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The linearized Boltzmann equation: Sound-wave propagation in a rarefied gas

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Abstract. An analytical version of the discrete-ordinates method (the ADO method) is used to establish concise and particularly accurate solutions to the problem of sound-wave propagation in a rarefied gas. The analysis and the numerical work are based on a rigorous form of the linearized Boltzmann equation (for rigid-sphere interactions), and in contrast to many other works formulated (for an infinite medium) without a boundary condition, the solution reported here satisfies a boundary condition that models a diffusely-reflecting vibrating plate. In addition and in order to investigate the effect of kinetic models, solutions are developed for the BGK model, the S model, the Gross–Jackson model, as well as for the (newly defined) MRS model and the CES model. While the developed numerical results are compared to available experimental data, emphasis in this work is placed on the solutions of the problem of sound-wave propagation as described by the linearized Boltzmann equation and the five considered kinetic models.

Keywords. Rarefied gas dynamics, sound-wave propagation, discrete ordinates.

1. Introduction

Sound-wave propagation in a gas is known to be adequately described by classical (Navier-Stokes) theory, provided the mean-free path between particle collisions in the gas is much smaller than the sound wavelength and the distance between the transmitter and the receiver. For wavelengths of the order of a mean-free path, a reasonable physical description can still be obtained using simplified kinetic-theory approaches developed by Wang Chang and Uhlenbeck (the so-called Burnett [1] and super-Burnett [2] theories). However, for wavelengths shorter than a mean-free path and/or small distances between transmitter and receiver, these theories fail, and a rigorous kinetic-theory approach based on the Boltzmann equation is required [3–5].

Earlier attempts at getting better theoretical predictions for short wavelengths based on polynomial-expansion solutions of the Boltzmann equation [6, 7] were not successful (see a detailed discussion by Cercignani [4, 5]) until a work by Sirovich and Thurber [8] reported (in graphical form) good agreement between theory and experiments for mean-free path to wavelength ratios as high as 10. These authors

[8] used kinetic models to represent the collision term in the linearized Boltzmann equation for Maxwell and rigid-sphere gases and analytic continuation of the dispersion relation to determine "effective" sound modes for frequencies beyond the critical frequency for which no discrete mode exists. However, since only the dispersion relation for a plane wave was analyzed (i.e., no boundary condition was considered in Ref. 8), a question was raised as to the adequacy of such description [9, 10].

Improved problem formulations that include a boundary condition at the transmitter (usually modeled as a fixed vibrating plate) are available in the literature. Buckner and Ferziger [11] used the method of elementary solutions [12, 13] to solve a kinetic model for a Maxwell gas in a half space bounded by a diffusely reflecting, vibrating plate. Good agreement with experimental results was reported in graphical form over a wide range of frequency, despite the fact that these authors did not use the correct boundary condition to generate their numerical results (see footnote on p. 2320 of Ref. 11). Thomas and Siewert [14] and Loyalka and Cheng [15] have solved the half-space linearized problem for the BGK model [16] of the Boltzmann equation using the correct diffuse boundary condition. The work in Ref. 14 was based on the method of elementary solutions [12, 13, 17, 18], whereas in Ref. 15 the problem was reformulated in terms of a system of linear integral equations which was solved by a numerical differencing technique. While we consider that Thomas and Siewert [14] provided the first rigorous solution to a correctly formulated version of the sound-wave problem based on the BGK model, two following papers [19, 20] must also be mentioned. It was reported [15] that the results [14, 15] based on the correct boundary condition are in better agreement with experimental results than the BGK results of Sirovich and Thurber [8] and those of Buckner and Ferziger [11]. The method of Ref. 15 has also been applied to (what was called) a Gross–Jackson kinetic model [21] for Maxwell molecules [22]. In a following work, Banankhah and Loyalka [23] reported a study of sound-wave propagation in a polyatomic gas.

In this work, we use a modern version [24] of the discrete-ordinates method [25] to solve the problem of sound-wave propagation as described by the linearized Boltzmann equation (for rigid-sphere interactions). The considered problem is modeled as a half space that is bounded by a vibrating plate. In addition to a solution defined by a rigorous description based on the linearized Boltzmann equation, we pay special attention in this work to the predictions of various kinetic models, so we report solutions and numerical results derived from the classical BGK model [16], the S model [26], the Gross–Jackson model [21], the MRS model, and the CES model [27]. The results found from the five kinetic models and the LBE are also compared to the experiments of Schotter [28].

2. A formulation of the problem

To begin, we consider that the particle distribution function we seek is a solution of the Boltzmann equation written as

$$\Big(\frac{1}{v_0}\frac{\partial}{\partial t}+c_x\frac{\partial}{\partial x}\Big)f(x,\boldsymbol{c},t)=J(f)(x,\boldsymbol{c},t), \tag{2.1}$$

where the spatial variable x is measured (for example) in cm, t is the time variable, and

$$\boldsymbol{c}=\boldsymbol{v}/v_0, \quad \text{with} \quad v_0=(2kT_0/m)^{1/2}, \tag{2.2a,b}$$

is a dimensionless velocity vector. Here, $\boldsymbol{v}$ is the velocity variable, k is the Boltzmann constant, m is the mass of a gas particle, and T_0 is a reference temperature. The term $J(f)$ in Eq. (2.1) is used to denote the collision operator that (at this point, at least) can be nonlinear in f. We consider that the gas occupies a one-dimensional half-space $x>0$, and so we seek a solution of Eq. (2.1) that satisfies a boundary condition at $x=0$ and that is bounded as x tends to infinity.

We note that x is the distance from a plate that is vibrating with a frequency ω and a velocity in the x direction the maximum magnitude of which is u_0 (in units of v_0). And so assuming diffuse reflection at the plate, we follow Loyalka and Cheng [15] and express the required boundary condition as

$$f(0,c_x,c_y,c_z,t)=n_m\pi^{-3/2}\exp\{-[c^2-2w(t)c_x+w^2(t)]\}, \quad c_x>0, \tag{2.3}$$

where c_x, c_y, and c_z are the components and c the magnitude of $\boldsymbol{c}$, and

$$w(t)=u_0\mathrm{e}^{i\omega t}. \tag{2.4}$$

Here, since there is no loss of particles as they collide with the plate, the constant n_m is to be defined by the conservation condition [15]

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[c_x-w(t)]f(0,\boldsymbol{c},t)\mathrm{d}c_x\mathrm{d}c_y\mathrm{d}c_z=0. \tag{2.5}$$

We can now use Eq. (2.3) in Eq. (2.5) to find

$$n_m\pi^{-3/2}\mathrm{e}^{-w^2(t)}=\frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}[c_x+w(t)]f(0,-c_x,c_y,c_z,t)\mathrm{d}c_x\mathrm{d}c_y\mathrm{d}c_z, \tag{2.6}$$

and we can use n_m from Eq. (2.6) in Eq. (2.3) to obtain the boundary condition

$$f(0,c_x,c_y,c_z,t)=\frac{2}{\pi}\exp\{-c^2+2w(t)c_x\}\mathcal{D}\{f\}(0,t), \tag{2.7}$$

for $c_x>0$. Here

$$\mathcal{D}\{f\}(0,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}[c_x'+w(t)]f(0,-c_x',c_y',c_z',t)\mathrm{d}c_x'\mathrm{d}c_y'\mathrm{d}c_z'. \tag{2.8}$$

We next use the expansions

$$\exp\{2w(t)c_x\}=\sum_{n=0}^{\infty}\frac{[2w(t)c_x]^n}{n!} \tag{2.9}$$

and

$$f(x,\boldsymbol{c},t) = n_0\pi^{-3/2}\mathrm{e}^{-c^2}\big[1 + h(x,\boldsymbol{c})w(t) + \sum_{n=2}^{\infty} h_n(x,\boldsymbol{c})w^n(t)\big] \tag{2.10}$$

to find from Eq. (2.7) the boundary condition for the first Fourier mode of the perturbation from the Maxwellian distribution, viz.

$$h(0,c_x,c_y,c_z) - \frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} \mathrm{e}^{-c'^2} h(0,-c_x',c_y',c_z')c_x'\mathrm{d}c_x'\mathrm{d}c_y'\mathrm{d}c_z' = R(c_x), \tag{2.11}$$

for $c_x > 0$ and all c_y and c_z. Here

$$R(c_x) = 2c_x + \pi^{1/2}. \tag{2.12}$$

Note that n_0 is used in Eq. (2.10) to denote the equilibrium number density. If we now take note of Eq. (2.10) and replace the collision term in Eq. (2.1) by a linear form (basic to rigid-sphere interactions), then we can write the resulting equation for $h(x,\boldsymbol{c})$ as [29, 30]

$$\left(c\mu\frac{\partial}{\partial x} + i\omega/v_0\right) h(x,\boldsymbol{c}) = \sigma_0^2 n_0 \pi^{1/2}\mathcal{L}\{h\}(x,\boldsymbol{c}), \tag{2.13}$$

where σ_0 is the scattering diameter of a gas particle, and where

$$\mathcal{L}\{h\}(x,\boldsymbol{c}) = -\nu(c)h(x,\boldsymbol{c}) + \int_0^{\infty}\int_{-1}^{1}\int_0^{2\pi} \mathrm{e}^{-c'^2}K(\boldsymbol{c}' : \boldsymbol{c})h(x,\boldsymbol{c}')c'^2\mathrm{d}\chi'\mathrm{d}\mu'\mathrm{d}c'. \tag{2.14}$$

Here

$$\nu(c) = \frac{2c^2+1}{c}\int_0^c \mathrm{e}^{-x^2}\mathrm{d}x + \mathrm{e}^{-c^2} \tag{2.15}$$

is the collision frequency, and we express the scattering kernel in the Pekeris form [29], viz.

$$K(\boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{4\pi}\sum_{l=0}^{\infty}\sum_{m=0}^{l}(2l+1)(2-\delta_{0,m})P_l^m(\mu')P_l^m(\mu)k_l(c',c)\cos m(\chi'-\chi), \tag{2.16}$$

where the component functions $k_l(c',c)$ are available from the paper of Pekeris and Alterman [30], and where the *normalized* Legendre functions are given (in terms of the Legendre polynomials) by

$$P_l^m(\mu) = \left[\frac{(l-m)\,!}{(l+m)\,!}\right]^{1/2}(1-\mu^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}\mu^m}P_l(\mu), \quad l \geq m. \tag{2.17}$$

Note that we now express the dimensionless velocity vectors $\boldsymbol{c}$ and $\boldsymbol{c}'$ in spherical coordinates $\{c,\mu,\chi\}$ and $\{c',\mu',\chi'\}$. While explicit forms for the scattering functions $k_l(c',c)$ are available for some values of l, we can follow Pekeris and Alterman

[30] and write (in our notation)

$$k_l(c',c) = 2\int_0^\pi [(2/R)\mathrm{e}^{r^2} - R]P_l(\cos\theta)\sin\theta \mathrm{d}\theta, \tag{2.18}$$

where $R = |\boldsymbol{c}' - \boldsymbol{c}|$, $r = (1/R)c'c\sin\theta$ and θ is the angle between $\boldsymbol{c}'$ and $\boldsymbol{c}$.

