Available online at www.sciencedirect.com
ScienceDirect
European Journal of Mechanics B/Fluids 26 (2007) 749–778

The viscous-slip, diffusion-slip, and thermal-creep problems for a binary mixture of rigid spheres described by the linearized Boltzmann equation

R.D.M. Garcia [a,*], C.E. Siewert [b]

[a] *HSH Scientific Computing, Rua Carlos de Campos, 286, São José dos Campos, SP 12242-540, Brazil*
[b] *Mathematics Department, North Carolina State University, Raleigh, NC 27695-8205, USA*

Received 11 July 2006; received in revised form 16 December 2006; accepted 18 December 2006

Available online 11 January 2007

Abstract

An analytical version of the discrete-ordinates method (the ADO method) is used with recently established analytical expressions for the rigid-sphere scattering kernels to develop concise and particularly accurate solutions to the viscous-slip, the diffusion-slip, and the half-space thermal-creep problems for a binary gas mixture described by the linearized Boltzmann equation. In addition to a computation of the viscous-slip, the diffusion-slip, and the thermal-slip coefficients, for the case of Maxwell boundary conditions for each of the two species, the velocity, heat-flow, and shear-stress profiles are established for each species of particles. Numerical results are reported for two binary mixtures (Ne–Ar and He–Xe) with various molar concentrations.
© 2007 Elsevier Masson SAS. All rights reserved.

Keywords: Rarefied gas dynamics; Binary mixtures; Rigid spheres; Viscous slip; Thermal creep; Linearized Boltzmann equation

1. Introduction

While the classical problems of viscous slip (Kramers' problem) and thermal creep in the general field of rarefied gas dynamics [1–4] have been extensively studied for the case of a single-species gas (see, for example, Refs. [5–11] and the references therein), there are relatively few works (for example, [12–22]) devoted to these problems and/or the diffusion-slip problem for gas mixtures. Almost all of the works on gas mixtures are based on model equations [12–17], or there are works that report estimates of the slip coefficients obtained from the application of low-order solution techniques to the linearized Boltzmann equation [18–20]. Most closely related to our work here are Refs. [21,22] that consider a full treatment of the collision term in the linearized Boltzmann equation (LBE) for a binary mixture of rigid-sphere gas particles. In Ref. [21], Yasuda, Takata, and Aoki implement a numerical procedure applied to a fully discretized form of the LBE and thus solve well Kramers' problem for a binary mixture of rigid spheres. In Ref. [22], Takata, Yasuda, Kosuge, and Aoki use the same numerical approach to solve the thermal-creep and the diffusion-slip problems, again for a binary mixture of rigid spheres described by the linearized Boltzmann equation. It can be noted

* Corresponding author.
E-mail address: rdmgarcia@uol.com.br (R.D.M. Garcia).

0997-7546/$ – see front matter © 2007 Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.euromechflu.2006.12.002

that Refs. [21,22] report numerical results for spatial-dependent quantities [velocity and heat-flow profiles] as well as the basic slip coefficients.

In this work we develop and evaluate numerically solutions for the three half-space problems of viscous-slip, thermal-creep, and diffusion-slip, as defined by the recently reported explicit forms of the rigid-sphere collision kernels for binary gas mixtures [23] and the analytical discrete-ordinates (ADO) method [24]. The developed solutions depend (aside from some normalizations) only on the mass and diameter ratios and the relative equilibrium concentration of the two species of particles. We also allow a free choice of the accommodation coefficient for each species at the confining surface of the considered half space. Our approach relies on a continuous treatment of both the space and speed variables that has proved to be particularly efficient and accurate. And so, in addition to reporting the basic slip coefficients, we list (for some specific data cases) the velocity, heat-flow, and shear-stress profiles for each of the two species. It should be noted that by reporting species-dependent results we make available mixture quantities that can have different definitions (for example: average velocity and bulk velocity) in terms of the relevant quantities for the individual components in the mixture.

