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The McCormack model for gas mixtures: Plane Couette flow

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An analytical version of the discrete-ordinates method is used to establish a concise and particularly accurate solution to the problem of plane Couette flow for a binary gas mixture described by the McCormack kinetic model. The solution yields, for the general (specular-diffuse) case of Maxwell boundary conditions for each of the two species, the velocity, heat-flow, and shear-stress profiles for both types of particles, as well as the particle-flow and heat-flow rates associated with each of the two species of gas particles. Highly accurate numerical results are reported for the case of a helium–argon mixture confined between molybdenum and tantalum plates. The algorithm is considered especially easy to use, and the developed (FORTRAN) code requires typically less than a second on a 2.2 GHz Pentium 4 machine to compute all quantities of interest with at least five figures of accuracy. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1845911]

I. INTRODUCTION

The flow of a rarefied gas between two infinite plates that are moving in parallel and opposite directions is a classical problem in rarefied gas dynamics known as plane Couette flow. Under the assumption that the plate velocities are small compared to the reference Maxwellian speed $(2kT_0/m)^{1/2}$, where k is the Boltzmann constant, T_0 is the (unperturbed) gas temperature, and m is the mass of a gas particle, the problem can be adequately modeled by the linearized Boltzmann equation.

There are numerous works dedicated to the study of linearized plane Couette flow of a single gas. A list of all these works would be too lengthy to report here, and thus we refer the reader to the books of Cercignani,[1–3] Williams,[4] and Ferziger and Kaper,[5] as well as the review papers by Sharipov and Seleznev[6] and Williams,[7] for general background material and a discussion of previous works on the single-gas case. In regard to gas mixtures, however, the literature on this problem is scarce. We have found only three works[8–10] on linearized plane Couette flow for gas mixtures, two of which[9,10] are related to the present work as they are also based on the discrete-ordinates method. The work in Ref. 9 relies on space discretization and iteration, while Ref. 10 uses the same analytical discrete-ordinates (ADO) method that we use here. These works[8–10] have addressed the special case of purely diffuse boundary conditions and are based on the relatively limited Hamel model,[11] and so here we develop a concise and accurate ADO solution for plane Couette flow of a binary gas mixture described by the physically more consistent McCormack model.[12] In addition, we consider general (specular-diffuse) Maxwell boundary conditions with a free choice of the accommodation coefficient for each species at each confining plate.

II. FORMULATION

In this work we base our analysis of a binary gas mixture on the McCormack model as introduced in an important paper[12] published in 1973. While we use this model as defined in Ref. 12, we employ a notation that is appropriate to the analysis and computations we report here. The ADO method[13] has been used in two recent works[14,15] to solve a collection of basic flow problems, defined for binary gas mixtures in terms of the McCormack model, for semi-infinite media[14] (Kramers' problem and the half-space problem of thermal creep) and plane channels[15] (Poiseuille flow, thermal-creep flow, and flow driven by density gradients). Other recent works based on the McCormack model for binary gas mixtures report ADO solutions for the temperature-jump problem[16] and the heat-transfer problem in a plane channel.[17] Our solution of the Couette flow problem for a gas mixture described by the McCormack model follows directly from the general analysis reported in Refs. 14 and 15, and so our presentation here is brief.

We consider that the required functions $h_\alpha(x,\mathbf{v})$ for the two types of particles (α=1 and 2) denote perturbations from Maxwellian distributions for each species, i.e.,

$$f_\alpha(x,\mathbf{v}) = f_{\alpha,0}(v)[1 + h_\alpha(x,\mathbf{v})], \tag{1}$$

where

$$f_{\alpha,0}(v) = n_\alpha(\lambda_\alpha/\pi)^{3/2}e^{-\lambda_\alpha v^2}, \quad \lambda_\alpha = m_\alpha/(2kT_0). \tag{2}$$