Now, if we let l_0 denote a mean-free path, we can introduce a dimensionless spatial variable and rewrite Eq. (2.13) as

$$\left(c\mu\frac{\partial}{\partial\tau} + s\right)h(\tau,\boldsymbol{c}) = \varepsilon_0\mathcal{L}\{h\}(\tau,\boldsymbol{c}), \tag{2.19}$$

where

$$\tau = x/l_0, \quad \varepsilon_0 = \sigma_0^2 n_0\pi^{1/2}l_0, \quad \text{and} \quad s = i\omega l_0/v_0. \tag{2.20a, b, c}$$

We choose to use a mean-free path based on viscosity, and so we take

$$l_0 = l_p = \frac{v_0\mu_*}{P_0}, \tag{2.21}$$

where μ_* is the viscosity and $P_0 = n_0kT_0$ is the reference pressure. With this choice of a mean-free path, it follows from Pekeris and Alterman [30] that we should use $\varepsilon_0 = \varepsilon_p$, where

$$\varepsilon_p = \frac{16}{15\pi^{1/2}}\int_0^\infty \mathrm{e}^{-c^2}B(c)c^4\mathrm{d}c. \tag{2.22}$$

Here $B(c)$ is the solution of the integral equation

$$\nu(c)B(c) - \int_0^\infty \mathrm{e}^{-c'^2}B(c')k_2(c',c)c'^2\mathrm{d}c' = c^2, \tag{2.23}$$

for $c \in [0,\infty)$. Thus, for a given value of s, we seek a bounded (as τ tends to infinity) solution of Eq. (2.19) that satisfies the boundary condition

$$h(0,c,\mu,\chi) - \frac{2}{\pi}\int_0^\infty\int_0^1\int_0^{2\pi} \mathrm{e}^{-c'^2}h(0,c',-\mu',\chi')\mu'c'^3\mathrm{d}\chi'\mathrm{d}\mu'\mathrm{d}c' = R(c\mu), \tag{2.24}$$

for $\mu \in (0,1]$, all c, and all χ.

In this work, we seek the xx-component of the pressure tensor expressed as

$$P_{xx}(\tau,t) = m\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty f(\tau,\boldsymbol{v},t){v_x}^2\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z, \tag{2.25}$$

and so, making use of Eq. (2.10), we find, to first order in $w(t)$, that we can rewrite Eq. (2.25) as

$$P_{xx}(\tau,t) = P_0[1 + \mathcal{P}(\tau)w(t)], \tag{2.26}$$

where (again) $P_0 = n_0kT_0$, and where

$$\mathcal{P}(\tau) = \frac{2}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2}h(\tau,\boldsymbol{c})\mu^2c^4\mathrm{d}\chi\mathrm{d}\mu\mathrm{d}c. \tag{2.27}$$

As we wish to compute $\mathcal{P}(\tau)$, we rewrite Eq. (2.27) as

$$\mathcal{P}(\tau) = \frac{4}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \phi(\tau, c, \mu) \mu^2 c^4 \mathrm{d}\mu \mathrm{d}c, \tag{2.28}$$

where

$$\phi(\tau, c, \mu) = \frac{1}{2\pi} \int_0^{2\pi} h(\tau, \boldsymbol{c}) \mathrm{d}\chi \tag{2.29}$$

is an azimuthal average. We find the defining equations for $\phi(\tau, c, \mu)$ from Eqs. (2.19) and (2.24). Thus, for a given value of s, we must solve

$$\big[c\mu \frac{\partial}{\partial \tau} + \sigma(c)\big] \phi(\tau, c, \mu) = \varepsilon_p \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c'^2} k(c', \mu' : c, \mu) \phi(\tau, c', \mu') c'^2 \mathrm{d}\mu' \mathrm{d}c' \tag{2.30}$$

subject to

$$\phi(0, c, \mu) - 4 \int_0^\infty \int_0^1 \mathrm{e}^{-c'^2} \phi(0, c', -\mu') \mu' c'^3 \mathrm{d}\mu' \mathrm{d}c' = 2c\mu + \pi^{1/2}, \tag{2.31}$$

for $\mu \in (0, 1]$ and all c. To obtain Eq. (2.30) we have defined

$$\sigma(c) = s + \varepsilon_p \nu(c) \tag{2.32}$$

and

$$k(c', \mu' : c, \mu) = \int_0^{2\pi} K(\boldsymbol{c}' : \boldsymbol{c}) \mathrm{d}\chi. \tag{2.33}$$

It follows that

$$k(c', \mu' : c, \mu) = \frac{1}{2} \sum_{l=0}^{\infty} (2l + 1) P_l(\mu') P_l(\mu) k_l(c', c). \tag{2.34}$$

We note that Eq. (2.26) is written in terms of the true time variable t; however it is convenient now to introduce a dimensionless time variable

$$t_* = v_0 t / l_p \tag{2.35}$$

and rewrite Eq. (2.26) as

$$P_{xx}(\tau, t_*) = P_0[1 + \mathcal{P}(\tau) u_0 \mathrm{e}^{i\omega_* t_*}], \tag{2.36}$$

where

$$\omega_* = \omega l_p / v_0 \quad \text{and} \quad s = i\omega_*. \tag{2.37a,b}$$

And so we seek $\mathcal{P}(\tau)$ for specified values of ω_*.

3. Kinetic models

In this section we discuss the various kinetic models for which we develop our (ADO) solutions and for which we report numerical results. As these models can all be obtained by approximating the kernel functions $k_l(c', c)$, we note first of all three identities that must be satisfied for any model in order to accommodate the conservation laws: mass, energy, and momentum. Thus we must have:

$$\nu(c) = \int_0^\infty \mathrm{e}^{-c'^2} k_0(c', c) c'^2 \mathrm{d}c', \tag{3.1a}$$

$$\nu(c)c = \int_0^\infty \mathrm{e}^{-c'^2} k_1(c', c) c'^3 \mathrm{d}c', \tag{3.1b}$$

and

$$\nu(c)c^2 = \int_0^\infty \mathrm{e}^{-c'^2} k_0(c', c) c'^4 \mathrm{d}c', \tag{3.1c}$$

for $c \in [0, \infty)$. In addition to basic quantities already defined in the previous section of this work, we also report here, for the various models considered, values of $A(c)$ and

$$\varepsilon_t = \frac{16}{15\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2} A(c) c^5 \mathrm{d}c, \tag{3.2}$$

where $A(c)$ is the solution of the integral equation

$$\nu(c)A(c) - \int_0^\infty \mathrm{e}^{-c'^2} A(c') k_1(c', c) c'^2 \mathrm{d}c' = c(c^2 - 5/2), \tag{3.3a}$$

for $c \in [0, \infty)$, subject to the constraint

$$\int_0^\infty \mathrm{e}^{-c^2} A(c) c^3 \mathrm{d}c = 0. \tag{3.3b}$$

As noted previously [30, 31], should we wish to use a mean-free path based on thermal conductivity, rather than viscosity, then we should use in the previous section ε_t in place of ε_p.

3.1. The BGK model

The often used BGK model [16] can be obtained from our general form of the linearized Boltzmann equation simply by approximating the kernel functions $k_l(c', c)$. Thus here we use

$$k_0(c', c) = \frac{4}{\pi^{1/2}} [1 + (2/3)(c'^2 - 3/2)(c^2 - 3/2)], \tag{3.4}$$

$$k_1(c', c) = \frac{8c'c}{3\pi^{1/2}}, \tag{3.5}$$

and $k_l(c', c) = 0, l > 1$, to find

$$\nu(c) = 1, \quad \varepsilon_t = 1, \quad \varepsilon_p = 1, \quad \text{and} \quad \varepsilon_p/\varepsilon_t = 1. \tag{3.6a,b,c,d}$$

3.2. The S model

The S model [26] has

$$k_0(c',c) = \frac{4}{\pi^{1/2}}[1 + (2/3)(c'^2 - 3/2)(c^2 - 3/2)], \tag{3.7}$$

$$k_1(c',c) = \frac{8c'c}{3\pi^{1/2}}[1 + (2/15)(c'^2 - 5/2)(c^2 - 5/2)], \tag{3.8}$$

and $k_l(c',c) = 0, l > 1$, with

$$\nu(c) = 1, \quad \varepsilon_t = 3/2, \quad \varepsilon_p = 1, \quad \text{and} \quad \varepsilon_p/\varepsilon_t = 2/3. \tag{3.9a,b,c,d}$$

3.3. The GJ model

The GJ model, defined in an important paper by Gross and Jackson [21], has

$$k_0(c',c) = \frac{4}{\pi^{1/2}}[1 + (2/3)(c'^2 - 3/2)(c^2 - 3/2)], \tag{3.10}$$

$$k_1(c',c) = \frac{8c'c}{3\pi^{1/2}}[1 + (2/9)(c'^2 - 5/2)(c^2 - 5/2)], \tag{3.11}$$

$$k_2(c',c) = \frac{16c'^2c^2}{45\pi^{1/2}}, \tag{3.12}$$

and $k_l(c',c) = 0, l > 2$, with

$$\nu(c) = 1, \quad \varepsilon_t = 9/4, \quad \varepsilon_p = 3/2, \quad \text{and} \quad \varepsilon_p/\varepsilon_t = 2/3. \tag{3.13a,b,c,d}$$

3.4. The MRS model

The MRS model has been obtained (as discussed in Appendix A of this work) from a special case of the McCormack model [32] (restricted to a single-species gas of rigid spheres). Here

$$k_0(c',c) = \frac{4}{\pi^{1/2}}[1 + (2/3)(c'^2 - 3/2)(c^2 - 3/2)], \tag{3.14}$$

$$k_1(c',c) = \frac{8c'c}{3\pi^{1/2}}[1 + (2\beta/5)(c'^2 - 5/2)(c^2 - 5/2)], \tag{3.15}$$

$$k_2(c',c) = \frac{16\varpi c'^2c^2}{15\pi^{1/2}}, \tag{3.16}$$

and $k_l(c',c) = 0, l > 2$, with

$$\nu(c) = 1, \quad \varepsilon_t = (15/32)2^{1/2}, \quad \varepsilon_p = (5/16)2^{1/2}, \quad \text{and} \quad \varepsilon_p/\varepsilon_t = 2/3. \tag{3.17a,b,c,d}$$

In addition

$$\varpi = 1 - (8/5)2^{1/2} \quad \text{and} \quad \beta = 1 - (16/15)2^{1/2}. \tag{3.18a,b}$$

