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Some exact results basic to the linearized Boltzmann equations for a binary mixture of rigid spheres

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Abstract. Six exact solutions (related to the conservation of number, energy and momentum) of the linearized Boltzmann equations for a binary mixture of rigid spheres, for the case of isotropic scattering in the center-of-mass system, are reported. The verification of the reported exact solutions (collisional invariants) is based on a recently reported explicit formulation of the linearized Boltzmann equation for a binary mixture of rigid spheres. Elementary analysis is used also to establish a basic flow condition.

Keywords. Rarefied gas dynamics, binary mixtures, rigid spheres.

1. Introduction

In a recent work, Garcia, Siewert and Williams [1] reported explicit forms of the collision operators required to establish the linearized Boltzmann equation for a binary mixture of rigid spheres that are assumed to scatter isotropically in the center-of-mass system. And so here we report the way in which we have verified that six exact solutions (corresponding to conservation of number, energy and momentum) and a flow condition are valid. While the established results are exactly what could have been foreseen, we have found that considerable work was required to show explicitly that the expected results are correct. In the process of establishing the results reported herein, we have also gained more confidence that the explicit, but complicated, forms for the collision operators reported in Ref. [1] are correct.

To start this work, we write the coupled, linearized Boltzmann equations, for the considered binary mixture of rigid spheres, in the form reported by Garcia, Siewert and Williams [1], viz.

$$\boldsymbol{c}\cdot\nabla_r h_\alpha(\boldsymbol{r},\boldsymbol{c}) + \varpi_\alpha(c) h_\alpha(\boldsymbol{r},\boldsymbol{c}) = \sum_{\beta=1}^{2}\int \mathrm{e}^{-c'^2} K_{\alpha,\beta}(\boldsymbol{c}':\boldsymbol{c}) h_\beta(\boldsymbol{r},\boldsymbol{c}')\mathrm{d}^3c', \quad \alpha = 1,2, \tag{1}$$

where $\boldsymbol{r}$, with Cartesian coordinates $\{x,y,z\}$, is the spatial variable and $\boldsymbol{c}$, with coordinates $\{c_x, c_y, c_z\}$ and magnitude c, denotes the dimensionless velocity vector. In addition

$$K_{1,1}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_1\sigma_{1,1}\pi^{1/2}\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) + n_2\sigma_{1,2}\pi^{1/2}\mathcal{F}_{1,2}(\boldsymbol{c}' : \boldsymbol{c}), \tag{2}$$

$$K_{1,2}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_2\sigma_{1,2}\pi^{1/2}\mathcal{G}_{1,2}(\boldsymbol{c}' : \boldsymbol{c}), \tag{3}$$

$$K_{2,1}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_1\sigma_{2,1}\pi^{1/2}\mathcal{G}_{2,1}(\boldsymbol{c}' : \boldsymbol{c}), \tag{4}$$

and

$$K_{2,2}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_2\sigma_{2,2}\pi^{1/2}\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) + n_1\sigma_{2,1}\pi^{1/2}\mathcal{F}_{2,1}(\boldsymbol{c}' : \boldsymbol{c}). \tag{5}$$

Here

$$\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{\pi}\Big(\frac{2}{|\boldsymbol{c}' - \boldsymbol{c}|}\exp\Big\{\frac{|\boldsymbol{c}' \times \boldsymbol{c}|^2}{|\boldsymbol{c}' - \boldsymbol{c}|^2}\Big\} - |\boldsymbol{c}' - \boldsymbol{c}|\Big) \tag{6}$$

is the basic kernel for a single-species gas used by Pekeris [2],

$$\mathcal{F}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c}) = \mathcal{F}(a_{\alpha,\beta}; \boldsymbol{c}' : \boldsymbol{c}), \tag{7}$$

and

$$\mathcal{G}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c}) = \mathcal{G}(a_{\alpha,\beta}; \boldsymbol{c}' : \boldsymbol{c}), \tag{8}$$

where

$$a_{\alpha,\beta} = (m_\beta/m_\alpha)^{1/2}, \tag{9}$$

$$\mathcal{F}(a; \boldsymbol{c}' : \boldsymbol{c}) = \frac{(a^2+1)^2}{a^3\pi|\boldsymbol{c}' - \boldsymbol{c}|}\exp\Big\{a^2\frac{|\boldsymbol{c}' \times \boldsymbol{c}|^2}{|\boldsymbol{c}' - \boldsymbol{c}|^2} - \frac{(1-a^2)^2(c'^2 + c^2)}{4a^2} - \frac{(a^4-1)\boldsymbol{c}' \cdot \boldsymbol{c}}{2a^2}\Big\}, \tag{10}$$

and

$$\mathcal{G}(a; \boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{a\pi}\big|\boldsymbol{c}' - a\boldsymbol{c}\big|[J(a; \boldsymbol{c}' : \boldsymbol{c}) - 1]. \tag{11}$$

In addition,

$$\varpi_\alpha(c) = \varpi_\alpha^{(1)}(c) + \varpi_\alpha^{(2)}(c), \tag{12}$$

with

$$\varpi_\alpha^{(\beta)}(c) = 4\pi^{1/2}n_\beta\sigma_{\alpha,\beta}a_{\beta,\alpha}\nu(a_{\alpha,\beta}c) \tag{13}$$

and

$$\nu(c) = \frac{2c^2+1}{c}\int_0^c \mathrm{e}^{-x^2}\mathrm{d}x + \mathrm{e}^{-c^2}. \tag{14}$$