Here m_α and n_α denote, respectively, the particle mass and the equilibrium density of the αth species, x is the spatial variable (measured, for example, in centimeters), $\mathbf{v}$, with components v_x, v_y, v_z and magnitude v, is the particle velocity, and T_0 is the reference temperature. It follows from McCormack's work[12] that the perturbations satisfy the coupled equations, for $\alpha=1,2$,

$$c_x\frac{\partial}{\partial x}h_\alpha(x,\mathbf{c})+\omega_\alpha\gamma_\alpha h_\alpha(x,\mathbf{c})=\omega_\alpha\gamma_\alpha\mathcal{L}_\alpha\{h_1,h_2\}(x,\mathbf{c}), \tag{3}$$

where $\mathbf{c}$, with components c_x,c_y,c_z and magnitude c, is a dimensionless velocity variable, $\omega_\alpha=\lambda_\alpha^{1/2}$, and γ_α denotes the collision frequency for the αth species. Here we write the integral operators as

$$\mathcal{L}_\alpha\{h_1,h_2\}(x,\mathbf{c})=\frac{1}{\pi^{3/2}}\sum_{\beta=1}^{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-c'^2}h_\beta(x,\mathbf{c}')\times K_{\beta,\alpha}(\mathbf{c}',\mathbf{c})dc_x'dc_y'dc_z', \tag{4}$$

where the kernels $K_{\beta,\alpha}(\mathbf{c}',\mathbf{c})$ are listed explicitly in Refs. 14 and 15. As shown in detail in these works, a dimensionless spatial variable τ defined in terms of a viscosity-based mean-free path l_0, originally introduced by Sharipov and Kalempa,[18] can be used to restate the problem in a more convenient way. Thus, following Refs. 14 and 15, we rewrite Eq. (3) as

$$c_x\frac{\partial}{\partial \tau}h_\alpha(\tau,\mathbf{c})+\sigma_\alpha h_\alpha(\tau,\mathbf{c})=\sigma_\alpha\mathcal{L}_\alpha\{h_1,h_2\}(\tau,\mathbf{c}), \tag{5}$$

where $\sigma_\alpha=\gamma_\alpha\omega_\alpha l_0$, or, more explicitly,

$$\sigma_\alpha=\gamma_\alpha[(n_1/\gamma_1+n_2/\gamma_2)/(n_1+n_2)](m_\alpha/m)^{1/2}. \tag{6}$$

Here the mass average is defined as

$$m=(n_1m_1+n_2m_2)/(n_1+n_2). \tag{7}$$

In this work, we consider the problem of plane Couette flow between plates that are located at $\tau=-a$ and $\tau=a$ and that are moving with specified velocities in the z direction, and so we seek solutions of Eq. (5) that are valid for all $\tau\in(-a,a)$ and that satisfy the Maxwell boundary conditions[7]

$$h_\alpha(-a,c_x,c_y,c_z)-(1-a_\alpha)h_\alpha(-a,-c_x,c_y,c_z)-a_\alpha\mathcal{I}_-\{h_\alpha\}(-a)=2a_\alpha r_\alpha u_{w,1}c_z \tag{8a}$$

and

$$h_\alpha(a,-c_x,c_y,c_z)-(1-b_\alpha)h_\alpha(a,c_x,c_y,c_z)-b_\alpha\mathcal{I}_+\{h_\alpha\}(a)=2b_\alpha r_\alpha u_{w,2}c_z \tag{8b}$$

for $c_x>0$ and all c_y and c_z. Note that $h_\alpha(\tau,\mathbf{c})\Leftrightarrow h_\alpha(\tau,c_x,c_y,c_z)$ and that we use a_1 and a_2 to denote the two accommodation coefficients basic to the plate located at $\tau=-a$ and b_1 and b_2 to denote the two accommodation coefficients for the plate located at $\tau=a$. In addition, $r_\alpha=(m_\alpha/m)^{1/2}$, and we have used

$$\mathcal{I}_\mp\{h_\alpha\}(\tau)=\frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-c'^2}h_\alpha(\tau,\mp c_x',c_y',c_z')\times c_x'dc_x'dc_y'dc_z' \tag{9}$$

to denote the diffuse terms in Eqs. (8). In writing Eqs. (8) we have used $v_0=(2kT_0/m)^{1/2}$ to express the wall velocities in dimensionless units. In other words, $u_{w,1}v_0$ and $u_{w,2}v_0$ are the velocities (in the z direction) given to the two confining plates.