2. Basic formulation

Before starting our work that is specific to the viscous-slip, diffusion-slip, and thermal-creep flow problems, we review here our analytical formulation of the linearized Boltzmann equation for a binary mixture of rigid spheres. This formulation was started in Ref. [23] and was further developed in Refs. [25–27]. Considering what has gone before this work, we write the coupled linearized Boltzmann equation for variations only in the z direction (perpendicular to the confining surface) for the considered binary mixture of rigid spheres as

$$c\mu\frac{\partial}{\partial z}\boldsymbol{H}(z,\boldsymbol{c})+\varepsilon_0\boldsymbol{V}(c)\boldsymbol{H}(z,\boldsymbol{c})=\varepsilon_0\int \mathrm{e}^{-c'^2}\boldsymbol{\mathcal{K}}(\boldsymbol{c}':\boldsymbol{c})\boldsymbol{H}(z,\boldsymbol{c}')\,\mathrm{d}^3c', \tag{2.1}$$

where

$$\boldsymbol{H}(z,\boldsymbol{c})=\begin{bmatrix}h_1(z,\boldsymbol{c})\\ h_2(z,\boldsymbol{c})\end{bmatrix}. \tag{2.2}$$

Since Eq. (2.1) is written in terms of a dimensionless velocity variable $\boldsymbol{c}$, we note that the basic velocity distribution functions are available from

$$f_\alpha(z,\boldsymbol{v})=f_{\alpha,0}(v)\big[1+h_\alpha\big(z,\lambda_\alpha^{1/2}\boldsymbol{v}\big)\big],\quad \alpha=1,2, \tag{2.3}$$

where $\lambda_\alpha=m_\alpha/(2kT_0)$ and where

$$f_{\alpha,0}(v)=n_\alpha\left(\frac{\lambda_\alpha}{\pi}\right)^{3/2}\mathrm{e}^{-\lambda_\alpha v^2} \tag{2.4}$$

is the Maxwellian distribution for n_α particles of mass m_α in equilibrium at temperature T_0. Here, k is the Boltzmann constant. It can be noted from Eq. (2.3) that, at this point, the particle distribution functions $f_\alpha(z,\boldsymbol{v})$ have been linearized about the absolute Maxwellian distributions $f_{\alpha,0}(v)$. While this linearization will be used for the viscous-slip problem, we will, in a following part of this work, use linearizations about local Maxwellian distributions for the problems of thermal creep and diffusion slip. Continuing, we note that we use spherical coordinates $\{c,\theta,\phi\}$, with $\mu=\cos\theta$, to describe the dimensionless velocity vector, so that

$$\boldsymbol{H}(z,\boldsymbol{c})\Leftrightarrow\boldsymbol{H}(z,c,\mu,\phi).$$

In addition,

$$\boldsymbol{V}(c)=\frac{1}{\varepsilon_0}\boldsymbol{\Sigma}(c) \tag{2.5}$$

and

$$\boldsymbol{\mathcal{K}}(\boldsymbol{c}':\boldsymbol{c})=\frac{1}{\varepsilon_0}\boldsymbol{K}(\boldsymbol{c}':\boldsymbol{c}), \tag{2.6}$$

where $\boldsymbol{\Sigma}(c)$ and $\boldsymbol{K}(\boldsymbol{c}' : \boldsymbol{c})$ are as defined in Ref. [26]: first of all,

$$\boldsymbol{\Sigma}(c) = \begin{bmatrix} \varpi_1(c) & 0 \\ 0 & \varpi_2(c) \end{bmatrix}, \tag{2.7}$$

with

$$\varpi_\alpha(c) = \varpi_\alpha^{(1)}(c) + \varpi_\alpha^{(2)}(c) \tag{2.8}$$

and

$$\varpi_\alpha^{(\beta)}(c) = 4\pi^{1/2} n_\beta \sigma_{\alpha,\beta} a_{\beta,\alpha} \nu(a_{\alpha,\beta} c). \tag{2.9}$$

Here

$$\nu(c) = \frac{2c^2+1}{c} \int_0^c \mathrm{e}^{-x^2}\,\mathrm{d}x + \mathrm{e}^{-c^2}, \tag{2.10}$$

$$a_{\alpha,\beta} = \left(\frac{m_\beta}{m_\alpha}\right)^{1/2}, \quad \alpha, \beta = 1, 2, \tag{2.11}$$

and, to be clear, we note that we use $\sigma_{\alpha,\beta}$ to denote the differential-scattering cross section, which for the case of rigid-sphere scattering that is isotropic in the center-of-mass system, we write as [4]

$$\sigma_{\alpha,\beta} = \frac{1}{4}\left(\frac{d_\alpha + d_\beta}{2}\right)^2, \tag{2.12}$$

where d_1 and d_2 are the atomic diameters of the two types of gas particles. We continue to follow Ref. [26] and write