3.5. The CES model

The CES model, introduced (within the context of rigid-sphere interactions) by Barichello and Siewert [27], has

$$k_0(c',c) = \nu(c')\nu(c)[\varpi_{01} + \varpi_{02}(c'^2 - 7/4)(c^2 - 7/4)], \tag{3.19}$$

$$k_1(c',c) = \varpi_{11}c'\nu(c')c\nu(c) + \varpi_{12}\Delta_1(c')\Delta_1(c), \tag{3.20}$$

$$k_2(c',c) = \varpi_2\Delta_2(c')\Delta_2(c), \tag{3.21}$$

and $k_l(c',c) = 0, l > 2$, with $\nu(c)$ given by Eq. (2.15) and

$$\varepsilon_t = 0.679630049..., \quad \varepsilon_p = 0.449027806..., \quad \text{and} \quad \varepsilon_p/\varepsilon_t = 0.660694457.... \tag{3.22a,b,c}$$

We note that the values of ε_t and ε_p used in the CES model are the exact values as defined by the linearized Boltzmann equation (for rigid-sphere interactions). In addition,

$$\Delta_1(c) = \nu(c)[a_*c - A(c)] + c(c^2 - 5/2), \quad \Delta_2(c) = c^2 - \nu(c)B(c), \tag{3.23a,b}$$

$$\varpi_{01} = 0.797884561..., \quad \varpi_{02} = 0.425538432..., \quad \varpi_{11} = 0.455934035..., \tag{3.23c,d,e}$$

$$\varpi_{12} = 0.586873122..., \quad a_* = 0.221880745..., \quad \text{and} \quad \varpi_2 = 2.16400346.... \tag{3.23f,g,h}$$

To conclude this section, we note that for the BGK model, the S model, the GJ model and the MRS model, the Chapman–Enskog functions are given by

$$A(c) = \varepsilon_t c(c^2 - 5/2) \quad \text{and} \quad B(c) = \varepsilon_p c^2, \tag{3.24a,b}$$

where appropriate values of ε_t and ε_p should be used for each model. On the other hand, for the CES model, as for the linearized Boltzmann equation, the Chapman–Enskog functions $A(c)$ and $B(c)$ are solutions of Eqs. (2.23) and (3.3) in which the true rigid-sphere kernels are used. Computations of these functions and the resulting values of ε_t and ε_p have been discussed, for example, in Refs. 33 and 34.

4. A discrete-ordinates solution of the linearized Boltzmann equation

For our work with the linearized Boltzmann equation, we truncate the summation in Eq. (2.34) after $L + 1$ terms, and we then use the exact rigid-sphere kernel functions $k_l(c',c)$ for $l = 0, 1, ..., L$. It therefore follows that $\nu(c)$ is as listed in Eq. (2.15) and that ε_p is as given in Eq. (3.22b). Restating our problem, we note that, for a given value of $s = i\omega_*$, we require a solution of

$$\left[c\mu\frac{\partial}{\partial\tau} + \sigma(c)\right]\phi(\tau,c,\mu) = \varepsilon_p\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c'^2}k(c',\mu' : c,\mu)\phi(\tau,c',\mu')c'^2\mathrm{d}\mu'\mathrm{d}c' \tag{4.1}$$

subject to the boundary condition

$$\phi(0,c,\mu) - 4\int_0^\infty \int_0^1 \mathrm{e}^{-c'^2}\phi(0,c',-\mu')\mu' c'^3 \mathrm{d}\mu' \mathrm{d}c' = 2c\mu + \pi^{1/2}, \tag{4.2}$$

for $\mu \in (0,1]$ and all c, so that we can compute

$$\mathcal{P}(\tau) = \frac{4}{\pi^{1/2}}\int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2}\phi(\tau,c,\mu)\mu^2 c^4 \mathrm{d}\mu \mathrm{d}c \tag{4.3}$$

in order to complete Eq. (2.36). Here

$$\sigma(c) = s + \varepsilon_p \nu(c) \tag{4.4}$$

and

$$k(c',\mu' : c,\mu) = \frac{1}{2}\sum_{l=0}^{L}(2l+1)P_l(\mu')P_l(\mu)k_l(c',c). \tag{4.5}$$

In regard to our numerical work with the linearized Boltzmann equation, we have used $L = 8$ in Eq. (4.5).

While the five (BGK, S, GJ, MRS, CES) kinetic models we consider in this work can all be solved well [35, 36] in ways that do not require any approximation in the (dimensionless) speed variable c, we will use a Legendre expansion (truncated after $K + 1$ terms) of the form

$$\phi(\tau,c,\mu) = \sum_{k=0}^{K} P_k(2\mathrm{e}^{-c} - 1)g_k(\tau,\mu) \tag{4.6}$$

for our work with the linearized Boltzmann equation. We note that in proposing a solution of the form given by Eq. (4.6), we are able to deal with the linearized Boltzmann equation, but by using the relevant approximations required to define the kinetic models, we can also include these models in the solution developed here. However, since the model equations can be solved without using Eq. (4.6), we include in Appendix A of this work some details about what can be considered faster and more accurate solutions for the BGK, S, GJ, and MRS models. A similar approach [36] could be used for the CES model [27], but it was not pursued in this work.

To find defining equations for the functions $g_k(\tau,\mu)$ required in Eq. (4.6), we substitute that expression into Eq. (4.1), multiply the resulting equation by

$$W_i(c) = c^2\mathrm{e}^{-c^2}P_i(2\mathrm{e}^{-c} - 1), \quad i = 0, 1, 2, .., K, \tag{4.7}$$

and integrate over all c to obtain the coupled system

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{A}\boldsymbol{G}(\tau,\mu) + (s\boldsymbol{F} + \varepsilon_p\boldsymbol{S})\boldsymbol{G}(\tau,\mu) = \varepsilon_p\sum_{l=0}^{L}\boldsymbol{B}_l P_l(\mu)\int_{-1}^{1}P_l(\mu')\boldsymbol{G}(\tau,\mu')\mathrm{d}\mu'. \tag{4.8}$$

Here the $K+1$ vector-valued function $\boldsymbol{G}(\tau,\mu)$ has components $g_k(\tau,\mu)$, and the constant matrices are given by

$$\boldsymbol{A} = \int_0^\infty \mathrm{e}^{-c^2} \boldsymbol{P}^T(c)\boldsymbol{P}(c)c^3\mathrm{d}c, \tag{4.9}$$

$$\boldsymbol{F} = \int_0^\infty \mathrm{e}^{-c^2} \boldsymbol{P}^T(c)\boldsymbol{P}(c)c^2\mathrm{d}c, \tag{4.10}$$

$$\boldsymbol{S} = \int_0^\infty \mathrm{e}^{-c^2} \boldsymbol{P}^T(c)\boldsymbol{P}(c)\nu(c)c^2\mathrm{d}c, \tag{4.11}$$

and

$$\boldsymbol{B}_l = \frac{2l+1}{2}\int_0^\infty\int_0^\infty \mathrm{e}^{-c'^2}\mathrm{e}^{-c^2}k_l(c',c)\boldsymbol{P}^T(c')\boldsymbol{P}(c)c'^2c^2\mathrm{d}c'\mathrm{d}c, \tag{4.12}$$

where the superscript T is used to denote the transpose operation, and where

$$\boldsymbol{P}(c) = \big[P_0(2\mathrm{e}^{-c}-1), P_1(2\mathrm{e}^{-c}-1), \cdots, P_K(2\mathrm{e}^{-c}-1)\big]. \tag{4.13}$$

Now in regard to the boundary condition subject to which we must solve Eq. (4.8), we use Eq. (4.6) in Eq. (4.2), multiply the resulting equation by $W_i(c)$, and integrate over all c to obtain

$$\boldsymbol{F}\boldsymbol{G}(0,\mu) - 4\boldsymbol{J}\int_0^1 \boldsymbol{G}(0,-\mu')\mu'\mathrm{d}\mu' = 2\mu\boldsymbol{P}_1^T + \pi^{1/2}\boldsymbol{P}_0^T, \tag{4.14}$$

for $\mu \in (0,1]$. Here

$$\boldsymbol{J} = \boldsymbol{P}_0^T\boldsymbol{P}_1, \tag{4.15}$$

where, in general,

$$\boldsymbol{P}_n = \int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{P}(c)c^{n+2}\mathrm{d}c. \tag{4.16}$$

And so, we now must solve Eq. (4.8) subject to the boundary condition given as Eq. (4.14); however, in order to make use of a previous work [31], we multiply Eq. (4.8) by $\boldsymbol{A}^{-1}$ and Eq. (4.14) by $\boldsymbol{F}^{-1}$ to obtain the final forms we solve, viz.

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{G}(\tau,\mu) + \varepsilon_p\boldsymbol{\Sigma}\boldsymbol{G}(\tau,\mu) = \varepsilon_p\sum_{l=0}^{L}\boldsymbol{C}_l P_l(\mu)\int_{-1}^1 P_l(\mu')\boldsymbol{G}(\tau,\mu')\mathrm{d}\mu' \tag{4.17}$$

and

$$\boldsymbol{G}(0,\mu) - 4\boldsymbol{D}\int_0^1 \boldsymbol{G}(0,-\mu')\mu'\mathrm{d}\mu' = 2\mu\boldsymbol{Q}_1 + \pi^{1/2}\boldsymbol{Q}_0, \tag{4.18}$$

for $\mu \in (0,1]$. Here

$$\boldsymbol{\Sigma} = \boldsymbol{A}^{-1}[(s/\varepsilon_p)\boldsymbol{F} + \boldsymbol{S}], \tag{4.19}$$

$$\boldsymbol{C}_l = \boldsymbol{A}^{-1}\boldsymbol{B}_l, \quad \boldsymbol{D} = \boldsymbol{F}^{-1}\boldsymbol{J}, \quad \text{and} \quad \boldsymbol{Q}_n = \boldsymbol{F}^{-1}\boldsymbol{P}_n^T. \tag{4.20a,b,c}$$

And so now we continue by developing our analytical discrete-ordinates solution of the transport problem defined by Eqs. (4.17) and (4.18).