Since Eq. (1) is written in terms of a dimensionless velocity variable $\boldsymbol{c}$, we note that the basic velocity distribution functions are available from

$$f_\alpha(\boldsymbol{r}, \boldsymbol{v}) = f_{\alpha,0}(v)[1 + h_\alpha(\boldsymbol{r}, \lambda_\alpha^{1/2}\boldsymbol{v})], \tag{15}$$

where $\lambda_\alpha = m_\alpha/(2kT_0)$ and where

$$f_{\alpha,0}(v) = n_\alpha(\lambda_\alpha/\pi)^{3/2}\mathrm{e}^{-\lambda_\alpha v^2} \tag{16}$$

is the Maxwellian distribution for n_α particles of mass m_α in equilibrium at temperature T_0. Note that k is the Boltzmann constant. To complete our starting equations, we note from Ref. [1] that

$$J(a;\boldsymbol{c}':\boldsymbol{c}) = \frac{(a+1/a)^2}{2\Delta(a;\boldsymbol{c}':\boldsymbol{c})}\exp\Big\{\frac{-2C(a;\boldsymbol{c}':\boldsymbol{c})}{(a-1/a)^2}\Big\}\sinh\Big\{\frac{2\Delta(a;\boldsymbol{c}':\boldsymbol{c})}{(a-1/a)^2}\Big\}, \quad a \neq 1, \tag{17a}$$

and

$$J(a;\boldsymbol{c}':\boldsymbol{c}) = \frac{1}{|\boldsymbol{c}'-\boldsymbol{c}|^2}\exp\Big\{\frac{|\boldsymbol{c}'\times\boldsymbol{c}|^2}{|\boldsymbol{c}'-\boldsymbol{c}|^2}\Big\}, \quad a = 1, \tag{17b}$$

where, to write Eq. (17a), we have used the definitions [1]

$$\Delta(a;\boldsymbol{c}':\boldsymbol{c}) = \big\{C^2(a;\boldsymbol{c}':\boldsymbol{c}) + (a-1/a)^2|\boldsymbol{c}'\times\boldsymbol{c}|^2\big\}^{1/2} \tag{18}$$

and

$$C(a;\boldsymbol{c}':\boldsymbol{c}) = {c'}^2 + c^2 - (a+1/a)\boldsymbol{c}'\cdot\boldsymbol{c}. \tag{19}$$

Finally, and to be clear, we note that we use $\sigma_{\alpha,\beta}$ to denote the differential-scattering cross section, which for the case of rigid-sphere scattering that is isotropic in the center-of-mass system, we write, after consultation with Chapman and Cowling [3], as

$$\sigma_{\alpha,\beta} = \frac{1}{4}\Big(\frac{d_\alpha + d_\beta}{2}\Big)^2, \tag{20}$$

where d_1 and d_2 are the atomic diameters of the two types of gas particles.

2. Exact solutions

To start this section, we first rewrite Eq. (1) as

$$\boldsymbol{c}\cdot\nabla_r\boldsymbol{H}(\boldsymbol{r},\boldsymbol{c}) + \boldsymbol{\Sigma}(c)\boldsymbol{H}(\boldsymbol{r},\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\boldsymbol{K}(\boldsymbol{c}':\boldsymbol{c})\boldsymbol{H}(\boldsymbol{r},\boldsymbol{c}')\mathrm{d}^3c', \tag{21}$$

where

$$\boldsymbol{H}(\boldsymbol{r},\boldsymbol{c}) = \begin{bmatrix} h_1(\boldsymbol{r},\boldsymbol{c}) \\ h_2(\boldsymbol{r},\boldsymbol{c}) \end{bmatrix} \tag{22}$$

and

$$\boldsymbol{K}(\boldsymbol{c}':\boldsymbol{c}) = \begin{bmatrix} K_{1,1}(\boldsymbol{c}':\boldsymbol{c}) & K_{1,2}(\boldsymbol{c}':\boldsymbol{c}) \\ K_{2,1}(\boldsymbol{c}':\boldsymbol{c}) & K_{2,2}(\boldsymbol{c}':\boldsymbol{c}) \end{bmatrix}. \tag{23}$$

Here the elements of the scattering matrix $\boldsymbol{K}(\boldsymbol{c}':\boldsymbol{c})$ are given by Eqs. (2–5), and

$$\boldsymbol{\Sigma}(c) = \begin{bmatrix} \varpi_1(c) & 0 \\ 0 & \varpi_2(c) \end{bmatrix}. \tag{24}$$

In a recent work [4] in which the McCormack model [5] was used to describe a binary mixture of rigid spheres, six exact solutions related to the conservation of number, energy and momentum were reported (in our current notation) as

$$\boldsymbol{H}_1(\boldsymbol{r},\boldsymbol{c}) = \boldsymbol{H}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{H}_2(\boldsymbol{r},\boldsymbol{c}) = \boldsymbol{H}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{25a,b}$$

$$\boldsymbol{H}_3(\boldsymbol{r},\boldsymbol{c}) = \boldsymbol{H}_3(c) = c^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{H}_4(\boldsymbol{r},\boldsymbol{c}) = \boldsymbol{H}_4(\boldsymbol{c}) = c\mu \begin{bmatrix} 1 \\ a_{1,2} \end{bmatrix}, \tag{25c,d}$$

$$\boldsymbol{H}_5(\boldsymbol{r},\boldsymbol{c}) = \boldsymbol{H}_5(\boldsymbol{c}) = c(1-\mu^2)^{1/2}\cos\phi \begin{bmatrix} 1 \\ a_{1,2} \end{bmatrix}, \tag{25e}$$

and

$$\boldsymbol{H}_6(\boldsymbol{r},\boldsymbol{c}) = \boldsymbol{H}_6(\boldsymbol{c}) = c(1-\mu^2)^{1/2}\sin\phi \begin{bmatrix} 1 \\ a_{1,2} \end{bmatrix}. \tag{25f}$$