$$\boldsymbol{K}(\boldsymbol{c}' : \boldsymbol{c}) = \begin{bmatrix} K_{1,1}(\boldsymbol{c}' : \boldsymbol{c}) & K_{1,2}(\boldsymbol{c}' : \boldsymbol{c}) \\ K_{2,1}(\boldsymbol{c}' : \boldsymbol{c}) & K_{2,2}(\boldsymbol{c}' : \boldsymbol{c}) \end{bmatrix}, \tag{2.13}$$

where

$$K_{1,1}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_1\sigma_{1,1}\pi^{1/2}\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) + n_2\sigma_{1,2}\pi^{1/2}\mathcal{F}_{1,2}(\boldsymbol{c}' : \boldsymbol{c}), \tag{2.14}$$

$$K_{1,2}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_2\sigma_{1,2}\pi^{1/2}\mathcal{G}_{1,2}(\boldsymbol{c}' : \boldsymbol{c}), \tag{2.15}$$

$$K_{2,1}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_1\sigma_{2,1}\pi^{1/2}\mathcal{G}_{2,1}(\boldsymbol{c}' : \boldsymbol{c}), \tag{2.16}$$

and

$$K_{2,2}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_2\sigma_{2,2}\pi^{1/2}\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) + n_1\sigma_{2,1}\pi^{1/2}\mathcal{F}_{2,1}(\boldsymbol{c}' : \boldsymbol{c}). \tag{2.17}$$

Here

$$\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{\pi}\left(\frac{2}{|\boldsymbol{c}' - \boldsymbol{c}|}\exp\left\{\frac{|\boldsymbol{c}' \times \boldsymbol{c}|^2}{|\boldsymbol{c}' - \boldsymbol{c}|^2}\right\} - |\boldsymbol{c}' - \boldsymbol{c}|\right) \tag{2.18}$$

is the basic kernel for a single-species gas used by Pekeris [28]. In addition,

$$\mathcal{F}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c}) = \mathcal{F}(a_{\alpha,\beta}; \boldsymbol{c}' : \boldsymbol{c}) \tag{2.19}$$

and

$$\mathcal{G}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c}) = \mathcal{G}(a_{\alpha,\beta}; \boldsymbol{c}' : \boldsymbol{c}), \tag{2.20}$$

where [26]

$$\mathcal{F}(a; \boldsymbol{c}' : \boldsymbol{c}) = \frac{(a^2+1)^2}{a^3\pi|\boldsymbol{c}' - \boldsymbol{c}|}\exp\left\{a^2\frac{|\boldsymbol{c}' \times \boldsymbol{c}|^2}{|\boldsymbol{c}' - \boldsymbol{c}|^2} - \frac{(1-a^2)^2(c'^2 + c^2)}{4a^2} - \frac{(a^4-1)\boldsymbol{c}' \cdot \boldsymbol{c}}{2a^2}\right\}, \tag{2.21}$$

and

$$\mathcal{G}(a; \boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{a\pi}|\boldsymbol{c}' - a\boldsymbol{c}|\big[J(a; \boldsymbol{c}' : \boldsymbol{c}) - 1\big], \tag{2.22}$$

with

$$J(a;\boldsymbol{c}':\boldsymbol{c}) = \frac{(a+1/a)^2}{2\Delta(a;\boldsymbol{c}':\boldsymbol{c})}\exp\left\{\frac{-2C(a;\boldsymbol{c}':\boldsymbol{c})}{(a-1/a)^2}\right\}\sinh\left\{\frac{2\Delta(a;\boldsymbol{c}':\boldsymbol{c})}{(a-1/a)^2}\right\}, \quad a\neq 1, \tag{2.23a}$$

or

$$J(a;\boldsymbol{c}':\boldsymbol{c}) = \frac{1}{|\boldsymbol{c}'-\boldsymbol{c}|^2}\exp\left\{\frac{|\boldsymbol{c}'\times\boldsymbol{c}|^2}{|\boldsymbol{c}'-\boldsymbol{c}|^2}\right\}, \quad a=1. \tag{2.23b}$$

To write Eq. (2.23a), we have used the definitions [26]

$$\Delta(a;\boldsymbol{c}':\boldsymbol{c}) = \left\{C^2(a;\boldsymbol{c}':\boldsymbol{c}) + \left(a-\frac{1}{a}\right)^2|\boldsymbol{c}'\times\boldsymbol{c}|^2\right\}^{1/2} \tag{2.24}$$

and

$$C(a;\boldsymbol{c}':\boldsymbol{c}) = c'^2 + c^2 - \left(a+\frac{1}{a}\right)\boldsymbol{c}'\cdot\boldsymbol{c}. \tag{2.25}$$