Since the boundary condition listed as Eq. (4.18) is defined only over the "half range," $\mu \in [0,1]$, and since an integration over this same half range is required in Eq. (4.18), we base our discrete-ordinates solution on a quadrature defined for this half range. We let $\{\mu_n, w_n\}$ denote the nodes and weights of such a quadrature scheme, so that we can approximate Eq. (4.17) by

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{G}(\tau,\mu)+\varepsilon_p\boldsymbol{\Sigma}\boldsymbol{G}(\tau,\mu)=\varepsilon_p\sum_{l=0}^{L}P_l(\mu)\boldsymbol{C}_l\sum_{n=1}^{N}w_n\boldsymbol{G}_{l,n}(\tau), \tag{4.21}$$

where to compact our notation we have introduced

$$\boldsymbol{G}_{l,n}(\tau)=P_l(\mu_n)[\boldsymbol{G}(\tau,\mu_n)+(-1)^l\boldsymbol{G}(\tau,-\mu_n)]\,. \tag{4.22}$$

Following Ref. 31 and seeking solutions of Eq. (4.21) of the form

$$\boldsymbol{G}(\tau,\mu)=\boldsymbol{\Phi}(\nu,\mu)\mathrm{e}^{-\varepsilon_p\tau/\nu}, \tag{4.23}$$

we find ultimately that we can express our discrete-ordinates solution of a collocated version of Eq. (4.21) as

$$\boldsymbol{G}(\tau,\pm\mu_i)=\sum_{j=1}^{J}\big[A_j\boldsymbol{\Phi}(\nu_j,\pm\mu_i)\mathrm{e}^{-\varepsilon_p\tau/\nu_j}+B_j\boldsymbol{\Phi}(\nu_j,\mp\mu_i)\mathrm{e}^{\varepsilon_p\tau/\nu_j}\big], \tag{4.24}$$

for $i=1,2,...,N$, and where $J=N(K+1)$. Here the elementary solutions $\boldsymbol{\Phi}(\nu_j,\pm\mu_i)$ and the separation constants ν_j (where ν_j is taken to have a positive real part) are, in general, complex. These elementary solutions and separation constants are as defined in Ref. 31 after we note that the $\boldsymbol{\Sigma}$ matrix listed in Eq. (4.19) has an imaginary component that is not present in Ref. 31. The $\boldsymbol{\Sigma}$ matrix in Ref. 31 is as given in Eq. (4.19) for the case $s=0$. In addition, the arbitrary constants A_j and B_j are to be determined from the boundary conditions of our problem. Now since our solution is to be bounded as τ tends to infinity, we take the constants B_j to be zero so that we have

$$\boldsymbol{G}(\tau,\pm\mu_i)=\sum_{j=1}^{J}A_j\boldsymbol{\Phi}(\nu_j,\pm\mu_i)\mathrm{e}^{-\varepsilon_p\tau/\nu_j}. \tag{4.25}$$

At this point we substitute Eq. (4.25) into the boundary condition written as

$$\boldsymbol{G}(0,\mu_i)-4\boldsymbol{D}\sum_{n=1}^{N}w_n\mu_n\boldsymbol{G}(0,-\mu_n)=2\mu_i\boldsymbol{Q}_1+\pi^{1/2}\boldsymbol{Q}_0, \tag{4.26}$$

for $i=1,2,...,N$, and solve the resulting system of linear algebraic equations to find the constants A_j. In this way the solution $\boldsymbol{G}(\tau,\pm\mu_i)$, for $i=1,2,...,N$, is established.

To complete our work in this section, we use Eqs. (4.6) and (4.25) in Eq. (4.3) to find our final result, viz.

$$\mathcal{P}(\tau)=\frac{4}{\pi^{1/2}}\sum_{j=1}^{J}A_jX_j\mathrm{e}^{-\varepsilon_p\tau/\nu_j}, \tag{4.27}$$

Table 1. The amplitude $|\mathcal{P}(\tau)|$ for the case $\omega_* = 0.5$

τ	BGK	S	GJ	MRS	CES	LBE
0.0	1.822	1.816	1.813	1.823	1.820	1.826
0.2	1.802	1.796	1.793	1.804	1.800	1.806
0.4	1.778	1.771	1.768	1.779	1.774	1.781
0.6	1.750	1.744	1.740	1.752	1.746	1.752
0.8	1.721	1.714	1.709	1.723	1.715	1.722
1.0	1.691	1.682	1.677	1.692	1.682	1.690
1.2	1.659	1.649	1.644	1.660	1.649	1.657
1.4	1.627	1.616	1.610	1.628	1.614	1.624
1.6	1.595	1.582	1.575	1.595	1.580	1.590
1.8	1.563	1.548	1.541	1.562	1.545	1.556
2.0	1.531	1.515	1.506	1.529	1.510	1.522
3.0	1.378	1.350	1.337	1.370	1.341	1.359
5.0	1.110	1.061	1.042	1.087	1.048	1.075

where

$$X_j = \boldsymbol{P}_2 \sum_{n=1}^{N} w_n \mu_n^2 [\boldsymbol{\Phi}(\nu_j, \mu_n) + \boldsymbol{\Phi}(\nu_j, -\mu_n)]. \tag{4.28}$$

5. Numerical results

Having completed our analysis of the considered sound-wave problem, we are ready to report the results obtained from a numerical implementation of our solutions for the linearized Boltzmann equation and for the five kinetic models, viz, the BGK, the S, the GJ, the MRS and the CES models. First of all, we list in Tables 1–8 our numerical results for the amplitude

$$|\mathcal{P}(\tau)| = \left[\mathcal{P}_R^2(\tau) + \mathcal{P}_I^2(\tau)\right]^{1/2} \tag{5.1}$$

and (the negative of) the phase

$$\vartheta(\tau) = \arg \mathcal{P}(\tau) = \tan^{-1} \frac{\mathcal{P}_I(\tau)}{\mathcal{P}_R(\tau)} \tag{5.2}$$

of the pressure perturbation at various positions, for selected values of the reduced frequency ω_*. Here $\mathcal{P}_R(\tau)$ and $\mathcal{P}_I(\tau)$ stand, respectively, for the real and imaginary parts of $\mathcal{P}(\tau)$, and we use continuous values of the arctan function. We note that our results are thought to be accurate in all figures shown and were obtained using $19 \le K \le 29$ and $30 \le N \le 40$ in the ADO solution for the linearized Boltzmann equation that was developed in Section 4. The numerical results for the BGK, S, GJ and MRS models were confirmed using $N = 100$ in the ADO solution for the (projected) balance equations reported in Appendix A.

As no model emerges as the best from a comparison with the LBE results in Tables 1–8, we decided to perform an additional comparison with experimental results available in the literature for the sound attenuation and dispersion. Earlier

Table 2. The negative of the phase $\vartheta(\tau)$ for the case $\omega_* = 0.5$

τ	BGK	S	GJ	MRS	CES	LBE
0.0	–8.318(–2)	–8.594(–2)	–8.565(–2)	–8.517(–2)	–8.770(–2)	–8.688(–2)
0.2	2.608(–2)	2.367(–2)	2.418(–2)	2.407(–2)	2.170(–2)	2.215(–2)
0.4	1.346(–1)	1.326(–1)	1.334(–1)	1.327(–1)	1.305(–1)	1.305(–1)
0.6	2.420(–1)	2.407(–1)	2.416(–1)	2.406(–1)	2.383(–1)	2.380(–1)
0.8	3.485(–1)	3.478(–1)	3.489(–1)	3.475(–1)	3.452(–1)	3.446(–1)
1.0	4.540(–1)	4.540(–1)	4.553(–1)	4.536(–1)	4.512(–1)	4.502(–1)
1.2	5.586(–1)	5.593(–1)	5.607(–1)	5.588(–1)	5.562(–1)	5.549(–1)
1.4	6.624(–1)	6.638(–1)	6.653(–1)	6.633(–1)	6.603(–1)	6.588(–1)
1.6	7.656(–1)	7.676(–1)	7.690(–1)	7.671(–1)	7.636(–1)	7.619(–1)
1.8	8.680(–1)	8.706(–1)	8.720(–1)	8.704(–1)	8.661(–1)	8.643(–1)
2.0	9.699(–1)	9.730(–1)	9.743(–1)	9.730(–1)	9.678(–1)	9.660(–1)
3.0	1.473	1.477	1.477	1.480	1.467	1.466
5.0	2.463	2.460	2.450	2.477	2.430	2.441

Table 3. The amplitude $|\mathcal{P}(\tau)|$ for the case $\omega_* = 1.0$

τ	BGK	S	GJ	MRS	CES	LBE
0.0	1.923	1.920	1.916	1.925	1.925	1.928
0.2	1.880	1.876	1.872	1.882	1.881	1.885
0.4	1.821	1.816	1.811	1.823	1.820	1.825
0.6	1.753	1.746	1.741	1.754	1.750	1.756
0.8	1.681	1.672	1.665	1.681	1.675	1.682
1.0	1.607	1.595	1.588	1.605	1.598	1.607
1.2	1.533	1.518	1.510	1.529	1.520	1.531
1.4	1.460	1.442	1.433	1.454	1.444	1.457
1.6	1.389	1.368	1.358	1.380	1.370	1.384
1.8	1.320	1.295	1.285	1.308	1.298	1.313
2.0	1.253	1.225	1.215	1.238	1.228	1.245
3.0	9.583(–1)	9.174(–1)	9.080(–1)	9.271(–1)	9.278(–1)	9.461(–1)
5.0	5.448(–1)	4.959(–1)	4.990(–1)	4.915(–1)	5.322(–1)	5.347(–1)

works by Greenspan [37, 38] and Meyer and Sessler [39] deduce these parameters from measurements, performing linear fits in plots of the logarithm of the amplitude and the phase over a range of distances between the transmitter and the receiver, but do not give information on the ranges of distances that were used in their experiments (the work of Meyer and Sessler gives the ranges in graphical form only for the measurements made in air). Since the attenuation and dispersion become more and more dependent on the distance between the transmitter and the receiver as ω_* is increased past one, we believe that the point-wise definitions of these quantities adopted by Schotter [28] are more meaningful and provide a more adequate basis for comparisons between theoretical models and experiments. So, in this work, we define the attenuation as

$$\eta(\tau) = -\frac{(5/6)^{1/2}}{\omega_*}\frac{\mathrm{d}}{\mathrm{d}\tau}\ln|\mathcal{P}(\tau)| \tag{5.3}$$

Table 4. The negative of the phase $\vartheta(\tau)$ for the case $\omega_* = 1.0$

τ	BGK	S	GJ	MRS	CES	LBE
0.0	−7.426(−2)	−7.660(−2)	−7.709(−2)	−7.471(−2)	−7.604(−2)	−7.423(−2)
0.2	1.321(−1)	1.301(−1)	1.299(−1)	1.318(−1)	1.301(−1)	1.316(−1)
0.4	3.343(−1)	3.330(−1)	3.327(−1)	3.348(−1)	3.322(−1)	3.335(−1)
0.6	5.322(−1)	5.315(−1)	5.311(−1)	5.336(−1)	5.297(−1)	5.310(−1)
0.8	7.261(−1)	7.261(−1)	7.253(−1)	7.288(−1)	7.231(−1)	7.245(−1)
1.0	9.168(−1)	9.172(−1)	9.158(−1)	9.208(−1)	9.126(−1)	9.144(−1)
1.2	1.105	1.105	1.103	1.110	1.099	1.101
1.4	1.290	1.291	1.287	1.297	1.282	1.286
1.6	1.474	1.473	1.469	1.482	1.462	1.467
1.8	1.655	1.654	1.648	1.665	1.640	1.647
2.0	1.836	1.833	1.825	1.847	1.816	1.825
3.0	2.724	2.708	2.683	2.741	2.666	2.695
5.0	4.474	4.399	4.323	4.506	4.283	4.378