Here we use the spherical coordinates $\{c,\theta,\phi\}$, with $\mu = \cos\theta$, to define the dimensionless vector $\boldsymbol{c}$ so that $c_z = c\mu$, $c_x = c(1-\mu^2)^{1/2}\cos\phi$ and $c_y = c(1-\mu^2)^{1/2}\sin\phi$. In the following section, we proceed to show that these equations define solutions also for the linearized Boltzmann equations considered in this work.

3. Proofs

While it may not be difficult to anticipate that the expressions listed as Eqs. (25) define exact solutions of Eq. (21), we have found that considerable work is required in order to prove (by direct substitution) that these solutions are correct.

3.1 Conservation of number and a flow condition

We wish to show that Eqs. (25a) and (25b) define valid solutions (that can be related to the conservation of number) of Eq. (21), and so we first define

$$\mathcal{P}^{(0)}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c})\mathrm{d}^3c', \tag{26}$$

$$\mathcal{F}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{F}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c})\mathrm{d}^3c', \tag{27}$$

and

$$\mathcal{G}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{G}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c})\mathrm{d}^3c', \tag{28}$$

where Eqs. (6–11) and (17–19) are to be used. And so to verify that $\boldsymbol{H}_1(\boldsymbol{r},\boldsymbol{c})$ and $\boldsymbol{H}_2(\boldsymbol{r},\boldsymbol{c})$ are valid solutions, we must show that

$$\varpi_1(c) = \pi^{1/2}[4n_1\sigma_{1,1}\mathcal{P}^{(0)}(\boldsymbol{c}) + n_2\sigma_{1,2}\mathcal{F}^{(0)}_{1,2}(\boldsymbol{c})] \tag{29a}$$

and

$$\varpi_2(c) = \pi^{1/2}[4n_2\sigma_{2,2}\mathcal{P}^{(0)}(\boldsymbol{c}) + n_1\sigma_{2,1}\mathcal{F}^{(0)}_{2,1}(\boldsymbol{c})]. \tag{29b}$$

If we use $\boldsymbol{c}$ as a reference direction and change the integration variable from $\boldsymbol{c}'$ to $\boldsymbol{w} = \boldsymbol{c}' - \boldsymbol{c}$, with $\mathrm{d}^3c' = \mathrm{d}^3w$, then we can easily find from Eqs. (26) and (27) that

$$\mathcal{P}^{(0)}(\boldsymbol{c}) = \nu(c) \tag{30}$$

and

$$\mathcal{F}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) = 4a_{\beta,\alpha}\nu(a_{\alpha,\beta}c), \tag{31}$$

where $\nu(c)$ is given by Eq. (14). We can now use Eqs. (30) and (31) to show that Eqs. (29) are valid and thus that $\boldsymbol{H}_1(\boldsymbol{r}, \boldsymbol{c})$ and $\boldsymbol{H}_2(\boldsymbol{r}, \boldsymbol{c})$ are solutions of Eq. (21).

Now, considering Eqs. (8) and (11), we write

$$\mathcal{G}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c}) = \mathcal{X}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c}) - \frac{1}{\pi}\big|a_{\beta,\alpha}\boldsymbol{c}' - \boldsymbol{c}\big|, \tag{32}$$

where

$$\mathcal{X}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{\pi}\big|a_{\beta,\alpha}\boldsymbol{c}' - \boldsymbol{c}\big|J(a_{\alpha,\beta}; \boldsymbol{c}' : \boldsymbol{c}). \tag{33}$$

To evaluate Eq. (28) we again use $\boldsymbol{c}$ as a reference direction, but this time we change the integration variable from $\boldsymbol{c}'$ to $\boldsymbol{w} = a_{\beta,\alpha}\boldsymbol{c}' - \boldsymbol{c}$, with $\mathrm{d}^3c' = a^3_{\alpha,\beta}\mathrm{d}^3w$, so that we can write, after we have integrated the second term in Eq. (32),

$$\mathcal{G}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) = \mathcal{X}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) - a_{\beta,\alpha}\nu(a_{\alpha,\beta}c), \tag{34}$$

where

$$\mathcal{X}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{X}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c})\mathrm{d}^3c'. \tag{35}$$

If we define $\boldsymbol{w}$, relative to the vector $\boldsymbol{c}$, in terms of the spherical coordinates $\{w, \theta_w, \phi_w\}$, with $\mu_w = \cos\theta_w$, then we find we can write, after an integration over the azimuthal angle ϕ_w,

$$\mathcal{X}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) = (1/2)a_{\alpha,\beta}(1 + a^2_{\alpha,\beta})^2 \int_0^\infty \int_{-1}^1 E[B(c, w, \mu_w), w]\frac{w^2}{B(c, w, \mu_w)}\mathrm{d}\mu_w\mathrm{d}w, \tag{36}$$

where

$$E(x, w) = \exp\{-A(x - w)^2\} - \exp\{-A(x + w)^2\}, \tag{37}$$

with

$$A = \frac{a^2_{\alpha,\beta}}{(1 - a^2_{\alpha,\beta})^2}, \tag{38}$$

and

$$B(c, w, \mu_w) = \{a^4_{\alpha,\beta}w^2 + 2a^2_{\alpha,\beta}(a^2_{\alpha,\beta} - 1)cw\mu_w + (1 - a^2_{\alpha,\beta})^2c^2\}^{1/2}. \tag{39}$$

