6. Two-Group Neutron Transport Theory in Spherical Geometry, *T. W. Schnatz, C. E. Siewert (NC St U)*

The singular eigenfunction expansion technique introduced by Case has been used extensively in the areas of neutron transport theory and radiative transfer to construct rigorous solutions to a certain class of model problems.[1-5] This method has enjoyed particular success for energy-dependent problems,[5] for time-dependent theory,[6] for anisotropic scattering models,[3] for reactor cell calculations,[7] and for several astrophysical applications.[4,8] Although Case's normal-mode expansion technique has been found suitable for a large number of applications, one of the major restrictions of the method is the difficulty with which the extension to non-planar geometries is made.

The purpose of the present paper is to blend the methods of Erdmann and Siewert[9] for spherical problems with the two-group analysis of Siewert and Shieh[10] to solve the isotropically emitting spherical-shell source problem for the two-group model in an infinite medium.

We thus seek a solution to the time-independent transport equation

$$\mu \frac{\partial}{\partial r} \Psi(r_0:r,\mu) + \frac{(1-\mu^2)}{r}\frac{\partial}{\partial \mu}\Psi(r_0:r,\mu) + \Sigma\Psi(r_0:r,\mu)$$
$$= C \int_{-1}^{1} \Psi(r_0:r,\mu')\, d\mu' + \frac{\delta(r-r_0)}{8\pi r^2} Q \quad , \qquad (1)$$

subject to the constraint that $\Psi(r_0:r,\mu)$ must be bounded for all r, since the considered medium must be non-multiplying. For the sake of brevity, the notation of Refs. 9 and 10 is used here without further definition. In addition, the components of Q, q_1, and q_2, are used to indicate the intensities of the two-group sources.

It is proposed that the homogeneous version of Eq. (1) has solutions of the form

$$\Psi_\eta(r,\mu) = \sum_{m=0}^{\infty}\left(\frac{2m+1}{2}\right) P_m(\mu)[A(\eta)k_m(r/\eta)$$
$$+ B(\eta)(-1)^m i_m(r/\eta)]\, G_m(\eta) \quad , \qquad (2)$$

with $A(\eta)$, $B(\eta)$ and, at this point, η being arbitrary.

If the proposed solution is substituted into the homogeneous transport equation, we observe that the G-vectors must be solutions of the recursion relation

$$(2m+1)\,\eta\Sigma G_m(\eta) = 2\eta C\, G_0(\eta)\,\delta_{0,m} + (m+1)\, G_{m+1}(\eta)$$
$$+ m\, G_{m-1}(\eta),\ m = 0,\ 1,\ 2 \ldots . \qquad (3)$$

It can be shown by consideration of the eigenvector equation for $F(\eta,\mu)$ experienced by Siewert and Shieh[10] that

$$G_m(\eta) \triangleq \int_{-1}^{1} P_m(\mu)\, F(\eta,\mu)\, d\mu \qquad (4)$$

is a solution to Eq. (3).

Since the normal modes are thus established, we construct from them a bounded solution to the homogeneous transport equation subject to the jump boundary condition

$$\mu[\Psi(r_0:r_0^+,\mu) - \Psi(r_0:r_0^-,\mu)] = (8\pi r_0^2)^{-1} Q \quad . \qquad (5)$$

The desired solution is written as

$$\Psi(r_0;r,\mu) = \sum_{m=0}^{\infty}\left(\frac{2m+1}{2}\right) P_m(\mu)\, R_m^+(r),\ r > r_0 \qquad (6a)$$

and

$$\Psi(r_0:r,\mu) = \sum_{m=0}^{\infty}\left(\frac{2m+1}{2}\right) P_m(\mu)\, R_m^-(r),\ r < r_0 \quad , \qquad (6b)$$

where

$$R_m^+(r) \triangleq \sum_i A(\eta_i)\, G_m(\eta_i)\, k_m(r/\eta_i)$$
$$+ \int_0^{1/\sigma} [A_1(\eta)\, G_{1,m}(\eta) + A_2(\eta)\, G_{2,m}(\eta)]$$
$$\times k_m(r/\eta)\, d\eta + \int_{1/\sigma}^{1} A_3(\eta)\, G_{3,m}(\eta)\, k_m(r/\eta)\, d\eta$$
$$(7a)$$

and

$$R_m^-(\eta_i) \triangleq \sum_i B(\eta_i)\, G_m(\eta_i)(-1)^m\, i_m(r/\eta_i)$$
$$+ \int_0^{1/\sigma}[B_1(\eta)\, G_{1,m}(\eta) + B_2(\eta)\, G_{2,m}(\eta)]$$
$$\times (-1)^m\, i_m(r/\eta)\, d\eta + \int_{1/\sigma}^{1} B_3(\eta)\, G_{3,m}(\eta)(-1)^m$$
$$\times i_m(r/\eta)\, d\eta \quad . \qquad (7b)$$

Substituting Eqs. (6) and (7) into the jump condition, we find that

$$S_m(r_0) \triangleq 4\pi r_0^2[R_m^+(r_0) - R_m^-(r_0)] \qquad (8)$$

must satisfy the constraint

$$S_m(r_0) = -\frac{2\cdot 4\cdot 6\cdot 8 \ldots (m-1)}{3\cdot 5\cdot 7\cdot 9 \ldots (m)}(-1)^{(m+1)/2}$$
$$\times \left[\frac{1-(-1)^m}{2}\right][1-\delta_{1,m}]Q + \delta_{1,m} Q$$
$$(9)$$

for all m. For the case m = 0, the above condition is met by taking

$$\frac{A(\xi)}{i_0(r_0/\xi)} = \frac{B(\xi)}{k_0(r_0/\xi)} \triangleq D(\xi)\ ,\ \xi = \eta_i \text{ or } \eta\epsilon(0,1) \quad . \qquad (10)$$

In a manner similar to that used in Ref. 9, we consider the following full-range expansion in terms of eigenvectors introduced by Siewert and Shieh.[10]

$$(2\pi^2)^{-1} Q = \mu \left\{ \sum_i D(\eta_i)[F_{i+}(\mu) - F_{i-}(\mu)]\eta_i^2 \right.$$
$$+ \int_0^{1/\sigma} D_1(\eta)[F_1^{(1)}(\eta,\mu) - F_1^{(1)}(-\eta,\mu)]\,\eta^2 d\eta$$
$$+ \int_0^{1/\sigma} D_2(\eta)[F_2^{(1)}(\eta,\mu) - F_2^{(1)}(-\eta,\mu)]\,\eta^2 d\eta$$
$$\left. + \int_{1/\sigma}^{1} D_3(\eta)[F^{(2)}(\eta,\mu) - F^{(2)}(-\eta,\mu)]\,\eta^2 d\eta \right\} \quad . \qquad (11)$$

The required expansion coefficients $D(\xi)$ are thus obtained from Eq. (11) by making use of the full-range orthogonality theorem and the necessary normalization integrals given in Ref. 10; because of the limited space available here, these results are not given explicitly, but as discussed, they follow most readily.

The proof that the expansion coefficients so determined are correct is accomplished by multiplying Eq. (11) by $\mu^{k-1}P_m(\mu)$ and integrating over all μ to show that Eq. (9) is satisfied for all m.

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