Table 5. The amplitude $|\mathcal{P}(\tau)|$ for the case $\omega_* = 2.0$

τ	BGK	S	GJ	MRS	CES	LBE
0.0	1.984	1.983	1.981	1.985	1.985	1.986
0.2	1.898	1.896	1.893	1.899	1.898	1.900
0.4	1.765	1.760	1.756	1.764	1.763	1.766
0.6	1.617	1.609	1.605	1.613	1.613	1.618
0.8	1.469	1.457	1.453	1.461	1.464	1.470
1.0	1.328	1.311	1.308	1.314	1.320	1.328
1.2	1.195	1.175	1.172	1.176	1.187	1.195
1.4	1.072	1.048	1.048	1.047	1.065	1.072
1.6	9.583(−1)	9.327(−1)	9.345(−1)	9.286(−1)	9.546(−1)	9.600(−1)
1.8	8.549(−1)	8.279(−1)	8.324(−1)	8.203(−1)	8.550(−1)	8.580(−1)
2.0	7.609(−1)	7.331(−1)	7.409(−1)	7.218(−1)	7.660(−1)	7.657(−1)
3.0	4.117(−1)	3.884(−1)	4.123(−1)	3.587(−1)	4.477(−1)	4.262(−1)
5.0	1.041(−1)	9.649(−2)	1.289(−1)	5.939(−2)	1.657(−1)	1.240(−1)

and the dispersion as

$$\frac{V_0}{V(\tau)} = -\frac{(5/6)^{1/2}}{\omega_*}\frac{\mathrm{d}}{\mathrm{d}\tau}\vartheta(\tau), \tag{5.4}$$

where the factor $(5/6)^{1/2}$ is the ratio between the velocity of sound at high pressures V_0 and the reference velocity v_0 defined by Eq. (2.2b). Explicit expressions that are more convenient for computing these quantities are

$$\eta(\tau) = -\frac{(5/6)^{1/2}}{\omega_*|\mathcal{P}(\tau)|^2}\left[\mathcal{P}_R(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{P}_R(\tau) + \mathcal{P}_I(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{P}_I(\tau)\right] \tag{5.5}$$

and

$$\frac{V_0}{V(\tau)} = -\frac{(5/6)^{1/2}}{\omega_*|\mathcal{P}(\tau)|^2}\left[\mathcal{P}_R(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{P}_I(\tau) - \mathcal{P}_I(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{P}_R(\tau)\right]. \tag{5.6}$$

We show in Tables 9–12 the attenuation and the dispersion as computed by Eqs. (5.5) and (5.6) for the five considered models and for the LBE with the rigid-sphere kernel. Our calculations were performed for 13 values of ω_* that cover the

Table 6. The negative of the phase $\vartheta(\tau)$ for the case $\omega_* = 2.0$

τ	BGK	S	GJ	MRS	CES	LBE
0.0	–4.748(–2)	–4.900(–2)	–5.001(–2)	–4.732(–2)	–4.780(–2)	–4.661(–2)
0.2	3.476(–1)	3.465(–1)	3.454(–1)	3.484(–1)	3.469(–1)	3.481(–1)
0.4	7.225(–1)	7.220(–1)	7.203(–1)	7.247(–1)	7.211(–1)	7.226(–1)
0.6	1.080	1.080	1.077	1.084	1.077	1.080
0.8	1.425	1.424	1.419	1.430	1.418	1.423
1.0	1.759	1.756	1.749	1.766	1.747	1.755
1.2	2.085	2.079	2.069	2.092	2.066	2.078
1.4	2.405	2.394	2.380	2.412	2.376	2.394
1.6	2.719	2.702	2.683	2.726	2.679	2.704
1.8	3.029	3.005	2.980	3.034	2.975	3.008
2.0	3.336	3.302	3.271	3.338	3.266	3.308
3.0	4.831	4.730	4.667	4.812	4.661	4.757
5.0	7.771	7.425	7.317	7.700	7.341	7.520

Table 7. The amplitude $|\mathcal{P}(\tau)|$ for the case $\omega_* = 5.2$

τ	BGK	S	GJ	MRS	CES	LBE
0.0	2.010	2.009	2.009	2.010	2.010	2.010
0.2	1.742	1.740	1.739	1.741	1.742	1.743
0.4	1.383	1.376	1.376	1.375	1.383	1.385
0.6	1.070	1.059	1.062	1.055	1.072	1.072
0.8	8.161(–1)	8.044(–1)	8.121(–1)	7.950(–1)	8.253(–1)	8.214(–1)
1.0	6.170(–1)	6.069(–1)	6.196(–1)	5.920(–1)	6.359(–1)	6.255(–1)
1.2	4.628(–1)	4.559(–1)	4.732(–1)	4.360(–1)	4.921(–1)	4.746(–1)
1.4	3.445(–1)	3.415(–1)	3.621(–1)	3.179(–1)	3.832(–1)	3.590(–1)
1.6	2.544(–1)	2.552(–1)	2.777(–1)	2.296(–1)	3.002(–1)	2.709(–1)
1.8	1.862(–1)	1.903(–1)	2.135(–1)	1.643(–1)	2.366(–1)	2.040(–1)
2.0	1.351(–1)	1.417(–1)	1.642(–1)	1.167(–1)	1.873(–1)	1.532(–1)
3.0	2.282(–2)	3.140(–2)	4.368(–2)	2.195(–2)	6.030(–2)	3.541(–2)
5.0	7.618(–4)	1.245(–3)	2.428(–3)	3.222(–3)	6.242(–3)	1.541(–3)

entire ω_* range in helium studied by Schotter [28]. In addition, as done by Schotter [28], for each value of ω_* we have considered two different values of the distance between the transmitter and the receiver which are specified as $10/(\pi^{1/2}\omega_*)$ and $20/(\pi^{1/2}\omega_*)$ in our notation. All of our results are thought to be accurate in the three figures shown. The entries labeled as "Experimental" were extracted from the graph in Fig. 4 of Ref. 28 and are expected to represent the plotted points with a maximum error of ± 2 in the last reported figure. Our results for the BGK, S, GJ and MRS models were obtained with the ADO solution reported in Appendix A, using $100 \le N \le 200$ for $\omega_* \le 5.19$, $200 \le N \le 400$ for $11.1 \le \omega_* \le 41.9$ and $400 \le N \le 800$ for $\omega_* = 83.9$. To generate the results reported in Tables 9–12 for the CES model, we have used the ADO solution of Section 4 with $K = 19$ and $N = 30$ for $\omega_* \le 5.19$ and $K = 49$ and $N = 60$ for $\omega_* \ge 11.1$. The ADO solution of Section 4 was also used to generate the LBE (rigid-sphere) results. For this purpose, we have used $K = 19$ and $N = 30$ for $\omega_* \le 2.72$ and $K = 39$ and $N = 50$ for $\omega_* = 5.19$ and 11.1. In regard to three largest values of ω_* in Tables 9–12, our LBE results were generated with $K = 49$ and $N = 60$ in a postprocessed

Table 8. The negative of phase $\vartheta(\tau)$ for the case $\omega_* = 5.2$

τ	BGK	S	GJ	MRS	CES	LBE
0.0	–1.997(–2)	–2.061(–2)	–2.128(–2)	–1.981(–2)	–2.001(–2)	–1.955(–2)
0.2	9.449(–1)	9.445(–1)	9.431(–1)	9.462(–1)	9.441(–1)	9.450(–1)
0.4	1.789	1.787	1.784	1.791	1.785	1.788
0.6	2.560	2.552	2.546	2.560	2.547	2.557
0.8	3.281	3.263	3.253	3.274	3.256	3.274
1.0	3.966	3.933	3.920	3.948	3.926	3.955
1.2	4.623	4.571	4.557	4.586	4.568	4.608
1.4	5.257	5.181	5.171	5.192	5.189	5.237
1.6	5.870	5.769	5.767	5.768	5.795	5.847
1.8	6.465	6.337	6.350	6.313	6.391	6.440
2.0	7.044	6.887	6.923	6.827	6.981	7.019
3.0	9.689	9.422	9.698	8.855	9.892	9.745
5.0	1.249(1)	1.356(1)	1.519(1)	1.217(1)	1.571(1)	1.458(1)

Table 9. The attenuation $\eta(\tau)$ at $\tau = 10/(\pi^{1/2}\omega_*)$

ω_*	BGK	S	GJ	MRS	CES	LBE	Experimental [28]
4.50(–2)	2.68(–2)	3.13(–2)	3.13(–2)	3.11(–2)	3.14(–2)	3.14(–2)	3.24(–2)
7.00(–2)	4.13(–2)	4.81(–2)	4.83(–2)	4.77(–2)	4.84(–2)	4.82(–2)	4.84(–2)
1.10(–1)	6.36(–2)	7.38(–2)	7.43(–2)	7.25(–2)	7.42(–2)	7.38(–2)	7.46(–2)
1.70(–1)	9.39(–2)	1.09(–1)	1.10(–1)	1.05(–1)	1.09(–1)	1.08(–1)	1.11(–1)
3.00(–1)	1.46(–1)	1.68(–1)	1.70(–1)	1.61(–1)	1.66(–1)	1.64(–1)	1.65(–1)
6.70(–1)	2.30(–1)	2.59(–1)	2.53(–1)	2.58(–1)	2.31(–1)	2.42(–1)	2.25(–1)
1.34	2.84(–1)	3.02(–1)	2.76(–1)	3.35(–1)	2.45(–1)	2.74(–1)	2.82(–1)
2.72	2.79(–1)	2.82(–1)	2.57(–1)	3.14(–1)	2.36(–1)	2.63(–1)	2.89(–1)
5.19	2.52(–1)	2.51(–1)	2.37(–1)	2.68(–1)	2.25(–1)	2.42(–1)	2.36(–1)
1.11(1)	2.29(–1)	2.28(–1)	2.22(–1)	2.35(–1)	2.16(–1)	2.25(–1)	2.28(–1)
2.16(1)	2.18(–1)	2.17(–1)	2.14(–1)	2.21(–1)	2.12(–1)	2.16(–1)	2.09(–1)
4.19(1)	2.12(–1)	2.12(–1)	2.10(–1)	2.14(–1)	2.09(–1)	2.11(–1)	2.08(–1)
8.39(1)	2.09(–1)	2.09(–1)	2.08(–1)	2.10(–1)	2.08(–1)	2.09(–1)	2.00(–1)

version of the ADO solution of Section 4 that was developed to overcome the slow convergence rate observed in the standard ADO solution for large values of the reduced frequency.