In order to avoid especially heavy notation we suppress, when convenient, some $\{\alpha, \beta\}$ dependence of our intermediate expressions. After changing the variable of integration μ_w to $B(c, w, \mu_w)$ in Eq. (36) and then putting $B(c, w, \mu_w) = x$ in the resulting equation , we can rewrite Eq. (36) as

$$\mathcal{X}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) = \frac{(1 + a^2_{\alpha,\beta})^2}{2a_{\alpha,\beta}c(a^2_{\alpha,\beta} - 1)} \int_0^\infty F(c, w) w \mathrm{d}w, \tag{40}$$

where

$$F(c, w) = \int_{a^2_{\alpha,\beta}w-(a^2_{\alpha,\beta}-1)c}^{a^2_{\alpha,\beta}w+(a^2_{\alpha,\beta}-1)c} E(x, w)\mathrm{d}x. \tag{41}$$

At this point we can use "integration by parts" [with $\mathrm{d}v = w\mathrm{d}w$ and $u = F(c, w)$] to deduce from Eq. (40) that

$$\mathcal{X}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) = a_{\beta,\alpha}\nu(a_{\alpha,\beta}c) \tag{42}$$

so that, finally,

$$\mathcal{G}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) = 0. \tag{43}$$

We can now use Eqs. (30), (31) and (43) to prove a desired flow condition. And so, we multiply Eq. (1) by $\exp\{-c^2\}\mathrm{d}^3c$ and integrate over all $\boldsymbol{c}$ to find, after we interchange $\boldsymbol{c}$ and $\boldsymbol{c}'$ in the resulting term with the repeated integrals,

$$\int \mathrm{e}^{-c^2}\boldsymbol{c} \cdot \nabla_r h_\alpha(\boldsymbol{r}, \boldsymbol{c})\mathrm{d}^3c = 0, \quad \alpha = 1, 2. \tag{44}$$

In obtaining Eq. (44), we have used, in addition to Eqs. (30), (31) and (43), the facts that

$$K_{\alpha,\alpha}(\boldsymbol{c}' : \boldsymbol{c}) = K_{\alpha,\alpha}(\boldsymbol{c} : \boldsymbol{c}') \tag{45}$$

and

$$\mathcal{G}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c}) = a_{\beta,\alpha}\mathcal{G}_{\beta,\alpha}(\boldsymbol{c} : \boldsymbol{c}'). \tag{46}$$

3.2 Conservation of energy

To show, by direct substitution, that $\boldsymbol{H}_3(\boldsymbol{r}, \boldsymbol{c})$ is a solution of Eq. (21) some additional integrals must be evaluated. First of all, we define

$$\mathcal{P}^{(2)}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c})c'^2\mathrm{d}^3c' \tag{47}$$

and

$$\mathcal{F}^{(2)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{F}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c})c'^2\mathrm{d}^3c', \tag{48}$$

which we can integrate to find

$$\mathcal{P}^{(2)}(\boldsymbol{c}) = c^2\nu(c) \tag{49}$$

and

$$\mathcal{F}^{(2)}_{\alpha,\beta}(\boldsymbol{c}) = 2[\pi^{1/2}\mathrm{erf}(a_{\alpha,\beta}c)p_1(c) + \mathrm{e}^{-(a_{\alpha,\beta}c)^2}p_2(c)]/[ca^2_{\alpha,\beta}(a^2_{\alpha,\beta} + 1)^2], \tag{50}$$

where

$$p_1(c) = 2a_{\alpha,\beta}^2(a_{\alpha,\beta}^4 + 1)c^4 + (9a_{\alpha,\beta}^4 - 2a_{\alpha,\beta}^2 + 1)c^2 + 4a_{\alpha,\beta}^2 + 1 \tag{51}$$

and

$$p_2(c) = 2a_{\alpha,\beta}(a_{\alpha,\beta}^4 + 1)c^3 + 2a_{\alpha,\beta}(4a_{\alpha,\beta}^2 - 1)c. \tag{52}$$

We can now introduce Eq. (32) into

$$\mathcal{G}_{\alpha,\beta}^{(2)}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{G}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c})c'^2 \mathrm{d}^3c' \tag{53}$$

to obtain

$$\mathcal{G}_{\alpha,\beta}^{(2)}(\boldsymbol{c}) = \mathcal{X}_{\alpha,\beta}^{(2)}(\boldsymbol{c}) - \frac{a_{\beta,\alpha}^2}{4c}\big[(5 + 6a_{\alpha,\beta}^2c^2)\pi^{1/2}\mathrm{erf}(a_{\alpha,\beta}c) + 6a_{\alpha,\beta}c\,\mathrm{e}^{-(a_{\alpha,\beta}c)^2}\big] \tag{54}$$

where

$$\mathcal{X}_{\alpha,\beta}^{(2)}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{X}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c})c'^2 \mathrm{d}^3c'. \tag{55}$$