In our notation, $c\mu$ is the component of the (dimensionless) velocity vector in the positive z direction, and so if we let

$$c_x = c\left(1-\mu^2\right)^{1/2}\cos\phi \tag{2.26}$$

denote the component of velocity in the direction x (parallel to the confining surface) of the flow, then we can express the velocity, the shear-stress, and the heat-flow profiles for the considered flow problems as

$$\boldsymbol{U}(z) = \frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2}\boldsymbol{H}(z,\boldsymbol{c})c^3\left(1-\mu^2\right)^{1/2}\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.27}$$

$$\boldsymbol{P}(z) = \frac{2}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2}\boldsymbol{H}(z,\boldsymbol{c})c^4\mu\left(1-\mu^2\right)^{1/2}\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.28}$$

and

$$\boldsymbol{Q}(z) = \frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2}\boldsymbol{H}(z,\boldsymbol{c})\left(c^2-\frac{5}{2}\right)c^3\left(1-\mu^2\right)^{1/2}\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.29}$$

where the components of $\boldsymbol{U}(z)$, $\boldsymbol{P}(z)$, and $\boldsymbol{Q}(z)$ are, respectively, the functions $U_\alpha(z)$, $P_\alpha(z)$, and $Q_\alpha(z)$, for $\alpha = 1, 2$, that are used in Appendix A to define macroscopic quantities for a binary mixture.

While it may not be clear until later in this work when the specific problems of viscous-slip, thermal-creep and diffusion-slip flow will be defined mathematically, we note that an expansion (for the considered problems) of $\boldsymbol{H}(z,\boldsymbol{c})$ in a Fourier series requires only one term – viz., one proportional to $\cos\phi$. And so, we introduce the dimensionless spatial variable

$$\tau = z\varepsilon_0, \tag{2.30}$$

where

$$\varepsilon_0 = (n_1+n_2)\pi^{1/2}\left(\frac{n_1d_1+n_2d_2}{n_1+n_2}\right)^2, \tag{2.31}$$

and write

$$\boldsymbol{H}(\tau/\varepsilon_0,\boldsymbol{c}) = \boldsymbol{\Psi}(\tau,c,\mu)\left(1-\mu^2\right)^{1/2}\cos\phi, \tag{2.32}$$

where $\boldsymbol{\Psi}(\tau,c,\mu)$ is the (vector-valued) function to be determined. We now let $z = \tau/\varepsilon_0$ in Eqs. (2.27)–(2.29) and consider that

$$\boldsymbol{U}(\tau) = \frac{1}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \boldsymbol{\Psi}(\tau, c, \mu) c^3 \left(1 - \mu^2\right) \mathrm{d}\mu \, \mathrm{d}c, \tag{2.33}$$

$$\boldsymbol{P}(\tau) = \frac{2}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \boldsymbol{\Psi}(\tau, c, \mu) c^4 \left(1 - \mu^2\right) \mu \, \mathrm{d}\mu \, \mathrm{d}c, \tag{2.34}$$

and

$$\boldsymbol{Q}(\tau) = \frac{1}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \boldsymbol{\Psi}(\tau, c, \mu) \left(c^2 - \frac{5}{2}\right) c^3 \left(1 - \mu^2\right) \mathrm{d}\mu \, \mathrm{d}c \tag{2.35}$$

are the quantities to be computed. It should be noted that to avoid excessive notation, we have, in writing Eqs. (2.33)–(2.35), followed the (often-used) procedure of not always introducing new labels for dependent quantities (in this case $\boldsymbol{U}$, $\boldsymbol{P}$, and $\boldsymbol{Q}$) when the independent variable is changed.

We can now use Eq. (2.32) in Eq. (2.1), multiply the resulting equation by $\cos\phi$, integrate over all ϕ, and use the Legendre expansion of the scattering kernel $\boldsymbol{\mathcal{K}}(\boldsymbol{c}' : \boldsymbol{c})$ that was introduced in Ref. [26] to find

$$c\mu \frac{\partial}{\partial \tau} \boldsymbol{\Psi}(\tau, c, \mu) + \boldsymbol{V}(c) \boldsymbol{\Psi}(\tau, c, \mu) = \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c'^2} f(\mu', \mu) \boldsymbol{\mathcal{K}}(c', \mu' : c, \mu) \boldsymbol{\Psi}(\tau, c', \mu') c'^2 \, \mathrm{d}\mu' \, \mathrm{d}c', \tag{2.36}$$