If we sought to compute the complete distribution functions $h_\alpha(\tau,\mathbf{c})$, then we would have to work explicitly with Eqs. (5) and (8); however, since we seek here only the velocity profiles, the heat-flow profiles and the shear-stress profiles,

$$u_\alpha(\tau)=\frac{1}{\pi^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-c^2}h_\alpha(\tau,\mathbf{c})c_zdc_xdc_ydc_z, \tag{10}$$

$$q_\alpha(\tau)=\frac{1}{\pi^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-c^2}h_\alpha(\tau,\mathbf{c})(c^2-5/2)c_zdc_xdc_ydc_z, \tag{11}$$

and

$$p_\alpha(\tau)=\frac{1}{\pi^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-c^2}h_\alpha(\tau,\mathbf{c})c_xc_zdc_xdc_ydc_z, \tag{12}$$

we can work only with certain moments of Eqs. (5) and (8). Continuing to follow Refs. 14 and 15, we first multiply Eq. (5) by

$$\phi_1(c_y,c_z)=(1/\pi)e^{-(c_y^2+c_z^2)}c_z \tag{13}$$

and integrate the resulting equation over all c_y and all c_z. We then repeat this procedure using

$$\phi_2(c_y,c_z)=(1/\pi)e^{-(c_y^2+c_z^2)}(c_y^2+c_z^2-2)c_z \tag{14}$$

and define, for $\alpha=1$ and 2,

$$g_{2\alpha-1}(\tau,c_x)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_1(c_y,c_z)h_\alpha(\tau,\mathbf{c})dc_ydc_z \tag{15a}$$

and

$$g_{2\alpha}(\tau,c_x)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_2(c_y,c_z)h_\alpha(\tau,\mathbf{c})dc_ydc_z \tag{15b}$$

to find from these projections four coupled balance equations that we can write in matrix notation as

$$\xi\frac{\partial}{\partial\tau}\mathbf{G}(\tau,\xi)+\mathbf{\Sigma}\mathbf{G}(\tau,\xi)=\mathbf{\Sigma}\int_{-\infty}^{\infty}\psi(\xi')\mathbf{K}(\xi',\xi)\mathbf{G}(\tau,\xi')d\xi'. \tag{16}$$

Here the components of $\mathbf{G}(\tau,\xi)$ are $g_\alpha(\tau,\xi)$, for $\alpha=1$, 2, 3, and 4, and we now use ξ in place of c_x. In addition, we define

$$\mathbf{\Sigma}=\mathrm{diag}\{\sigma_1,\sigma_1,\sigma_2,\sigma_2\} \tag{17}$$

and

$$\psi(\xi)=\pi^{-1/2}e^{-\xi^2}, \tag{18}$$

and we note that the elements of the kernel $\mathbf{K}(\xi',\xi)$ are given explicitly in Refs. 14 and 15. To deduce the boundary conditions relevant to Eq. (16), we project Eqs. (8) against $\phi_1(c_y,c_z)$ and $\phi_2(c_y,c_z)$ to find

$$\mathbf{G}(-a,\xi)-\mathbf{S}_1\mathbf{G}(-a,-\xi)=u_{w,1}[a_1r_1\ 0\ a_2r_2\ 0]^T \tag{19a}$$

and

$$\mathbf{G}(a,-\xi) - \mathbf{S}_2\mathbf{G}(a,\xi) = u_{w,2}[b_1r_1\ 0\ b_2r_2\ 0]^T \tag{19b}$$

for $\xi > 0$. Here, T denotes the transpose operation,

$$\mathbf{S}_1 = \text{diag}\{1-a_1, 1-a_1, 1-a_2, 1-a_2\} \tag{20a}$$

and

$$\mathbf{S}_2 = \text{diag}\{1-b_1, 1-b_1, 1-b_2, 1-b_2\}. \tag{20b}$$

To close this section we note that the McCormack model (for rigid-sphere interactions) requires only the ratio of the two particle masses m_1/m_2, the ratio of the number densities n_1/n_2, and the ratio of the particle diameters d_1/d_2. Once these parameters are specified we seek to find the profiles listed in Eqs. (10)–(12) for selected values of the half-distance between plates a measured in mean-free paths, the two plate velocities $u_{w,1}$ and $u_{w,2}$, and the four accommodation coefficients a_1, a_2, b_1, b_2. Using Eqs. (13)–(15), we find we can write these profiles as

$$u_\alpha(\tau) = \int_{-\infty}^{\infty} \psi(\xi) g_{2\alpha-1}(\tau,\xi) d\xi, \tag{21}$$

$$q_\alpha(\tau) = \int_{-\infty}^{\infty} \psi(\xi)[(\xi^2 - 1/2)g_{2\alpha-1}(\tau,\xi) + g_{2\alpha}(\tau,\xi)]d\xi, \tag{22}$$

and

$$p_\alpha(\tau) = \int_{-\infty}^{\infty} \psi(\xi) g_{2\alpha-1}(\tau,\xi)\xi d\xi. \tag{23}$$

Therefore, once Eq. (16) subject to Eqs. (19) is solved, we can find the desired profiles from Eqs. (21)–(23).