We find we can rewrite Eq. (55) as

$$\mathcal{X}_{\alpha,\beta}^{(2)}(\boldsymbol{c}) = \frac{(1 + a_{\alpha,\beta}^2)^2}{2a_{\alpha,\beta}c(a_{\alpha,\beta}^2 - 1)^2}\int_0^\infty T(c, w)w\mathrm{d}w, \tag{56}$$

where

$$T(c, w) = \int_{a_{\alpha,\beta}^2 w - (a_{\alpha,\beta}^2 - 1)c}^{a_{\alpha,\beta}^2 w + (a_{\alpha,\beta}^2 - 1)c} E(x, w)[x^2 - a_{\alpha,\beta}^2 w^2 + (a_{\alpha,\beta}^2 - 1)c^2]\mathrm{d}x. \tag{57}$$

We have evaluated the repeated integral defined by Eq. (56) and used the result with Eq. (54) to find

$$\mathcal{G}_{\alpha,\beta}^{(2)}(\boldsymbol{c}) = [\pi^{1/2}\mathrm{erf}(a_{\alpha,\beta}c)p_3(c) + \mathrm{e}^{-(a_{\alpha,\beta}c)^2}p_4(c)]/[2ca_{\alpha,\beta}^2(a_{\alpha,\beta}^2 + 1)^2], \tag{58}$$

where

$$p_3(c) = 4a_{\alpha,\beta}^4c^4 + 4a_{\alpha,\beta}^2(1 - 2a_{\alpha,\beta}^2)c^2 - 4a_{\alpha,\beta}^2 - 1 \tag{59}$$

and

$$p_4(c) = 4a_{\alpha,\beta}^3c^3 + 2a_{\alpha,\beta}c(1 - 4a_{\alpha,\beta}^2). \tag{60}$$

Combining Eqs. (50) and (58), we find

$$\mathcal{F}_{\alpha,\beta}^{(2)}(\boldsymbol{c}) + 4\mathcal{G}_{\alpha,\beta}^{(2)}(\boldsymbol{c}) = 4c^2a_{\beta,\alpha}\nu(a_{\alpha,\beta}c), \tag{61}$$

which we can use, along with Eq. (49), to confirm that the conditions

$$\varpi_1(c)c^2 = \pi^{1/2}\big\{4n_1\sigma_{1,1}\mathcal{P}^{(2)}(\boldsymbol{c}) + n_2\sigma_{1,2}\big[\mathcal{F}_{1,2}^{(2)}(\boldsymbol{c}) + 4\mathcal{G}_{1,2}^{(2)}(\boldsymbol{c})\big]\big\} \tag{62a}$$

and

$$\varpi_2(c)c^2 = \pi^{1/2}\big\{4n_2\sigma_{2,2}\mathcal{P}^{(2)}(\boldsymbol{c}) + n_1\sigma_{2,1}\big[\mathcal{F}_{2,1}^{(2)}(\boldsymbol{c}) + 4\mathcal{G}_{2,1}^{(2)}(\boldsymbol{c})\big]\big\} \tag{62b}$$

are satisfied, and so we conclude that $\boldsymbol{H}_3(\boldsymbol{r}, \boldsymbol{c})$ is a solution of Eq. (21).

3.3. Conservation of momentum

In order to confirm that $\boldsymbol{H}_4(\boldsymbol{r},\boldsymbol{c})$ is a solution of Eq. (21), we consider the integrals

$$\mathcal{P}^{(1)}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{P}(\boldsymbol{c}':\boldsymbol{c})c'\mu'\mathrm{d}^3c', \tag{63}$$

$$\mathcal{F}^{(1)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{F}_{\alpha,\beta}(\boldsymbol{c}':\boldsymbol{c})c'\mu'\mathrm{d}^3c', \tag{64}$$

and

$$\mathcal{G}^{(1)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{G}_{\alpha,\beta}(\boldsymbol{c}':\boldsymbol{c})c'\mu'\mathrm{d}^3c'. \tag{65}$$

Again, we use $\boldsymbol{c}$ as a reference direction and change the integration variable from $\boldsymbol{c}'$ to $\boldsymbol{w} = \boldsymbol{c}' - \boldsymbol{c}$, with $\mathrm{d}^3c' = \mathrm{d}^3w$. We also let $\boldsymbol{k}$ denote a unit vector in the positive z direction, so that with $\boldsymbol{c}$ and $\boldsymbol{c}'$ referred to $\boldsymbol{k}$, we can write

$$c'\mu' = \boldsymbol{k}\cdot\boldsymbol{c}', \quad c\mu = \boldsymbol{k}\cdot\boldsymbol{c}, \quad \text{and} \quad c'\mu' = c\mu + \boldsymbol{k}\cdot\boldsymbol{w}. \tag{66a,b,c}$$

In this way, we can rewrite Eqs. (63), (64), and (65) as

$$\mathcal{P}^{(1)}(\boldsymbol{c}) = c\mu\mathcal{P}^{(0)}(\boldsymbol{c}) + \int \mathrm{e}^{-(\boldsymbol{w}+\boldsymbol{c})^2}\mathcal{P}(\boldsymbol{w}+\boldsymbol{c}:\boldsymbol{c})(\boldsymbol{k}\cdot\boldsymbol{w})\mathrm{d}^3w, \tag{67}$$

$$\mathcal{F}^{(1)}_{\alpha,\beta}(\boldsymbol{c}) = c\mu\mathcal{F}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) + \int \mathrm{e}^{-(\boldsymbol{w}+\boldsymbol{c})^2}\mathcal{F}_{\alpha,\beta}(\boldsymbol{w}+\boldsymbol{c}:\boldsymbol{c})(\boldsymbol{k}\cdot\boldsymbol{w})\mathrm{d}^3w, \tag{68}$$