where

$$f(\mu', \mu) = \left(\frac{1 - \mu'^2}{1 - \mu^2} \right)^{1/2}. \tag{2.37}$$

In addition

$$\boldsymbol{\mathcal{K}}(c', \mu' : c, \mu) \cos\phi' = \int_0^{2\pi} \boldsymbol{\mathcal{K}}(\boldsymbol{c}' : \boldsymbol{c}) \cos\phi \, \mathrm{d}\phi, \tag{2.38}$$

which we can express, in the notation of Ref. [26], as

$$\boldsymbol{\mathcal{K}}(c', \mu' : c, \mu) = \frac{1}{2} \sum_{n=1}^\infty (2n + 1) P_n^1(\mu') P_n^1(\mu) \boldsymbol{\mathcal{K}}_n(c', c), \tag{2.39}$$

where $P_n^1(x)$ is used to denote one of the normalized associated Legendre functions. More explicitly,

$$P_l^m(\mu) = \left[\frac{(l - m)!}{(l + m)!} \right]^{1/2} \left(1 - \mu^2\right)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_l(\mu), \tag{2.40}$$

where $P_l(\mu)$ is the Legendre polynomial. In addition

$$\boldsymbol{\mathcal{K}}_n(c', c) = \begin{bmatrix} \mathcal{K}_n^{(1,1)}(c', c) & \mathcal{K}_n^{(1,2)}(c', c) \\ \mathcal{K}_n^{(2,1)}(c', c) & \mathcal{K}_n^{(2,2)}(c', c) \end{bmatrix}, \tag{2.41}$$

with

$$\mathcal{K}_n^{(1,1)}(c', c) = p_1 \mathcal{P}^{(n)}(c', c) + \frac{g_2}{4} \mathcal{F}^{(n)}(a_{1,2}; c', c), \tag{2.42a}$$

$$\mathcal{K}_n^{(1,2)}(c', c) = g_2 \mathcal{G}^{(n)}(a_{1,2}; c', c), \tag{2.42b}$$

$$\mathcal{K}_n^{(2,1)}(c', c) = g_1 \mathcal{G}^{(n)}(a_{2,1}; c', c), \tag{2.42c}$$

and

$$\mathcal{K}_n^{(2,2)}(c', c) = p_2 \mathcal{P}^{(n)}(c', c) + \frac{g_1}{4} \mathcal{F}^{(n)}(a_{2,1}; c', c). \tag{2.42d}$$

We also can write

$$\boldsymbol{V}(c) = \begin{bmatrix} v_1(c) & 0 \\ 0 & v_2(c) \end{bmatrix}, \tag{2.43}$$

where now

$$v_1(c) = p_1 \nu(c) + g_2 a_{2,1} \nu(a_{1,2} c) \tag{2.44a}$$

and

$$v_2(c) = p_2 \nu(c) + g_1 a_{1,2} \nu(a_{2,1} c). \tag{2.44b}$$

In writing Eqs. (2.42) and (2.44), we have used

$$p_\alpha = c_\alpha \left(\frac{n d_\alpha}{n_1 d_1 + n_2 d_2} \right)^2, \quad \alpha = 1, 2, \tag{2.45a}$$

and

$$g_\alpha = c_\alpha \left(\frac{n d_{\text{avg}}}{n_1 d_1 + n_2 d_2} \right)^2, \quad \alpha = 1, 2, \tag{2.45b}$$

where

$$c_\alpha = n_\alpha / n, \quad n = n_1 + n_2, \quad \text{and} \quad d_{\text{avg}} = (d_1 + d_2)/2. \tag{2.46a–c}$$

In order to avoid too much repetition, we do not list here our expressions for the Legendre moments $\mathcal{P}^{(n)}(c', c)$, $\mathcal{F}^{(n)}(a; c', c)$, and $\mathcal{G}^{(n)}(a; c', c)$ since they are explicitly given in Ref. [26].