III. AN ANALYTICAL DISCRETE-ORDINATES SOLUTION

A general ADO solution to Eq. (16) has been fully developed and documented in Ref. 14, and so we omit the details of the derivation in this presentation. To start, we split the integral in Eq. (16) into two half-range integrals—one over $(0,\infty)$ and the other over $(-\infty,0)$—and we then change ξ to $-\xi$ in the latter. Doing this, we only have to deal with one half-range integration interval, $(0,\infty)$. Next, using a Gaussian quadrature set of order N with nodes $\{\xi_i\}$ and weights $\{w_i\}$ to approximate integrals over $(0,\infty)$, we can follow Ref. 14 and express our approximate solution for $\mathbf{G}(\tau,\xi)$ at the discrete ordinates $\pm\xi_i$, $i=1,2,\ldots,N$, as

$$\mathbf{G}(\tau,\pm\xi_i) = A_1\mathbf{G}_+ + B_1\mathbf{G}_-(\tau,\pm\xi_i) + \sum_{j=2}^{4N} [A_j\mathbf{\Phi}(\nu_j,\pm\xi_i) \times e^{-(a+\tau)/\nu_j} + B_j\mathbf{\Phi}(\nu_j,\mp\xi_i)e^{-(a-\tau)/\nu_j}], \tag{24}$$

where the separation constants $\{\nu_j\}$ and the elementary solutions $\{\mathbf{\Phi}(\nu_j,\pm\xi_i)\}$ are determined[14] from the solution of an eigensystem of order $4N$. Since one of the eigenvalues of that eigensystem approaches zero as N is increased, the corresponding elementary solutions are replaced with the exact solutions

$$\mathbf{G}_+ = \begin{bmatrix} 1 \\ 0 \\ s \\ 0 \end{bmatrix} \tag{25a}$$

and

$$\mathbf{G}_-(\tau,\xi) = \begin{bmatrix} \sigma_1\tau - \xi \\ 0 \\ s\sigma_1(\tau - \xi/\sigma_2) \\ 0 \end{bmatrix}, \tag{25b}$$

where $s=(m_2/m_1)^{1/2}$. Finally, the constants $A_1, B_1, \{A_j, B_j\}$ can be determined by solving the $8N\times 8N$ system of linear algebraic equations that is obtained when Eq. (24) is substituted into discrete-ordinates versions of the boundary conditions. Once these constants are available, we can compute the velocity, heat-flow and shear-stress profiles from discrete-ordinates approximations of Eqs. (21)–(23). On defining the vector-valued functions $\mathbf{u}(\tau)$, $\mathbf{q}(\tau)$, and $\mathbf{p}(\tau)$ with $u_\alpha(\tau)$, $q_\alpha(\tau)$, and $p_\alpha(\tau)$ for $\alpha=1$ and 2 as components, respectively, we can write our discrete-ordinates approximations to the desired profiles as

$$\mathbf{u}(\tau) = (A_1 + B_1\sigma_1\tau)\begin{bmatrix} 1 \\ s \end{bmatrix} + \sum_{j=2}^{4N} \mathbf{X}(\nu_j)[A_je^{-(a+\tau)/\nu_j} + B_je^{-(a-\tau)/\nu_j}], \tag{26}$$

$$\mathbf{q}(\tau) = \sum_{j=2}^{4N} \mathbf{Y}(\nu_j)[A_je^{-(a+\tau)/\nu_j} + B_je^{-(a-\tau)/\nu_j}], \tag{27}$$