and

$$\mathcal{G}^{(1)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-(\boldsymbol{w}+\boldsymbol{c})^2}\mathcal{G}_{\alpha,\beta}(\boldsymbol{w}+\boldsymbol{c}:\boldsymbol{c})(\boldsymbol{k}\cdot\boldsymbol{w})\mathrm{d}^3w, \tag{69}$$

where we have used Eq. (43). Now, if we use θ_k, with $\mu_k = \cos\theta_k$, and ϕ_k to locate $\boldsymbol{k}$ with respect to the vector $\boldsymbol{c}$ and θ_w, with $\mu_w = \cos\theta_w$, and ϕ_w to locate $\boldsymbol{w}$ with respect to the vector $\boldsymbol{c}$, so that $d^3w = w^2\mathrm{d}\mu_w\mathrm{d}\phi_w\mathrm{d}w$, we can write

$$\boldsymbol{k}\cdot\boldsymbol{w} = w\mu_*, \tag{70}$$

where

$$\mu_* = \mu_w\mu_k + (1-\mu_w^2)^{1/2}(1-\mu_k^2)^{1/2}\cos(\phi_w - \phi_k). \tag{71}$$

Using Eqs. (70) and (71) in Eqs. (67), (68), and (69), we can carry out the integrations to find

$$\mathcal{P}^{(1)}(\boldsymbol{c}) = c\mu\mathcal{P}^{(0)}(\boldsymbol{c}), \tag{72}$$

$$\begin{aligned}\mathcal{F}^{(1)}_{\alpha,\beta}(\boldsymbol{c}) &= c\mu\mathcal{F}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) \\ &\quad + \Delta_1(\mu_k)[\pi^{1/2}\mathrm{erf}(a_{\alpha,\beta}c)p_5(c) + \mathrm{e}^{-(a_{\alpha,\beta}c)^2}p_6(c)]/[c^2a^2_{\alpha,\beta}(a^2_{\alpha,\beta}+1)],\end{aligned} \tag{73}$$

and

$$\mathcal{G}^{(1)}_{\alpha,\beta}(\boldsymbol{c}) = -\Delta_1(\mu_k)[\pi^{1/2}\mathrm{erf}(a_{\alpha,\beta}c)p_5(c) + \mathrm{e}^{-(a_{\alpha,\beta}c)^2}p_6(c)]/[4c^2a^3_{\alpha,\beta}(a^2_{\alpha,\beta}+1)], \tag{74}$$

where

$$\frac{1}{2\pi}\int_0^{2\pi}(\boldsymbol{k}\cdot\boldsymbol{w})\mathrm{d}\phi_w=\Delta_1(\mu_k)w\mu_w, \tag{75}$$

with

$$\Delta_1(\mu_k)=\mu_k. \tag{76}$$

In addition,

$$p_5(c)=1-4a_{\alpha,\beta}^2c^2(1+a_{\alpha,\beta}^2c^2), \tag{77}$$

and

$$p_6(c)=-2a_{\alpha,\beta}c(1+2a_{\alpha,\beta}^2c^2). \tag{78}$$

Combining Eqs. (30), (31), (72), (73) and (74), we can now verify that the conditions

$$\varpi_1(c)c\mu=\pi^{1/2}\big\{4n_1\sigma_{1,1}\mathcal{P}^{(1)}(\boldsymbol{c})+n_2\sigma_{1,2}\big[\mathcal{F}_{1,2}^{(1)}(\boldsymbol{c})+4a_{1,2}\mathcal{G}_{1,2}^{(1)}(\boldsymbol{c})\big]\big\} \tag{79a}$$

and

$$\varpi_2(c)c\mu=\pi^{1/2}\big\{4n_2\sigma_{2,2}\mathcal{P}^{(1)}(\boldsymbol{c})+n_1\sigma_{2,1}\big[\mathcal{F}_{2,1}^{(1)}(\boldsymbol{c})+4a_{2,1}\mathcal{G}_{2,1}^{(1)}(\boldsymbol{c})\big]\big\} \tag{79b}$$

are satisfied, and so we conclude that $\boldsymbol{H}_4(\boldsymbol{r},\boldsymbol{c})$ is a solution of Eq. (21).

To show that $\boldsymbol{H}_5(\boldsymbol{r},\boldsymbol{c})$ and $\boldsymbol{H}_6(\boldsymbol{r},\boldsymbol{c})$ are solutions to Eq. (21), we make use of two Cartesian reference frames: the first is defined by unit vectors $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$, while the second is defined by unit vectors $\{\boldsymbol{l},\boldsymbol{m},\boldsymbol{n}\}$. These reference frames are chosen so that

$$\boldsymbol{c}=c[\sin\theta\cos\phi\,\boldsymbol{i}+\sin\theta\sin\phi\,\boldsymbol{j}+\cos\theta\,\boldsymbol{k}] \tag{80}$$

and

$$\boldsymbol{c}=c\,\boldsymbol{n}. \tag{81}$$

In addition,

$$\boldsymbol{c}'=c'[\sin\theta'\cos\phi'\,\boldsymbol{i}+\sin\theta'\sin\phi'\,\boldsymbol{j}+\cos\theta'\,\boldsymbol{k}]. \tag{82}$$

Recalling that $\boldsymbol{w}=\boldsymbol{c}'-\boldsymbol{c}$, we can write

$$\boldsymbol{k}\times\boldsymbol{w}\cdot\boldsymbol{i}=c'(1-\mu'^2)^{1/2}\cos\phi'-c(1-\mu^2)^{1/2}\cos\phi \tag{83a}$$

and

$$\boldsymbol{k}\times\boldsymbol{w}\cdot\boldsymbol{j}=c'(1-\mu'^2)^{1/2}\sin\phi'-c(1-\mu^2)^{1/2}\sin\phi. \tag{83b}$$

