note that equation (12) correctly meets boundary condition (b). The various eigenvectors $\mathbf{\Phi}(\xi,\mu)$ are given in several forms by Siewert & Zweifel (**8, 11**) and Siewert & Shieh (**12**); for the sake of clarity we write explicitly the variants used here:

$$\mathbf{\Phi}_1(\eta,\mu) = \begin{bmatrix}c_{12}\delta(\sigma\eta-\mu)\\-c_{11}\delta(\eta-\mu)\end{bmatrix}, \quad \eta\in(0, 1/\sigma), \tag{13a}$$

$$\mathbf{\Phi}_2(\eta,\mu) = \begin{bmatrix}c_{12}\eta\,\dfrac{P}{\sigma\eta-\mu}+\delta(\sigma\eta-\mu)[-2\eta c_{12}T(\sigma\eta)]\\[2ex] c_{22}\eta\,\dfrac{P}{\eta-\mu}+\delta(\eta-\mu)[1-2\eta c_{22}T(\eta)]\end{bmatrix}, \quad \eta\in(0, 1/\sigma), \tag{13b}$$

and

$$\mathbf{\Phi}_3(\eta,\mu) = \begin{bmatrix}\dfrac{c_{12}\eta}{\sigma\eta-\mu}\\[2ex] c_{22}\eta\,\dfrac{P}{\eta-\mu}+\delta(\eta-\mu)[1-2\eta c_{11}T(1/\sigma\eta)-2\eta c_{22}T(\eta)]\end{bmatrix}, \eta\in(1/\sigma, 1). \tag{13c}$$

Note that we have used σ for σ_1 since there are only two elements in the doublet model. The symbol P is used in the above equations to indicate that all ensuing integrals over η or μ are to be evaluated in the Cauchy principal-value sense, and

$\delta(x)$ denotes the Dirac delta function. Recalling that det $\mathbf{C} = 0$, we write the discrete solution as

$$\mathbf{\Phi}(\eta_0, \mu) = \begin{bmatrix} \dfrac{c_{12}\eta_0}{\sigma\eta_0 - \mu} \\ \dfrac{c_{22}\eta_0}{\eta_0 - \mu} \end{bmatrix}, \tag{14}$$

where η_0 is the positive zero of the dispersion function,

$$\Omega(z) = 1 - 2c_{11}zT(1/\sigma z) - 2c_{22}zT(1/z). \tag{15}$$

Here and throughout we note that σ is greater than unity, and we use the notation $T(x) = \tanh^{-1} x$.

If we now apply to equation (12) the condition of zero re-entrant radiation, we see that the expansion coefficients are to be determined from the equation

$$-\mathbf{I}_p(0, \mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu) + \int_0^{1/\sigma} [A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]\, d\eta + \int_{1/\sigma}^1 A_3(\eta)\mathbf{\Phi}_3(\eta, \mu)\, d\eta, \quad \mu \in (0, 1). \tag{16}$$

The half-range completeness theorem proved by Siewert & Zweifel (**8**), in addition to being restricted to det $\mathbf{C} = 0$, assumed explicit expressions for the various elements of the **C**-matrix; their case, in fact, was for the conservative model which led to $\eta_0 \to \infty$. Since the existing expansion theorem can be easily modified to include the problem considered here, we do not prove it but simply state that the vectors $\mathbf{\Phi}(\eta_0, \mu)$ and $\mathbf{\Phi}_i(\eta, \mu)$, $i = 1, 2$ and 3, $\eta \in (0, 1)$, span the space for two-component vector functions which satisfy a Hölder condition (**13**). The right-hand side of equation (16) is thus a valid expansion of the function $-\mathbf{I}_p(0, \mu)$, and by properly constructing an adjoint basis set, all expansion coefficients may be obtained immediately by taking scalar products. Siewert & Zweifel (**8**) also proved a half-range orthogonality theorem for the case of det $\mathbf{C} = 0$; for the problem considered here, we prefer to utilize the relations:

$$\int_0^1 \tilde{\mathbf{\Phi}}_i{}^\dagger(\xi', \mu)\mathbf{H}(\mu)\mathbf{\Phi}_j(\xi, \mu)\, d\mu = \xi H(\xi)N_i(\xi)\delta(\xi - \xi')\, \delta_{i,j},\ \xi, \xi' \in (0, 1), \quad i, j = 1, 2 \text{ or } 3, \tag{17a}$$

$$\int_0^1 \tilde{\mathbf{\Phi}}^\dagger(\eta_0, \mu)\mathbf{H}(\mu)\mathbf{\Phi}_i(\xi, \mu)\, d\mu = 0, \quad \xi \in (0, 1), \quad i = 1, 2 \text{ or } 3, \tag{17b}$$

$$\int_0^1 \tilde{\mathbf{\Phi}}_i{}^\dagger(\xi', \mu)\mathbf{H}(\mu)\mathbf{\Phi}(\eta_0, \mu)\, d\mu = 0, \quad \xi' \in (0, 1), \quad i = 1, 2 \text{ or } 3. \tag{17c}$$

and

$$\int_0^1 \tilde{\mathbf{\Phi}}^\dagger(\eta_0, \mu)\mathbf{H}(\mu)\mathbf{\Phi}(\eta_0, \mu)\, d\mu = \eta_0 H(\eta_0)N(\eta_0). \tag{17d}$$

Here the superscript tilde denotes the transpose operation, the dagger is used to indicate adjoint vectors, and the matrix $\mathbf{H}(\mu)$ is given by

$$\mathbf{H}(\mu) = \begin{bmatrix} \mu H(\mu/\sigma) & 0 \\ 0 & \mu H(\mu) \end{bmatrix}, \tag{18}$$

where $H(\mu)$ is Chandrasekhar's H-function (**3**) for 'characteristic function' $\psi(\mu) = c_{22}+c_{11}\Theta(\mu)$ with

$$\Theta(\mu) = 1, \quad \mu \in (-1/\sigma, 1/\sigma).$$
$$= 0 \text{ otherwise},$$

i.e.

$$H(\mu) = 1+\mu H(\mu)\int_0^1 \psi(\mu')H(\mu')\frac{d\mu'}{\mu'+\mu}. \tag{19}$$

An analytic expression for $H(\mu)$ is immediately available by noting

$$\frac{1}{X(-\mu)} = (\eta_0+\mu)\sqrt{\Omega(\infty)}\,H(\mu), \tag{20}$$

where $X(-\mu)$ is known (**8**), and

$$\Omega(\infty) = 1-2\frac{c_{11}}{\sigma}-2c_{22}. \tag{21}$$

Further,

$$N_i(\xi) = [1-2\xi c_{11}T(\sigma\xi)-2\xi c_{22}T(\xi)]^2+\pi^2\xi^2(c_{11}+c_{22})^2, \quad i = 1 \text{ or } 2, \tag{22a}$$

$$N_3(\xi) = [1-2\xi c_{11}T(1/\sigma\xi)-2\xi c_{22}T(\xi)]^2+\pi^2\xi^2 c_{22}{}^2, \tag{22b}$$

and

$$N(\eta_0) = c_{22}\eta_0\frac{d}{dz}\Omega(z)\bigg|_{z=\eta_0}, \tag{22c}$$

or

$$N(\eta_0) = c_{22}\left[2c_{11}\eta_0{}^2\frac{\sigma}{(\sigma\eta_0)^2-1}+2c_{22}\eta_0{}^2\frac{1}{\eta_0{}^2-1}-1\right]. \tag{22d}$$