The postprocessed formulas for $\mathcal{P}(\tau)$ were derived by using the standard ADO solution on the right-hand side of Eq. (4.1), approximating the integral over μ' with the half-range quadrature scheme and solving the resulting equation, viz.

$$\left[c\mu\frac{\partial}{\partial\tau} + \sigma(c)\right]\phi(\tau, c, \mu) = \frac{\varepsilon_p}{2}\sum_{l=0}^{L}(2l+1)P_l(\mu)\int_0^\infty \mathrm{e}^{-c'^2}k_l(c', c)\boldsymbol{P}(c')c'^2\mathrm{d}c'\sum_{n=1}^{N}w_n\boldsymbol{G}_{l,n}(\tau), \quad (5.7)$$

and its counterpart with μ changed to $-\mu$. The resulting expressions for $\phi(\tau, c, \mu)$

Table 10. The dispersion $V_0/V(\tau)$ at $\tau = 10/(\pi^{1/2}\omega_*)$

ω_*	BGK	S	GJ	MRS	CES	LBE	Experimental [28]
4.50(−2)	0.998	0.998	0.998	0.998	0.998	0.998	0.994
7.00(−2)	0.996	0.995	0.995	0.995	0.994	0.994	0.989
1.10(−1)	0.990	0.988	0.988	0.989	0.986	0.987	0.973
1.70(−1)	0.979	0.974	0.972	0.977	0.969	0.971	0.937
3.00(−1)	0.947	0.935	0.929	0.948	0.922	0.929	0.898
6.70(−1)	0.859	0.832	0.806	0.874	0.793	0.821	0.784
1.34	0.745	0.706	0.679	0.750	0.674	0.707	0.691
2.72	0.633	0.604	0.596	0.614	0.599	0.618	0.588
5.19	0.578	0.561	0.561	0.562	0.565	0.574	0.536
1.11(1)	0.550	0.542	0.543	0.542	0.546	0.550	0.496
2.16(1)	0.540	0.536	0.536	0.535	0.538	0.540	0.482
4.19(1)	0.535	0.533	0.533	0.533	0.534	0.535	0.458
8.39(1)	0.533	0.532	0.532	0.531	0.532	0.533	0.462

Table 11. The attenuation $\eta(\tau)$ at $\tau = 20/(\pi^{1/2}\omega_*)$

ω_*	BGK	S	GJ	MRS	CES	LBE	Experimental [28]
4.50(−2)	2.68(−2)	3.13(−2)	3.13(−2)	3.11(−2)	3.14(−2)	3.14(−2)	3.24(−2)
7.00(−2)	4.13(−2)	4.81(−2)	4.83(−2)	4.77(−2)	4.84(−2)	4.82(−2)	4.84(−2)
1.10(−1)	6.36(−2)	7.38(−2)	7.43(−2)	7.25(−2)	7.42(−2)	7.38(−2)	7.46(−2)
1.70(−1)	9.39(−2)	1.09(−1)	1.10(−1)	1.05(−1)	1.09(−1)	1.08(−1)	1.11(−1)
3.00(−1)	1.46(−1)	1.68(−1)	1.70(−1)	1.61(−1)	1.66(−1)	1.64(−1)	1.65(−1)
6.70(−1)	2.29(−1)	2.60(−1)	2.52(−1)	2.48(−1)	2.27(−1)	2.44(−1)	2.25(−1)
1.34	2.94(−1)	3.32(−1)	2.73(−1)	3.12(−1)	2.27(−1)	2.89(−1)	2.82(−1)
2.72	3.54(−1)	3.26(−1)	2.54(−1)	5.42(−1)	2.10(−1)	2.87(−1)	2.89(−1)
5.19	2.94(−1)	2.61(−1)	2.30(−1)	3.07(−1)	2.01(−1)	2.54(−1)	2.48(−1)
1.11(1)	2.38(−1)	2.22(−1)	2.11(−1)	2.36(−1)	1.96(−1)	2.23(−1)	2.28(−1)
2.16(1)	2.15(−1)	2.07(−1)	2.02(−1)	2.13(−1)	1.94(−1)	2.09(−1)	2.18(−1)
4.19(1)	2.04(−1)	2.00(−1)	1.97(−1)	2.03(−1)	1.93(−1)	2.01(−1)	1.99(−1)
8.39(1)	1.98(−1)	1.96(−1)	1.95(−1)	1.97(−1)	1.93(−1)	1.96(−1)	1.92(−1)

and $\phi(\tau, c, -\mu)$, $\mu > 0$, were then used in Eq. (4.3) to yield

$$\mathcal{P}(\tau) = \frac{4}{\pi^{1/2}}\bigg\{\int_0^\infty \mathrm{e}^{-c^2}\int_0^1 (2c\mu+\pi^{1/2}+\kappa)\,\mathrm{e}^{-\sigma(c)\tau/(c\mu)}\mu^2\mathrm{d}\mu\, c^4\mathrm{d}c+\sum_{l=0}^{L}\sum_{n=1}^{N} w_n P_l(\mu_n)$$
$$\times\sum_{j=1}^{J}\nu_j A_j[\mathbf{X}_{l,j}(\tau)+(-1)^l\mathbf{\Upsilon}_{l,j}\mathrm{e}^{-\varepsilon_p\tau/\nu_j}][\mathbf{\Phi}(\nu_j,\mu_n)+(-1)^l\mathbf{\Phi}(\nu_j,-\mu_n)]\bigg\}, \quad (5.8)$$

where

$$\kappa = 4\sum_{l=0}^{L}(-1)^l\sum_{n=1}^{N} w_n P_l(\mu_n)\sum_{j=1}^{J}\nu_j A_j\mathbf{\Gamma}_{l,j}[\mathbf{\Phi}(\nu_j,\mu_n)+(-1)^l\mathbf{\Phi}(\nu_j,-\mu_n)], \quad (5.9)$$

Table 12. The dispersion $V_0/V(\tau)$ at $\tau = 20/(\pi^{1/2}\omega_*)$

ω_*	BGK	S	GJ	MRS	CES	LBE	Experimental [28]
4.50(−2)	0.998	0.998	0.998	0.998	0.998	0.998	0.994
7.00(−2)	0.996	0.995	0.995	0.995	0.994	0.994	0.989
1.10(−1)	0.990	0.988	0.988	0.989	0.986	0.987	0.973
1.70(−1)	0.979	0.974	0.972	0.977	0.969	0.971	0.937
3.00(−1)	0.947	0.935	0.929	0.948	0.922	0.929	0.898
6.70(−1)	0.860	0.833	0.805	0.875	0.794	0.818	0.784
1.34	0.763	0.706	0.665	0.854	0.668	0.692	0.651
2.72	0.605	0.530	0.554	0.480	0.572	0.564	0.571
5.19	0.489	0.464	0.494	0.411	0.513	0.494	0.509
1.11(1)	0.451	0.443	0.460	0.422	0.472	0.458	0.428
2.16(1)	0.441	0.437	0.446	0.427	0.453	0.445	0.402
4.19(1)	0.436	0.435	0.439	0.430	0.443	0.438	0.414
8.39(1)	0.434	0.434	0.436	0.431	0.438	0.436	0.407

$$\mathsf{X}_{l,j}(\tau) = \frac{2l+1}{2}\int_0^\infty \mathrm{e}^{-c^2}\left[\frac{1}{\sigma(c)}\int_0^1 \mu^2 P_l(\mu) C(\tau : c\mu/\sigma(c), \nu_j/\varepsilon_p)\mathrm{d}\mu\right] \times \int_0^\infty \mathrm{e}^{-c'^2} k_l(c',c)\boldsymbol{P}(c')c'^2 \mathrm{d}c' c^4 \mathrm{d}c \tag{5.10}$$

and

$$\boldsymbol{\Upsilon}_{l,j} = \frac{2l+1}{2}\int_0^\infty \mathrm{e}^{-c^2}\left[\frac{1}{\sigma(c)}\int_0^1 \mu^2 P_l(\mu)\frac{\mathrm{d}\mu}{c\mu/\sigma(c)+\nu_j/\varepsilon_p}\right] \times \int_0^\infty \mathrm{e}^{-c'^2} k_l(c',c)\boldsymbol{P}(c')c'^2 \mathrm{d}c' c^4 \mathrm{d}c. \tag{5.11}$$

Here we define

$$\boldsymbol{\Gamma}_{l,j} = \frac{2l+1}{2}\int_0^\infty \mathrm{e}^{-c^2}\left[\frac{1}{\sigma(c)}\int_0^1 \mu P_l(\mu)\frac{\mathrm{d}\mu}{c\mu/\sigma(c)+\nu_j/\varepsilon_p}\right] \times \int_0^\infty \mathrm{e}^{-c'^2} k_l(c',c)\boldsymbol{P}(c')c'^2 \mathrm{d}c' c^3 \mathrm{d}c \tag{5.12}$$

and

$$C(\tau : x, y) = \frac{\mathrm{e}^{-\tau/x} - \mathrm{e}^{-\tau/y}}{x-y}. \tag{5.13}$$

We now record some observations made in regard to the numerical results displayed in Tables 9–12. First of all, we can see that the agreement between theory and experiment is in general better for the attenuation than for the dispersion (more about this is said in the next paragraph), and that, among all results based on theory, the LBE results are those that show the best agreement with experiment when the complete frequency range is considered. Concerning the models, we first note that in the classical (low-frequency) regime the attenuation computed with the BGK model is not as good as that from the other models, perhaps reflecting the fact that this model does not have the correct Prandtl number ($\varepsilon_p/\varepsilon_t$ ratio).

Nevertheless, in the transition and near free-molecular (high-frequency) regimes, the BGK results look reasonable. In regard to the other models, we note that all of them agree well with the LBE in the classical and the near free-molecular regimes. In the transition regime, the MRS model tends to overestimate the attenuation and dispersion (especially the attenuation), the CES model slightly underestimates the attenuation, the S model slightly overestimates the attenuation and the GJ model seems to be the one that gives the results that are closest to the LBE results.

As mentioned above, the agreement between theory and experiment is not as good for the dispersion as for the attenuation, and the differences become more apparent as the near free-molecular regime is approached. We attribute this to the fact that we have modeled the problem as a half space and so we are unable to consider the effect of the presence of the receiver on the pressure field, something that may be important in the near free-molecular regime, especially when the separation between the transmitter and the receiver is small, as is the case here. That this may indeed be the case is substantiated by the fact that our numerical results for the amplitude $|\mathcal{P}(\tau)|$, the phase $\vartheta(\tau)$, the attenuation $\eta(\tau)$, and the dispersion $V_0/V(\tau)$ at $\tau = 0$, obtained with $\omega_* = 200.0$ and any model, match very well the free-molecular limiting values of these quantities as $\tau \to 0$ that are reported in Ref. 40 for the case without receiver. Our results are also in qualitative agreement with the plots displayed in Figs. 1–4 of Ref. 40 for the case without receiver. We should mention that, in these comparisons, we have taken into account that the phase in Ref. 40 is equivalent to the negative of the phase in this work, that the amplitude in Ref. 40 is normalized as 1/2 of the amplitude in this work and that the labels "1" and "2" in Fig. 4 of Ref. 40 should be interchanged. As the differences between the results in the free-molecular regime for the cases with and without receiver are shown in Ref. 40 to be more significant as $\tau \to 0$, we believe this should also be true in the near free-molecular regime.