Continuing to use θ_k and ϕ_k to locate $\boldsymbol{k}$ with respect to the vector $\boldsymbol{c}$ and θ_w and ϕ_w to locate $\boldsymbol{w}$ with respect to the vector $\boldsymbol{c}$, we can write

$$\boldsymbol{k}\times\boldsymbol{w}=\alpha(\theta_k,\phi_k,\theta_w,\phi_w)\,\boldsymbol{l}+\beta(\theta_k,\phi_k,\theta_w,\phi_w)\,\boldsymbol{m}+\gamma(\theta_k,\phi_k,\theta_w,\phi_w)\,\boldsymbol{n}, \tag{84}$$

where

$$\alpha(\theta_k,\phi_k,\theta_w,\phi_w)=w[\cos\theta_w\sin\theta_k\sin\phi_k-\cos\theta_k\sin\theta_w\sin\phi_w], \tag{85a}$$

$$\beta(\theta_k,\phi_k,\theta_w,\phi_w)=w[\cos\theta_k\sin\theta_w\cos\phi_w-\cos\theta_w\sin\theta_k\cos\phi_k], \tag{85b}$$

and

$$\gamma(\theta_k,\phi_k,\theta_w,\phi_w) = w[\sin\theta_k \sin\theta_w \sin(\phi_w - \phi_k)]. \tag{85c}$$

Proceeding, we introduce

$$\mathcal{P}^{(c)}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c})c'(1-\mu'^2)^{1/2}\cos\phi'\mathrm{d}^3c', \tag{86a}$$

$$\mathcal{F}^{(c)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{F}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c})c'(1-\mu'^2)^{1/2}\cos\phi'\mathrm{d}^3c', \tag{86b}$$

$$\mathcal{G}^{(c)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{G}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c})c'(1-\mu'^2)^{1/2}\cos\phi'\mathrm{d}^3c', \tag{86c}$$

$$\mathcal{P}^{(s)}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c})c'(1-\mu'^2)^{1/2}\sin\phi'\mathrm{d}^3c', \tag{87a}$$

$$\mathcal{F}^{(s)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{F}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c})c'(1-\mu'^2)^{1/2}\sin\phi'\mathrm{d}^3c', \tag{87b}$$

and

$$\mathcal{G}^{(s)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-c'^2}\mathcal{G}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c})c'(1-\mu'^2)^{1/2}\sin\phi'\mathrm{d}^3c', \tag{87c}$$

which, after we note Eqs. (83), we can rewrite as

$$\mathcal{P}^{(c)}(\boldsymbol{c}) = c(1-\mu^2)^{1/2}\cos\phi\mathcal{P}^{(0)}(\boldsymbol{c}) + \int \mathrm{e}^{-(\boldsymbol{w}+\boldsymbol{c})^2}\mathcal{P}(\boldsymbol{w}+\boldsymbol{c} : \boldsymbol{c})(\boldsymbol{k}\times\boldsymbol{w}\cdot\boldsymbol{i})\mathrm{d}^3w, \tag{88a}$$

$$\mathcal{F}^{(c)}_{\alpha,\beta}(\boldsymbol{c}) = c(1-\mu^2)^{1/2}\cos\phi\mathcal{F}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) + \int \mathrm{e}^{-(\boldsymbol{w}+\boldsymbol{c})^2}\mathcal{F}_{\alpha,\beta}(\boldsymbol{w}+\boldsymbol{c} : \boldsymbol{c})(\boldsymbol{k}\times\boldsymbol{w}\cdot\boldsymbol{i})\mathrm{d}^3w, \tag{88b}$$

$$\mathcal{G}^{(c)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-(\boldsymbol{w}+\boldsymbol{c})^2}\mathcal{G}_{\alpha,\beta}(\boldsymbol{w}+\boldsymbol{c} : \boldsymbol{c})(\boldsymbol{k}\times\boldsymbol{w}\cdot\boldsymbol{i})\mathrm{d}^3w, \tag{88c}$$

$$\mathcal{P}^{(s)}(\boldsymbol{c}) = c(1-\mu^2)^{1/2}\sin\phi\mathcal{P}^{(0)}(\boldsymbol{c}) + \int \mathrm{e}^{-(\boldsymbol{w}+\boldsymbol{c})^2}\mathcal{P}(\boldsymbol{w}+\boldsymbol{c} : \boldsymbol{c})(\boldsymbol{k}\times\boldsymbol{w}\cdot\boldsymbol{j})\mathrm{d}^3w, \tag{89a}$$

$$\mathcal{F}^{(s)}_{\alpha,\beta}(\boldsymbol{c}) = c(1-\mu^2)^{1/2}\sin\phi\mathcal{F}^{(0)}_{\alpha,\beta}(\boldsymbol{c}) + \int \mathrm{e}^{-(\boldsymbol{w}+\boldsymbol{c})^2}\mathcal{F}_{\alpha,\beta}(\boldsymbol{w}+\boldsymbol{c} : \boldsymbol{c})(\boldsymbol{k}\times\boldsymbol{w}\cdot\boldsymbol{j})\mathrm{d}^3w, \tag{89b}$$