In addition, the adjoint vectors $\mathbf{\Phi}^\dagger(\eta_0, \mu)$ and $\mathbf{\Phi}_3{}^\dagger(\xi, \mu)$ are obtained respectively simply by interchanging c_{ij} with c_{ji} in equations (14) and (13c). Because of the degeneracy associated with that part of the eigenvalue spectrum for which $\eta \in (0, 1/\sigma)$, the adjoint vectors $\mathbf{\Phi}_i{}^\dagger(\xi, \mu)$, $i = 1$ or 2, take considerably more difficult forms:

$$\mathbf{\Phi}_1{}^\dagger(\xi, \mu) = M_{11}(\xi)\mathbf{G}_1(\xi, \mu)+M_{12}(\xi)\mathbf{G}_2(\xi, \mu) \tag{23a}$$

and

$$\mathbf{\Phi}_2{}^\dagger(\xi, \mu) = M_{21}(\xi)\mathbf{G}_1(\xi, \mu)+M_{22}(\xi)\mathbf{G}_2(\xi, \mu). \tag{23b}$$

Here

$$M_{11}(\eta) = \frac{1}{c_{11}c_{22}}\{c_{22}(c_{11}+c_{22})\pi^2\eta^2+[1-2\eta c_{22}T(\eta)]^2+4\eta^2 c_{11}c_{22}T^2(\sigma\eta)\}, \tag{24a}$$

$$M_{12}(\eta) = M_{21}(\eta) = \frac{1}{c_{22}}[1-2\eta c_{22}T(\eta)+2\eta c_{22}T(\sigma\eta)], \tag{24b}$$

and

$$M_{22}(\eta) = \frac{1}{c_{22}}(c_{11}+c_{22}). \tag{24c}$$

We note that $\mathbf{G}_1(\xi, \mu)$ and $\mathbf{G}_2(\xi, \mu)$ are obtained respectively by replacing c_{12} with c_{21} in equations (13a) and (13b).

Since the appropriate completeness theorem has been stated and the necessary orthogonality relations established, we now proceed to determine the unknown expansion coefficients from equation (16). Multiplying equation (16) successively by

$\tilde{\mathbf{\Phi}}\dagger(\eta_0, \mu)\mathbf{H}(\mu)$ and $\tilde{\mathbf{\Phi}}_i\dagger(\xi, \mu)\mathbf{H}(\mu)$, $i = 1, 2$ and 3, and integrating over the variable μ from 0–1, we utilize equations (17) to find immediately:

$$A(\eta_0) = -\frac{1}{\eta_0 H(\eta_0)N(\eta_0)} \int_0^1 \tilde{\mathbf{\Phi}}\dagger(\eta_0, \mu)\mathbf{H}(\mu)\mathbf{I}_p(0, \mu)\, d\mu, \tag{25a}$$

and

$$A_i(\eta) = -\frac{1}{\eta H(\eta)N_i(\eta)} \int_0^1 \tilde{\mathbf{\Phi}}_i\dagger(\eta, \mu)\mathbf{H}(\mu)\mathbf{I}_p(0, \mu)\, d\mu, \quad i = 1, 2 \text{ or } 3. \tag{25b}$$

The integrals appearing in equations (25) may be evaluated explicitly to yield our complete result for the radiation intensity:

$$\mathbf{I}(x, \mu) = \mathbf{I}_p(x, \mu) + A(\eta_0)\mathbf{\Phi}(\eta_0, \mu)\, e^{-x/\eta_0} + \int_0^{1/\sigma} [A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)] \times e^{-x/\eta}\, d\eta + \int_{1/\sigma}^1 A_3(\eta)\mathbf{\Phi}_3(\eta, \mu)\, e^{-x/\eta}\, d\eta, \tag{26}$$

where

$$A(\eta_0) = -\frac{1}{H(\eta_0)N(\eta_0)} \left\{(\alpha - \beta\eta_0) \left[\left(1 - 2\frac{c_{11}}{\sigma} - 2c_{22}\right)^{1/2} - (\sigma c_{21} - c_{11}) \times \int_0^{1/\sigma} H(\mu)\frac{\mu}{\mu - \eta_0}\, d\mu\right] + \beta\left[\sigma c_{21} \int_0^{1/\sigma} H(\mu)\mu\, d\mu + c_{22}\int_0^1 H(\mu)\mu\, d\mu\right]\right\}, \tag{27a}$$