We have also compared numerical results of our work with those reported by other authors for a couple of models that are studied in this paper. We have been able to confirm all (four) figures of the BGK results for the real and imaginary parts of the pressure perturbation reported by Thomas and Siewert [14] for $\omega_* = 0.5$ and 1.0. However, for $\omega_* = 5.2$, we have observed a progressive deterioration in the accuracy of their results as τ is increased past 5.6, so that for $\tau > 8$ some of the entries in Table II of Ref. 14 are good only to two figures. Note that the pressure perturbation is normalized in a different way in that work, so our results need to be multiplied by 1/4 to be appropriately compared with the results of Ref. 14. The discrepancies observed in Ref. 14 were resolved when Thomas [41] re-evaluated the solution of Ref. 14 with an increased number of quadrature points and found perfect agreement with our results for $\omega_* = 5.2$. In regard to similar BGK results reported by Loyalka and Cheng [15] for $\omega_* = 5.2$, we believe that they are accurate to two (sometimes three) figures, but only when the distances from the plate are not large, say $\tau < 2$. For larger distances, the accuracy of their results deteriorates, and there are points for which no correct figure in the real and imaginary parts of the pressure perturbation is given. The BGK results reported by Cheng and Loyalka [22] for $\omega_* = 2.0$ have a degree of accuracy similar to that of the results of Ref. 15. The situation is worse for the results based on the model they [22] call "Gross–Jackson (N=5)." These results [22] disagree completely from results of our S model which, in principle, we believe should be equivalent.

Finally, to give the reader an idea of the CPU time required to run our FORTRAN programs, we note that our implementation of the ADO solution described in Section 4 takes about 25 s on a AMD Athlon 64 3500+ computer for a rigid-sphere case with $K = 19$ and $N = 30$, using a quadrature of order 200 to calculate integrals over the c variable. As the computational effort associated with our calculation is, in great part, demanded by the eigensystem that has to be solved in order to determine the separation constants ν_j and the elementary solutions $\mathbf{\Phi}(\nu_j, \pm\mu_i)$, the CPU time scales approximately as J^3, where $J = N(K+1)$. This is without the postprocessing step defined by Eqs. (5.8) through (5.13). When postprocessing is required, the same case ($K = 19$ and $N = 30$) takes about 15 min, using a quadrature of order 100 to represent integrals over μ in the post-processing step, and the CPU time scales approximately as NJ^p, with p varying between 1 and 2. In addition, we note that our implementation of the ADO solution described in Appendix A takes about 1 s on the AMD computer to run any of the BGK, S, GJ or MRS cases with $N = 100$. The CPU time for this program scales somewhat faster than N^3.

6. Final comments

In this work we have solved (we believe well) the half-space version of the classical problem of sound-wave propagation in a rarefied gas. In contrast to older works on this subject, our (semi-analytical) solution of this problem is based on a rigorous form of the linearized Boltzmann equation for rigid-sphere interactions, and for the first time (and as opposed to log-log graphical presentations that are often used), the theoretical results have been compared to experimental measurements in a definitive way. In addition to reporting our solution based on the linearized Boltzmann equation for rigid-sphere scattering, we have carried out a systematic study of the BGK, the S, what we called the GJ (for Gross–Jackson), a newly defined MRS (for McCormack rigid spheres) and the CES kinetic models.

Since our solution requires only a matrix eigenvalue/eigenvector routine and a solver of linear algebraic equations, the algorithm is, in general, especially efficient, fast and easy to implement. However, since some of the experimental data was taken at very large values of the reduced frequency, we found it necessary to implement a form of "post processing" that we used along with our ADO (analytical discrete ordinates) method. While this post processing step increases the required computational time, we believe this procedure will prove to be of significant importance also for other rarefied-gas problems defined by dimensions that are very small (in terms of the mean-free path), but not small enough that "free molecular" equations can be used.

Finally, we would like to mention three reasons why, in our opinion, the ADO method that we have used in this work is so effective: (i) the half-range quadrature scheme allows a better treatment of the boundary conditions than a full-range scheme, (ii) the eigenvalue problem is formulated in a particularly useful way, and (iii) our results are continuous in the τ variable and thus are valid anywhere in the gas. This last point is especially important in this work since the definitions of "attenuation" and "dispersion" require a spatial derivative which is available at once from our solution.

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Appendix A. The McCormack model and special cases

In an important paper [32] published in 1973, McCormack used a method suggested by Gross and Jackson [21] to define a model for a mixture of two gases. While the McCormack model has been used recently [42,43] to solve several half-space problems for binary gas mixtures, we find it interesting to note here that when reduced to the special case of a single-species gas, the McCormack model contains a free parameter γ that can be defined so as to yield the S model or the GJ model, as well as other kinetic models. And so in this Appendix we investigate some choices of γ, and we develop an ADO solution that can be used for any choice of this free parameter. In discussing our ADO solution here, we find it convenient to work with the velocity vector written in rectangular coordinates rather than the spherical coordinate system used in the main body of this work. And so, looking to Ref. 42, we express the McCormack model, for a single-species gas, as

$$\left(c_x \frac{\partial}{\partial \tau} + s\right) h(\tau, \boldsymbol{c}) = \sigma \mathcal{L}_*\{h\}(\tau, \boldsymbol{c}), \tag{A.1}$$

where s is given by Eq. (2.37b) and

$$\mathcal{L}_*\{h\}(\tau, \boldsymbol{c}) = -h(\tau, \boldsymbol{c}) + \frac{1}{\pi^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-c'^2} h(\tau, \boldsymbol{c}') K(\boldsymbol{c}', \boldsymbol{c}) \mathrm{d}c'_x \mathrm{d}c'_y \mathrm{d}c'_z. \tag{A.2}$$

We continue to use τ to measure distances in terms of the mean-free path l_p given by Eq. (2.21), and

$$\sigma = \gamma \frac{\mu_*}{P_0}, \tag{A.3}$$

where γ is the free parameter in the model. In addition, the scattering kernel is given as

$$K(\boldsymbol{c}', \boldsymbol{c}) = K^{(1)}(\boldsymbol{c}', \boldsymbol{c}) + K^{(2)}(\boldsymbol{c}', \boldsymbol{c}) + K^{(3)}(\boldsymbol{c}', \boldsymbol{c}), \tag{A.4}$$

where

$$K^{(1)}(\boldsymbol{c}', \boldsymbol{c}) = 1 + 2\boldsymbol{c}' \cdot \boldsymbol{c} + (2/3)(c'^2 - 3/2)(c^2 - 3/2), \tag{A.5a}$$

$$K^{(2)}(\boldsymbol{c}', \boldsymbol{c}) = 2\varpi[(\boldsymbol{c}' \cdot \boldsymbol{c})^2 - (1/3)c'^2 c^2], \tag{A.5b}$$

and

$$K^{(3)}(\boldsymbol{c}', \boldsymbol{c}) = (4/5)\beta(c'^2 - 5/2)(c^2 - 5/2)\boldsymbol{c}' \cdot \boldsymbol{c}. \tag{A.5c}$$

The constants β and ϖ are given as

$$\beta = 1 - (16/15)n_0\Omega^{22}/\gamma \quad \text{and} \quad \varpi = 1 - (8/5)n_0\Omega^{22}/\gamma, \tag{A.6a,b}$$

where Ω^{22} is an "Ω integral" from Chapman and Cowling [44]. And so here we find that

$$\nu(c) = 1, \quad \varepsilon_t = \frac{1}{1-\beta}, \quad \varepsilon_p = \frac{1}{1-\varpi}, \quad \varepsilon_p/\varepsilon_t = 2/3, \tag{A.7a,b,c,d}$$

$$A(c) = \varepsilon_t c(c^2 - 5/2), \quad \text{and} \quad B(c) = \varepsilon_p c^2. \tag{A.7e,f}$$

We see that if we use

$$\gamma = (8/5)n_0\Omega^{22} \tag{A.8}$$

we obtain

$$\varpi = 0, \quad \beta = 1/3, \quad \varepsilon_t = 3/2, \quad \sigma = \varepsilon_p, \quad \text{and} \quad \varepsilon_p = 1, \tag{A.9a,b,c,d,e}$$

which define the S model. On the other hand, if we use

$$\gamma = (12/5)n_0\Omega^{22} \tag{A.10}$$

we obtain

$$\varpi = 1/3, \quad \beta = 5/9, \quad \varepsilon_t = 9/4, \quad \sigma = \varepsilon_p, \quad \text{and} \quad \varepsilon_p = 3/2, \tag{A.11a,b,c,d,e}$$

which define the GJ model.

While there is no choice of the parameter γ that will yield the BGK model, basic data for this model can be obtained from Eqs. (A.1–A.7f), with the exception of Eq. (A.7d), by taking $\sigma = 1$, $\varpi = 0$ and $\beta = 0$.

Continuing with our discussion of the McCormack model for the case of a single species gas, we note that: (i) there is one free parameter γ that is not specified and (ii) the physics of the scattering interaction is to be defined by the choice of the omega integral Ω^{22}. For the case of rigid-sphere interactions we can write [44]

$$\Omega^{22} = (2\pi)^{1/2} v_0 \sigma_0^2, \tag{A.12}$$

where v_0 is given by Eq. (2.2b) and, as used elsewhere in this work, σ_0 is the diameter of a gas particle. We can now use Eq. (A.12) with Eqs. (2.20b) and (2.21) to conclude that, for the case of rigid-sphere interactions,

$$\varepsilon_p = 2^{-1/2} n_0 \Omega^{22} \frac{\mu_*}{P_0}. \tag{A.13}$$

Now, in order to have Eq. (A.1) consistent with Eq. (2.21) for the case of rigid-sphere interactions, we must use

$$\gamma = 2^{-1/2} n_0 \Omega^{22} \tag{A.14}$$

so that we can have

$$\sigma = \varepsilon_p. \tag{A.15}$$

It is clear, for the case of rigid-sphere interactions, that the use of γ as defined by Eq. (A.14) yields a model that is consistent with the LBE. As this model is derived as a special case of the McCormack model, and since this model is based on the rigid-sphere interaction law, we choose to refer to this as the MRS model. So here we have

$$\varpi = 1 - (8/5)2^{1/2},\ \beta = 1 - (16/15)2^{1/2},\ \varepsilon_t = (15/32)2^{1/2},\ \text{and}\ \varepsilon_p = (5/16)2^{1/2}. \tag{A.16a,b,c,d}$$

Our focus in this work is on computing the xx component of the pressure tensor, and this quantity we can obtain from certain moments (integrals) of Eq. (A.1). And so, as was done in Ref. 43, we first multiply Eq. (A.1) by

$$\phi_1(c_y, c_z) = (1/\pi)\mathrm{e}^{-(c_y^2+c_z^2)} \tag{A.17}$$

and integrate over all c_y and all c_z. We then repeat this procedure using

$$\phi_2(c_y, c_z) = (1/\pi)\mathrm{e}^{-(c_y^2+c_z^2)}(c_y^2 + c_z^2 - 1). \tag{A.18}$$