and

$$\mathbf{p}(\tau) = -\frac{1}{2}B_1\begin{bmatrix} 1 \\ s\sigma_1/\sigma_2 \end{bmatrix} + \sum_{j=2}^{4N} \mathbf{Z}(\nu_j)[A_je^{-(a+\tau)/\nu_j} - B_je^{-(a-\tau)/\nu_j}], \tag{28}$$

where

$$\mathbf{X}(\nu_j) = \sum_{k=1}^{N} w_k\psi(\xi_k)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}[\mathbf{\Phi}(\nu_j,\xi_k) + \mathbf{\Phi}(\nu_j,-\xi_k)], \tag{29}$$

$$\mathbf{Y}(\nu_j) = \sum_{k=1}^{N} w_k\psi(\xi_k)\begin{bmatrix} \xi_k^2 - 1/2 & 1 & 0 & 0 \\ 0 & 0 & \xi_k^2 - 1/2 & 1 \end{bmatrix}[\mathbf{\Phi}(\nu_j,\xi_k) + \mathbf{\Phi}(\nu_j,-\xi_k)] \tag{30}$$

and

$$\mathbf{Z}(\nu_j) = \sum_{k=1}^{N} w_k\xi_k\psi(\xi_k)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}[\mathbf{\Phi}(\nu_j,\xi_k) - \mathbf{\Phi}(\nu_j,-\xi_k)]. \tag{31}$$

TABLE I. The velocity, heat-flow, and shear-stress profiles for a He–Ar mixture.

η	$u_1(-a+2\eta a)$	$u_2(-a+2\eta a)$	$q_1(-a+2\eta a)$	$q_2(-a+2\eta a)$	$p_1(-a+2\eta a)$	$p_2(-a+2\eta a)$
0.0	9.784 79(−2)	5.836 80(−1)	−1.970 38(−2)	−7.587 29(−2)	1.939 26(−2)	1.909 91(−1)
0.1	7.553 62(−2)	3.948 04(−1)	−1.403 18(−2)	−3.632 40(−2)	3.133 42(−2)	1.858 73(−1)
0.2	5.057 28(−2)	2.608 44(−1)	−1.044 75(−2)	−2.169 30(−2)	3.881 78(−2)	1.826 66(−1)
0.3	2.348 20(−2)	1.395 72(−1)	−7.338 97(−3)	−1.254 12(−2)	4.371 67(−2)	1.805 67(−1)
0.4	−4.885 65(−3)	2.395 09(−2)	−4.469 46(−3)	−5.730 45(−3)	4.681 84(−2)	1.792 37(−1)
0.5	−3.401 90(−2)	−8.916 35(−2)	−1.701 63(−3)	1.104 35(−4)	4.852 60(−2)	1.785 06(−1)
0.6	−6.358 21(−2)	−2.018 27(−1)	1.088 72(−3)	5.866 70(−3)	4.902 59(−2)	1.782 91(−1)
0.7	−9.333 07(−2)	−3.159 95(−1)	4.047 15(−3)	1.238 50(−2)	4.834 10(−2)	1.785 85(−1)
0.8	−1.230 85(−1)	−4.344 32(−1)	7.392 29(−3)	2.088 65(−2)	4.632 41(−2)	1.794 49(−1)
0.9	−1.528 42(−1)	−5.631 11(−1)	1.159 34(−2)	3.405 69(−2)	4.257 33(−2)	1.810 57(−1)
1.0	−1.858 83(−1)	−7.377 66(−1)	1.963 28(−2)	6.825 49(−2)	3.602 32(−2)	1.838 64(−1)

IV. NUMERICAL RESULTS

In order to demonstrate that our ADO solution for the problem of plane Couette flow yields highly accurate results with modest computational effort, we report detailed numerical results for a specific case based on the McCormack model for rigid-sphere interactions in a mixture of He and Ar and plates made of different materials (Mo for the plate located at $\tau=-a$ and Ta for the plate located at $\tau=a$). Thus, the gas parameters are

$$m_1 = 4.0026, \quad m_2 = 39.948, \quad d_2/d_1 = 1.665, \quad c_1 = 0.3,$$

where $c_\alpha = n_\alpha/n$, with $n=n_1+n_2$, whereas the plate parameters are

$$a_1 = 0.20, \quad a_2 = 0.67, \quad b_1 = 0.46, \quad b_2 = 0.78,$$

$$u_{w,1} = 1.0, \quad u_{w,2} = -1.0, \quad a = 1.5.$$

We note that the accommodation coefficients used in our calculation are the accommodation coefficients for tangential momentum, which we consider a natural choice for this problem. Specifically, we are using here the results of measurements performed by Lord[19] for He and Ar particles reflecting from Mo and Ta surfaces.