$$A_i(\eta) = -\frac{1}{H(\eta)N_i(\eta)} \Bigg\{M_{i1}(\eta)[(\alpha - \beta\eta)[\sigma c_{21} - c_{11}]H(\eta)] + M_{i2}(\eta)\left[(\alpha - \beta\eta)\left[\left(1 - 2\frac{c_{11}}{\sigma} - 2c_{22}\right)^{1/2} - (\sigma c_{21} - c_{11}) \times \left(2\eta H(\eta)T(\sigma\eta) - \int_0^{1/\sigma} H(\mu)\mu\frac{P}{\eta - \mu}\, d\mu\right)\right] + \beta\left[\sigma c_{21}\int_0^{1/\sigma} H(\mu)\mu\, d\mu + c_{22}\int_0^1 H(\mu)\mu\, d\mu\right]\right]\Bigg\},$$
$$i = 1 \text{ or } 2, \tag{27b}$$

and

$$A_3(\eta) = -\frac{1}{H(\eta)N_3(\eta)} \left\{(\alpha - \beta\eta) \left[\left(1 - 2\frac{c_{11}}{\sigma} - 2c_{22}\right)^{1/2} - (\sigma c_{21} - c_{11}) \int_0^{1/\sigma} H(\mu) \times \frac{\mu}{\mu - \eta}\, d\mu\right] + \beta\left[\sigma c_{21}\int_0^{1/\sigma} H(\mu)\mu\, d\mu + c_{22}\int_0^1 H(\mu)\mu\, d\mu\right]\right\}. \tag{27c}$$

It should be noted that the various moments of $\mathbf{I}(x, \mu)$,

$$\mathbf{I}_i(x) = \int_{-1}^1 \mathbf{I}(x, \mu)\mu^i\, d\mu, \quad i = 1, 2, 3 \ldots,$$

necessary for the evaluation of the mean intensity, the flux or the K-integral (3) are at once available since $\mathbf{I}_p(x, \mu)$ and the $\mathbf{\Phi}$-vectors are given explicitly by equations

(11), (13) and (14), and integrations of these functions are elementary tasks. As will be shown in the next section, the solution given by equation (26) reduces to a particularly simple result if only the exit distribution is sought.

4. SWITCHING RELATIONS AND THE S-MATRIX

In the previous section the expansion coefficients $A(\eta_0)$ and $A_i(\eta)$, $i = 1, 2$ and 3, were determined by taking scalar products of equation (16) with $\mathbf{\Phi}\dagger(\eta_0, \mu)\mathbf{H}(\mu)$ and $\mathbf{\Phi}_i\dagger(\eta, \mu)\mathbf{H}(\mu)$, $i = 1, 2$ and 3. If, however, only the surface quantity $\mathbf{I}(0, -\mu)$, $\mu \in (0, 1)$, is sought, the result can be accomplished without the necessity of determining these expansion coefficients.

Following a procedure introduced by McCormick & Kuščer (**14**) and used by Siewert (**15**), we consider the set of four switched scalar products:

$$(3, i) = \int_0^1 \tilde{\mathbf{\Phi}}_3\dagger(-\mu', \mu)\mathbf{H}(\mu)\mathbf{\Phi}_i(\eta, \mu)\, d\mu, \quad \eta \in (0, 1/\sigma), \quad \mu' \in (0, 1), \quad i = 1 \text{ and } 2, \tag{28a}$$

$$(3, 3) = \int_0^1 \tilde{\mathbf{\Phi}}_3\dagger(-\mu', \mu)\mathbf{H}(\mu)\mathbf{\Phi}_3(\eta, \mu)\, d\mu, \quad \eta \in (1/\sigma, 1), \quad \mu' \in (0, 1), \tag{28b}$$

and

$$(3, +) = \int_0^1 \tilde{\mathbf{\Phi}}_3\dagger(-\mu', \mu)\mathbf{H}(\mu)\mathbf{\Phi}(\eta_0, \mu)\, d\mu, \quad \mu' \in (0, 1). \tag{28c}$$