and

$$\mathcal{G}^{(s)}_{\alpha,\beta}(\boldsymbol{c}) = \int \mathrm{e}^{-(\boldsymbol{w}+\boldsymbol{c})^2}\mathcal{G}_{\alpha,\beta}(\boldsymbol{w}+\boldsymbol{c} : \boldsymbol{c})(\boldsymbol{k}\times\boldsymbol{w}\cdot\boldsymbol{j})\mathrm{d}^3w, \tag{89c}$$

where again we have used Eq. (43). At this point, we can use Eq. (84) to express $\boldsymbol{k}\times\boldsymbol{w}\cdot\boldsymbol{i}$ and $\boldsymbol{k}\times\boldsymbol{w}\cdot\boldsymbol{j}$ in terms of θ_k, ϕ_k, θ_w, and ϕ_w and integrate the resulting forms to find

$$\frac{1}{2\pi}\int_0^{2\pi}(\boldsymbol{k}\times\boldsymbol{w}\cdot\boldsymbol{i})\mathrm{d}\phi_w = \Delta_c(\theta_k,\phi_k)w\mu_w \tag{90a}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}(\boldsymbol{k}\times\boldsymbol{w}\cdot\boldsymbol{j})\mathrm{d}\phi_w = \Delta_s(\theta_k,\phi_k)w\mu_w, \tag{90b}$$

where

$$\Delta_c(\theta_k,\phi_k) = \sin\theta_k[(\boldsymbol{i}\cdot\boldsymbol{l})\sin\phi_k - (\boldsymbol{i}\cdot\boldsymbol{m})\cos\phi_k] \tag{91a}$$

and

$$\Delta_s(\theta_k,\phi_k) = \sin\theta_k[(\boldsymbol{j}\cdot\boldsymbol{l})\sin\phi_k - (\boldsymbol{j}\cdot\boldsymbol{m})\cos\phi_k]. \tag{91b}$$

Noting that $\Delta_c(\theta_k,\phi_k)w\mu_w$ and $\Delta_s(\theta_k,\phi_k)w\mu_w$ enter the current calculations in exactly the same way that $\Delta_1(\mu_k)w\mu_w$ entered the proof that $\boldsymbol{H}_4(\boldsymbol{r},\boldsymbol{c})$ is a solution, we conclude that the necessary cancellation takes place when combining the three Eqs. (88) [and the three Eqs. (89)] so that we obtain the desired result: the conditions

$$\varpi_1(c)c(1-\mu^2)^{1/2}\cos\phi = \pi^{1/2}\big\{4n_1\sigma_{1,1}\mathcal{P}^{(c)}(\boldsymbol{c}) + n_2\sigma_{1,2}\big[\mathcal{F}_{1,2}^{(c)}(\boldsymbol{c}) + 4a_{1,2}\mathcal{G}_{1,2}^{(c)}(\boldsymbol{c})\big]\big\}, \tag{92a}$$

$$\varpi_2(c)c(1-\mu^2)^{1/2}\cos\phi = \pi^{1/2}\big\{4n_2\sigma_{2,2}\mathcal{P}^{(c)}(\boldsymbol{c}) + n_1\sigma_{2,1}\big[\mathcal{F}_{2,1}^{(c)}(\boldsymbol{c}) + 4a_{2,1}\mathcal{G}_{2,1}^{(c)}(\boldsymbol{c})\big]\big\}, \tag{92b}$$

$$\varpi_1(c)c(1-\mu^2)^{1/2}\sin\phi = \pi^{1/2}\big\{4n_1\sigma_{1,1}\mathcal{P}^{(s)}(\boldsymbol{c}) + n_2\sigma_{1,2}\big[\mathcal{F}_{1,2}^{(s)}(\boldsymbol{c}) + 4a_{1,2}\mathcal{G}_{1,2}^{(s)}(\boldsymbol{c})\big]\big\}, \tag{93a}$$

and

$$\varpi_2(c)c(1-\mu^2)^{1/2}\sin\phi = \pi^{1/2}\big\{4n_2\sigma_{2,2}\mathcal{P}^{(s)}(\boldsymbol{c}) + n_1\sigma_{2,1}\big[\mathcal{F}_{2,1}^{(s)}(\boldsymbol{c}) + 4a_{2,1}\mathcal{G}_{2,1}^{(s)}(\boldsymbol{c})\big]\big\}, \tag{93b}$$

are satisfied, and $\boldsymbol{H}_5(\boldsymbol{r},\boldsymbol{c})$ and $\boldsymbol{H}_6(\boldsymbol{r},\boldsymbol{c})$ are solutions of Eq. (21).

4. Concluding remarks

In this work we have reported some solutions to an exact and explicit formulation of the (vector) linearized Boltzmann equation relevant to a binary mixture of rigid spheres that scatter isotropically in the center-of-mass system. The solutions (collisional invariants) listed as Eqs. (25) are consequences of the conservation of number, energy, and momentum. While these solutions are easily anticipated, we found that to prove (by direct substitution into the Boltzmann equation) the correctness of these solutions and to establish a standard flow condition were nontrivial tasks. Since we have worked with a recently reported explicit form of the linearized Boltzmann equation (for a binary mixture of rigid-spheres), we have demonstrated the usefulness (and provided additional evidence about the correctness) of the form of the Boltzmann equation established by Garcia, Siewert and Williams [1].

As in previous works on the linearized Boltzmann equation for a single-species gas [6] and on the McCormack model for binary mixtures [4], we believe that, in addition to the class of solutions listed in Eqs. (25), we can expect also to find (asymptotic) solutions that are linear in the spatial variables. While, at this point, we are not able to define these solutions explicitly for the considered linearized Boltzmann equation for a binary mixture of rigid spheres, we do intend to pursue such work.

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