Defining

$$g_1(\tau, c_x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_1(c_y, c_z) h(\tau, \boldsymbol{c}) \mathrm{d}c_y \mathrm{d}c_z \tag{A.19}$$

and

$$g_2(\tau, c_x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_2(c_y, c_z) h(\tau, \boldsymbol{c}) \mathrm{d}c_y \mathrm{d}c_z, \tag{A.20}$$

we find from these projections two coupled balance equations which we write (in matrix notation) as

$$\xi \frac{\partial}{\partial \tau}\boldsymbol{G}(\tau, \xi) + (s + \sigma)\boldsymbol{G}(\tau, \xi) = \sigma \int_{-\infty}^{\infty} \psi(\xi')\boldsymbol{K}(\xi', \xi)\boldsymbol{G}(\tau, \xi')\mathrm{d}\xi', \tag{A.21}$$

where the components of $\boldsymbol{G}(\tau, \xi)$ are $g_1(\tau, \xi)$ and $g_2(\tau, \xi)$, where we now use ξ in place of c_x, and where

$$\psi(\xi) = \pi^{-1/2}\mathrm{e}^{-\xi^2}. \tag{A.22}$$

It follows from Ref. 43 that (for the one-gas case) we can express the elements of $\boldsymbol{K}(\xi', \xi)$ as

$$k_{1,1}(\xi', \xi) = 1 + f_{1,1}(\xi', \xi)\xi'\xi + (2/3)(1 + 2\varpi)(\xi'^2 - 1/2)(\xi^2 - 1/2), \tag{A.23}$$

$$k_{1,2}(\xi', \xi) = (4/5)\beta(\xi^2 - 3/2)\xi'\xi + (2/3)(1 - \varpi)(\xi^2 - 1/2), \tag{A.24}$$

$$k_{2,1}(\xi', \xi) = (4/5)\beta(\xi'^2 - 3/2)\xi'\xi + (2/3)(1 - \varpi)(\xi'^2 - 1/2), \tag{A.25}$$

and

$$k_{2,2}(\xi',\xi) = (1/3)(2+\varpi) + (4/5)\beta\xi'\xi. \tag{A.26}$$

Here

$$f_{1,1}(\xi',\xi) = 2 + (4/5)\beta({\xi'}^2 - 3/2)(\xi^2 - 3/2). \tag{A.27}$$

The boundary condition we require here can be found by projecting Eq. (2.11) against Eqs. (A.17) and (A.18). In this way we find

$$\boldsymbol{G}(0,\xi) - 2\boldsymbol{D}\int_0^\infty \mathrm{e}^{-{\xi'}^2}\boldsymbol{G}(0,-\xi')\xi'\mathrm{d}\xi' = (2\xi + \pi^{1/2})\boldsymbol{R} \tag{A.28}$$

for $\xi > 0$. Here

$$\boldsymbol{D} = \mathrm{diag}\{1, 0\} \quad \text{and} \quad \boldsymbol{R} = \begin{bmatrix}1 & 0\end{bmatrix}^T. \tag{A.29a,b}$$

To complete our solution of the considered problem of sound-wave propagation we seek a bounded (as τ tends to infinity) solution of Eq. (A.21) that satisfies the boundary condition listed as Eq. (A.28), and so to start we seek solutions of Eq. (A.21) of the form

$$\boldsymbol{G}(\tau,\xi) = \boldsymbol{\Phi}(\nu,\xi)\mathrm{e}^{-(s+\sigma)\tau/\nu} \tag{A.30}$$

where the separation constants ν and the elementary solutions $\boldsymbol{\Phi}(\nu,\xi)$ are to be determined. Substituting Eq. (A.30) into Eq. (A.21), we find

$$(\nu-\xi)\boldsymbol{\Phi}(\nu,\xi) = \zeta\nu\int_0^\infty \psi(\xi')[\boldsymbol{K}(\xi',\xi)\boldsymbol{\Phi}(\nu,\xi') + \boldsymbol{K}(-\xi',\xi)\boldsymbol{\Phi}(\nu,-\xi')]\mathrm{d}\xi' \tag{A.31}$$

and

$$(\nu+\xi)\boldsymbol{\Phi}(\nu,-\xi) = \zeta\nu\int_0^\infty \psi(\xi')[\boldsymbol{K}(\xi',-\xi)\boldsymbol{\Phi}(\nu,\xi') + \boldsymbol{K}(-\xi',-\xi)\boldsymbol{\Phi}(\nu,-\xi')]\mathrm{d}\xi', \tag{A.32}$$

where

$$\zeta = \sigma/(s+\sigma). \tag{A.33}$$

Now, since

$$\boldsymbol{K}(\xi',-\xi) = \boldsymbol{K}(-\xi',\xi), \tag{A.34}$$

we conclude that

$$\boldsymbol{\Phi}(\nu,\xi) = \boldsymbol{\Phi}(-\nu,-\xi), \tag{A.35}$$

and so adding and subtracting Eqs. (A.31) and (A.32), one from the other, we find that

$$(1/\xi^2)\left[\boldsymbol{V}(\nu,\xi) - \zeta\int_0^\infty \psi(\xi')\boldsymbol{\mathcal{K}}(\xi',\xi)\boldsymbol{V}(\nu,\xi')\mathrm{d}\xi'\right] = \lambda\boldsymbol{V}(\nu,\xi) \tag{A.36}$$

and

$$\boldsymbol{U}(\nu,\xi) = (\nu/\xi)\left[\boldsymbol{V}(\nu,\xi) - \zeta\int_0^\infty \psi(\xi')\boldsymbol{K}_-(\xi',\xi)\boldsymbol{V}(\nu,\xi')\mathrm{d}\xi'\right], \tag{A.37}$$

where

$$\boldsymbol{U}(\nu,\xi)=\boldsymbol{\Phi}(\nu,\xi)+\boldsymbol{\Phi}(\nu,-\xi) \tag{A.38}$$

and

$$\boldsymbol{V}(\nu,\xi)=\boldsymbol{\Phi}(\nu,\xi)-\boldsymbol{\Phi}(\nu,-\xi). \tag{A.39}$$

Here

$$\lambda=1/\nu^2 \tag{A.40}$$

and

$$\mathcal{K}(\xi',\xi)=(\xi/\xi')\boldsymbol{K}_+(\xi',\xi)+\boldsymbol{K}_-(\xi',\xi)-\zeta\int_0^\infty \psi(\xi'')(\xi/\xi'')\boldsymbol{K}_+(\xi'',\xi)\boldsymbol{K}_-(\xi',\xi'')\mathrm{d}\xi'' \tag{A.41}$$

where

$$\boldsymbol{K}_+(\xi',\xi)=\boldsymbol{K}(\xi',\xi)+\boldsymbol{K}(-\xi',\xi) \tag{A.42}$$

and

$$\boldsymbol{K}_-(\xi',\xi)=\boldsymbol{K}(\xi',\xi)-\boldsymbol{K}(-\xi',\xi). \tag{A.43}$$

We now introduce a "half-range" quadrature scheme with weights and nodes $\{w_k,\xi_k\}$ and rewrite Eqs. (A.36) and (A.37) evaluated at the quadrature points as

$$(1/\xi_i^2)\left[\boldsymbol{V}(\nu_j,\xi_i)-\zeta\sum_{k=1}^{N} w_k\psi(\xi_k)\mathcal{K}(\xi_k,\xi_i)\boldsymbol{V}(\nu_j,\xi_k)\right]=\lambda_j\boldsymbol{V}(\nu_j,\xi_i) \tag{A.44}$$

and

$$\boldsymbol{U}(\nu_j,\xi_i)=(\nu_j/\xi_i)\left[\boldsymbol{V}(\nu_j,\xi_i)-\zeta\sum_{k=1}^{N} w_k\psi(\xi_k)\boldsymbol{K}_-(\xi_k,\xi_i)\boldsymbol{V}(\nu_j,\xi_k)\right], \tag{A.45}$$

for $i=1,2,...,N$. Equation (A.44) defines our eigenvalue problem, to which we have added the subscript j to label the eigenvalues and eigenvectors. Once this eigenvalue problem is solved, we have the elementary solutions from

$$\boldsymbol{\Phi}(\nu_j,\xi_i)=(1/2)[\boldsymbol{U}(\nu_j,\xi_i)+\boldsymbol{V}(\nu_j,\xi_i)] \tag{A.46}$$

and

$$\boldsymbol{\Phi}(\nu_j,-\xi_i)=(1/2)[\boldsymbol{U}(\nu_j,\xi_i)-\boldsymbol{V}(\nu_j,\xi_i)]. \tag{A.47}$$

Note that the separation constants defined by

$$\nu_j=\pm\lambda_j^{-1/2} \tag{A.48}$$

occur in $\pm$ pairs. From this point, we take ν_j to be the root (that has a positive real part) listed in Eq. (A.48). Once we have solved the eigenvalue problem defined by Eq. (A.44), we can write our general (discrete ordinates) solution to Eq. (A.21) as

$$\boldsymbol{G}(\tau,\pm\xi_i)=\sum_{j=1}^{2N}\left[A_j\boldsymbol{\Phi}(\nu_j,\pm\xi_i)\mathrm{e}^{-(s+\sigma)\tau/\nu_j}+B_j\boldsymbol{\Phi}(\nu_j,\mp\xi_i)\mathrm{e}^{(s+\sigma)\tau/\nu_j}\right], \tag{A.49}$$

for $i = 1, 2, ..., N$. Here the arbitrary constants $\{A_j\}$ and $\{B_j\}$ are to be determined. Since the solution must be bounded as τ tends to infinity, we take all $\{B_j\}$ to be zero to obtain

$$\boldsymbol{G}(\tau, \pm\xi_i) = \sum_{j=1}^{2N} A_j \boldsymbol{\Phi}(\nu_j, \pm\xi_i) \mathrm{e}^{-(s+\sigma)\tau/\nu_j} \tag{A.50}$$

for $i = 1, 2, ..., N$. To complete the solution we substitute Eq. (A.50) into a discrete-ordinates version of Eq. (A.28) to establish a system of linear algebraic equations we solve to find the constants $\{A_j\}$. Finally, we use Eq. (A.50) and write our result as

$$\mathcal{P}(\tau) = 2 \sum_{j=1}^{J} A_j X_j \mathrm{e}^{-(s+\sigma)\tau/\nu_j}, \tag{A.51}$$

where

$$X_j = \begin{bmatrix} 1 & 0 \end{bmatrix} \sum_{n=1}^{N} w_n \psi(\xi_n) \xi_n^2 [\boldsymbol{\Phi}(\nu_j, \xi_n) + \boldsymbol{\Phi}(\nu_j, -\xi_n)]. \tag{A.52}$$

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