We report in Table I our converged numerical results for the velocity, heat-flow, and shear-stress profiles. We note that these results were generated with a quadrature scheme based on the transformation $v(\xi)=e^{-\xi}$ to map $\xi\in[0,\infty)$ onto $v\in[0,1]$ and a linear mapping of the Gauss–Legendre scheme onto the interval [0, 1]. In regard to numerical linear algebra, we have used the EISPACK package[20] to solve the eigensystem that determines the separation constants and the elementary solutions, and LINPACK[21] to solve the linear system for the $8N$ unknowns $A_1, B_1, \{A_j, B_j\}$.

To establish some confidence in our results, we have observed numerical stability in all entries of Table I, as the order of the quadrature N was varied between 40 and 100, in increments of 20. Moreover, as a measure of the correctness of our computational implementation, we have verified that the tabulated shear-stress profiles satisfy the identity

$$c_1 p_1(\tau) + c_2 p_2(\tau) = p, \tag{32}$$

where the constant p is what we call total shear stress. We note that Eq. (32) can be formally deduced by taking zero-order moments [with $\psi(\xi)$ as the weighting function] of the first and third rows of Eq. (16), combining the resulting equations and integrating over space.

In addition to the velocity, heat-flow, and shear-stress profiles, we have computed in this work the particle-flow and heat-flow rates per unit area, defined for each species $(\alpha=1,2)$ as

$$U_\alpha = \frac{1}{2a}\int_{-a}^{a} u_\alpha(\tau)d\tau \tag{33}$$

and

$$Q_\alpha = \frac{1}{2a}\int_{-a}^{a} q_\alpha(\tau)d\tau. \tag{34}$$

Thus, we have used the discrete-ordinates approximations

$$\mathbf{U} = A_1\begin{bmatrix}1\\ s\end{bmatrix} + \frac{1}{2a}\sum_{j=2}^{4N} \nu_j \mathbf{X}(\nu_j)(A_j+B_j)[1-e^{-2a/\nu_j}] \tag{35}$$

and

$$\mathbf{Q} = \frac{1}{2a}\sum_{j=2}^{4N} \nu_j \mathbf{Y}(\nu_j)(A_j+B_j)[1-e^{-2a/\nu_j}], \tag{36}$$

where the vectors $\mathbf{U}$ and $\mathbf{Q}$ have, respectively, U_α and Q_α for $\alpha=1$ and 2 as components, to compute the particle-flow and heat-flow rates shown in Table II for several values of the half-distance between plates a.

In a recent work[22] the McCormack model was used to study the problem of Couette flow for a binary gas mixture in a plane channel for the special case of purely diffuse reflection at the walls. In that work[22] two interaction laws were used: one based on the rigid-sphere model and the other on a "so-called" realistic potential. It is interesting to note that it is reported,[22] in regard to the total shear stress, that the difference between the results for the two interaction potentials is very slight. On the other hand, we have found in this work that our results, all based on rigid-sphere interactions, can be

TABLE II. The particle-flow and heat-flow rates (per unit area) for a He–Ar mixture with various choices of the half-distance between plates.

a	$-U_1$	$-U_2$	$-Q_1$	$-Q_2$
0.001	1.674 31(−1)	1.387 39(−1)	6.341 88(−5)	4.193 70(−5)
0.01	1.566 94(−1)	1.383 03(−1)	9.839 20(−5)	2.020 99(−4)
0.1	1.144 48(−1)	1.337 20(−1)	7.404 89(−4)	7.927 37(−4)
0.5	6.508 88(−2)	1.151 67(−1)	1.942 00(−3)	1.109 28(−3)
1.0	4.596 74(−2)	9.866 80(−2)	1.779 15(−3)	8.110 98(−4)
2.0	3.053 86(−2)	7.699 56(−2)	1.135 38(−3)	4.190 33(−4)
5.0	1.609 15(−2)	4.620 30(−2)	3.440 38(−4)	1.115 66(−4)
10.0	9.156 07(−3)	2.756 42(−2)	1.035 30(−4)	3.314 95(−5)
20.0	4.936 13(−3)	1.522 12(−2)	2.838 31(−5)	9.083 89(−6)
50.0	2.074 37(−3)	6.490 03(−3)	4.818 17(−6)	1.542 03(−6)