With the above restrictions on the variables, the adjoint vector is no longer singular:

$$\mathbf{\Phi}_3\dagger(-\mu', \mu) = \begin{bmatrix} \dfrac{c_{21}\mu'}{\sigma\mu' + \mu} \\[2ex] \dfrac{c_{22}\mu'}{\mu' + \mu} \end{bmatrix}, \quad \mu, \mu' \in (0, 1). \tag{29}$$

Since the four integrals defined by equations (28) may be evaluated straightforwardly, only the results are given here. We find

$$(3, 1) = 0, \quad (3, +) = \frac{\mu'}{H(\mu')}\frac{c_{22}\eta_0}{\eta_0 + \mu'}, \tag{30a}$$

and

$$(3, 2) = (3, 3) = \frac{\mu'}{H(\mu')}\frac{c_{22}\eta}{\eta + \mu'}. \tag{30b}$$

The half-range completeness theorem mentioned previously states that a vector $\boldsymbol{\psi}(\mu)$ can be expanded in the form

$$\boldsymbol{\psi}(\mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu) + \int_0^{1/\sigma} [A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]\, d\eta + \int_{1/\sigma}^1 A_3(\eta)\mathbf{\Phi}_3(\eta, \mu)\, d\eta, \quad \mu \in (0, 1). \tag{31}$$

If we now multiply equation (31) by $\tilde{\mathbf{\Phi}}_3\dagger(-\mu', \mu)\mathbf{H}(\mu)$ and integrate over μ from zero to unity, there results

$$\int_0^1 \tilde{\mathbf{\Phi}}_3\dagger(-\mu', \mu)\mathbf{H}(\mu)\boldsymbol{\psi}(\mu)\, d\mu = A(\eta_0)(3, +) + \int_0^{1/\sigma} [A_1(\eta)(3, 1) + A_2(\eta)(3, 2)]\, d\eta + \int_{1/\sigma}^1 A_3(\eta)(3, 3)\, d\eta, \quad \mu' \in (0, 1). \tag{32}$$

We now substitute equations (30) into equation (32) to obtain

$$\frac{H(\mu')}{\mu'}\int_0^1 \tilde{\mathbf{\Phi}}_3{\dagger}(-\mu',\mu)\mathbf{H}(\mu)\boldsymbol{\psi}(\mu)\,d\mu = A(\eta_0)\frac{c_{22}\eta_0}{\eta_0+\mu'}+\int_0^{1/\sigma} A_2(\eta)\frac{c_{22}\eta}{\eta+\mu'}\,d\eta$$
$$+\int_{1/\sigma}^1 A_3(\eta)\frac{c_{22}\eta}{\eta+\mu'}\,d\eta, \quad \mu'\in(0,1). \quad (33)$$

Recalling equations (13) and (14) where the explicit forms of the $\mathbf{\Phi}$-vectors are given, we note that the right-hand side of equation (33) is the analytic continuation to negative μ' of the lower component of the right-hand side of equation (31) with μ changed to μ'. This result is now used to define the analytic continuation of $\boldsymbol{\psi}(\mu)$ to negative μ:

$$\psi_2(-\mu') = \frac{H(\mu')}{\mu'}\int_0^1 \tilde{\mathbf{\Phi}}_3{\dagger}(-\mu',\mu)\mathbf{H}(\mu)\boldsymbol{\psi}(\mu)\,d\mu, \quad \mu'\in(0,1); \quad (34a)$$

multiplying equation (34a) by $(1/\sigma)\times(c_{12}/c_{22})$ and changing μ' to μ'/σ, we also find

$$\psi_1(-\mu') = \frac{c_{12}}{c_{22}}\frac{H(\mu'/\sigma)}{\mu'}\int_0^1 \tilde{\mathbf{\Phi}}_3{\dagger}(-\mu'/\sigma,\mu)\mathbf{H}(\mu)\boldsymbol{\psi}(\mu)\,d\mu, \quad \mu'\in(0,1). \quad (34b)$$