greatly affected by the accommodation coefficients used to define combinations of specular and diffuse reflection boundary conditions. For this reason, we use the results reported in Table III for the ratio between the total shear stress p defined in Eq. (32) and the free-molecular total shear stress

$$p_{\mathrm{fm}} = \frac{1}{2\pi^{1/2}}(u_{w,1} - u_{w,2})\left[c_1 r_1 \frac{a_1 b_1}{a_1 + b_1 - a_1 b_1} + c_2 r_2 \frac{a_2 b_2}{a_2 + b_2 - a_2 b_2} \right] \tag{37}$$

for the specified He–Ar mixture to illustrate just how important the use of a general (specular/diffuse) boundary condition is. In regard to the numerical results reported in Table 2 of Ref. 22 for the total shear stress, we were able to confirm the four-digit results listed there.

As a final check of our work, we note that we have found agreement for the case of a single-species gas, a limiting case in this work that can be realized by taking either $c_1=0$ or $c_2=0$ or $m_1=m_2$ and $d_1=d_2$, with S-model[6] results obtained from a special case of the code written to establish the results based on the linearized Boltzmann equation (for rigid-sphere interactions) that are reported in Ref. 23. As noted,[17] the McCormack model reduces, for the special case of a single-species gas and for the explicit choices of the collision frequencies γ_α used in this and other works,[14–17] to the S model, not the BGK model.

TABLE III. The ratio p/p_{fm} for a He–Ar mixture with various choices of the accommodation coefficients and the half-distance between plates.

a	$a_1=b_1=0.4$ $a_2=b_2=0.7$	$a_1=b_1=0.6$ $a_2=b_2=0.8$	$a_1=b_1=0.8$ $a_2=b_2=0.9$	$a_1=b_1=1.0$ $a_2=b_2=1.0$
0.001	9.991 19(−1)	9.989 15(−1)	9.986 66(−1)	9.983 54(−1)
0.01	9.914 33(−1)	9.894 88(−1)	9.871 34(−1)	9.842 22(−1)
0.1	9.268 38(−1)	9.120 05(−1)	8.947 34(−1)	8.743 24(−1)
0.5	7.454 41(−1)	7.052 02(−1)	6.620 39(−1)	6.155 59(−1)
1.0	6.122 45(−1)	5.618 15(−1)	5.109 08(−1)	4.594 58(−1)
2.0	4.582 11(−1)	4.050 35(−1)	3.551 86(−1)	3.082 73(−1)
5.0	2.646 43(−1)	2.228 51(−1)	1.871 60(−1)	1.562 36(−1)
10.0	1.556 46(−1)	1.275 84(−1)	1.047 86(−1)	8.583 55(−2)
20.0	8.535 27(−2)	6.878 37(−2)	5.573 25(−2)	4.514 96(−2)
50.0	3.624 53(−2)	2.886 89(−2)	2.317 95(−2)	1.864 15(−2)

V. CONCLUDING REMARKS

In conclusion, we note that we regard our solution to the considered problem of plane Couette flow for a binary gas mixture as especially concise and easy to use. We have utilized in our formulation a general form of the Maxwell boundary condition at each plate, and we have reported what we believe to be highly accurate species-specific results for the velocity, heat-flow, and shear-stress profiles for a typical case. It should be noted that our formulas are continuous in the τ variable and thus are valid anywhere in the gas.

Since our solution requires only a matrix eigenvalue/eigenvector routine and a solver of linear algebraic equations, the algorithm is especially efficient, fast, and easy to implement. In fact, the developed (FORTRAN) code requires typically less than a second (on a 2.2 GHz mobile Pentium 4 machine) to yield all quantities of interest with what we believe to be five or six figures of accuracy.

Finally, we would like to mention the two reasons why, in our opinion, the ADO method that we have used in this work is so effective: (i) the half-range quadrature scheme allows a better treatment of the boundary conditions than a full-range scheme, and (ii) the eigenvalue problem is formulated in a particularly useful way.

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