Equations (34) may be written more conveniently in matrix notation:

$$\boldsymbol{\psi}(-\mu') = \frac{1}{2\mu'}\int_0^1 \mathbf{S}(\mu',\mu)\boldsymbol{\psi}(\mu)\,d\mu, \quad \mu'\in(0,1), \quad (35)$$

where the $\mathbf{S}$-matrix follows from above:

$$\mathbf{S}(\mu',\mu) = 2\mu\mu'\begin{bmatrix} \dfrac{c_{11}}{\sigma}\dfrac{1}{\mu+\mu'}H(\mu/\sigma)H(\mu'/\sigma) & c_{12}\dfrac{1}{\sigma\mu+\mu'}H(\mu)H(\mu'/\sigma) \\ c_{21}\dfrac{1}{\mu+\sigma\mu'}H(\mu/\sigma)H(\mu') & c_{22}\dfrac{1}{\mu+\mu'}H(\mu)H(\mu') \end{bmatrix}. \quad (36)$$

We note that this result for $\mathbf{S}(\mu',\mu)$ agrees with those obtained by other methods by Busbridge & Stibbs (5) and Shieh (**16**), though in the former paper the $\mathbf{S}$-matrix is defined slightly differently.

Thus when the right-hand side of the expansion

$$\boldsymbol{\psi}(\mu) = A(\eta_0)\mathbf{\Phi}(\eta_0,\mu)+\int_0^{1/\sigma}[A_1(\eta)\mathbf{\Phi}_1(\eta,\mu)+A_2(\eta)\mathbf{\Phi}_2(\eta,\mu)]\,d\eta$$
$$+\int_{1/\sigma}^1 A_3(\eta)\mathbf{\Phi}_3(\eta,\mu)\,d\eta, \quad \mu\in(0,1), \quad (37)$$

is continued to negative μ, the analogous continuation of $\boldsymbol{\psi}(\mu)$ is given by

$$\boldsymbol{\psi}(-\mu) = \frac{1}{2\mu}\int_0^1 \mathbf{S}(\mu,\mu')\boldsymbol{\psi}(\mu')\,d\mu', \quad \mu\in(0,1). \quad (38)$$

The preceding result can be used to find the exit distribution $\mathbf{I}(0,-\mu)$, $\mu\in(0,1)$, for the considered problem. We note from equations (12,) (16), (37) and (38) that

$$\mathbf{I}(0,-\mu) = \mathbf{I}_p(0,-\mu)-\frac{1}{2\mu}\int_0^1 \mathbf{S}(\mu,\mu')\mathbf{I}_p(0,\mu')\,d\mu', \quad \mu\in(0,1), \quad (39)$$

which can be simplified to yield known results (5):

$$I_1(0, -\mu) = (\alpha+\beta\mu/\sigma)H(\mu/\sigma)\left[\left(1-\frac{2c_{11}}{\sigma}-2c_{22}\right)^{1/2}-\left(\frac{c_{12}}{\sigma}-c_{22}\right)\times\int_0^1 \mu' H(\mu')\frac{d\mu'}{\mu'+\mu/\sigma}\right]$$
$$+\beta H(\mu/\sigma)\left[c_{11}\int_0^{1/\sigma}\mu' H(\mu')\,d\mu'+\frac{c_{12}}{\sigma}\int_0^1 \mu' H(\mu')\,d\mu'\right], \quad \mu\in(0,1), \quad (40a)$$

and

$$I_2(0, -\mu) = (\alpha+\beta\mu)H(\mu)\left[\left(1-\frac{2c_{11}}{\sigma}-2c_{22}\right)^{1/2}+(c_{11}-\sigma c_{21})\int_0^{1/\sigma}\mu' H(\mu')\frac{d\mu'}{\mu'+\mu}\right]$$
$$+\beta H(\mu)\left[\sigma c_{21}\int_0^{1/\sigma}\mu' H(\mu')\,d\mu'+c_{22}\int_0^1 \mu' H(\mu')\,d\mu'\right], \quad \mu\in(0,1). \quad (40b)$$

Numerical calculations for the H-function of interest here have been made, and are available upon request.

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