At the confining surface of the considered half-space, we use a combination of specular and diffuse reflection, and so, in regard to Eq. (2.1), we write the boundary condition as

$$\boldsymbol{H}(0, c, \mu, \phi) - (\boldsymbol{I} - \boldsymbol{\alpha}) \boldsymbol{H}(0, c, -\mu, \phi) - \frac{2}{\pi} \boldsymbol{\alpha} \int_0^\infty \int_0^1 \int_0^{2\pi} \mathrm{e}^{-c'^2} \boldsymbol{H}(0, c', -\mu', \phi') c'^3 \mu' \, \mathrm{d}\phi' \, \mathrm{d}\mu' \, \mathrm{d}c' = \boldsymbol{0}, \tag{2.47}$$

for $\mu \in (0, 1]$ and all c and all ϕ. Here

$$\boldsymbol{\alpha} = \text{diag}\{\alpha_1, \alpha_2\}, \tag{2.48}$$

where α_1 and α_2 are the accommodation coefficients to be used for the two species of gas particles. Taking note of Eq. (2.32), we find from Eq. (2.47) the boundary condition subject to which we must solve Eq. (2.36), viz.,

$$\boldsymbol{\Psi}(0, c, \mu) - (\boldsymbol{I} - \boldsymbol{\alpha}) \boldsymbol{\Psi}(0, c, -\mu) = \boldsymbol{0}, \tag{2.49}$$

for $\mu \in (0, 1]$ and all c. We use $\boldsymbol{I}$ to denote the 2×2 identity matrix.

While Eqs. (2.36) and (2.49) are basic to the three physical problems of interest here, some additional development will be done in a later section of this work in order to complete the formulation of the three specific problems of viscous-slip, thermal-creep, and diffusion-slip flow. More specifically and in addition to the wall condition listed as Eq. (2.49), we will be imposing conditions on the solution of Eq. (2.36) as τ tends to infinity. And, as noted already, for the problems of thermal-creep flow and diffusion-slip flow we will change the linearization from about absolute Maxwellian distributions to about local Maxwellian distributions. This linearization about local properties will introduce an inhomogeneous driving term in Eq. (2.36) for which we will have to develop a particular solution.

3. An expansion in the speed variable

In order to account for the c dependence of $\boldsymbol{\Psi}(\tau, c, \mu)$, we introduce the (approximate) representation

$$\boldsymbol{\Psi}(\tau, c, \mu) = \sum_{k=0}^{K} \Pi_k(c) \boldsymbol{G}_k(\tau, \mu), \tag{3.1}$$

where $\{\Pi_k(c)\}$ is a collection of basis functions to be specified. And so we substitute Eq. (3.1) into Eqs. (2.36) and (2.49), multiply the resulting equations by

$$c^2 \exp\{-c^2\}\Pi_l(c),$$

for $l = 0, 1, \ldots, K$, and integrate over all c to find

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{A}\boldsymbol{G}(\tau, \mu) + \boldsymbol{S}\boldsymbol{G}(\tau, \mu) = \int_{-1}^{1} \boldsymbol{C}(\mu' : \mu)\boldsymbol{G}(\tau, \mu')\, \mathrm{d}\mu' \tag{3.2}$$

and

$$\boldsymbol{F}\boldsymbol{G}(0, \mu) - \boldsymbol{H}\boldsymbol{G}(0, -\mu) = \boldsymbol{0}, \quad \mu \in (0, 1]. \tag{3.3}$$

Here

$$\boldsymbol{G}(\tau, \mu) = \big[\boldsymbol{G}_0(\tau, \mu), \boldsymbol{G}_1(\tau, \mu), \ldots, \boldsymbol{G}_K(\tau, \mu)\big]^{\mathrm{T}}, \tag{3.4}$$

where the superscript T is used to denote the transpose operation, and the $2(K + 1) \times 2(K + 1)$ matrices $\boldsymbol{A}$, $\boldsymbol{S}$ and $\boldsymbol{C}(\mu' : \mu)$ are given by

$$\boldsymbol{A} = \int_0^{\infty} \mathrm{e}^{-c^2} c^3 \boldsymbol{\Pi}^{\mathrm{T}}(c)\boldsymbol{\Pi}(c)\, \mathrm{d}c, \tag{3.5a}$$

$$\boldsymbol{S} = \int_0^{\infty} \mathrm{e}^{-c^2} c^2 \boldsymbol{\Pi}^{\mathrm{T}}(c)\boldsymbol{V}(c)\boldsymbol{\Pi}(c)\, \mathrm{d}c, \tag{3.5b}$$

and

$$\boldsymbol{C}(\mu' : \mu) = f(\mu', \mu) \int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-c^2}\mathrm{e}^{-c'^2} c^2 c'^2 \boldsymbol{\Pi}^{\mathrm{T}}(c)\boldsymbol{\mathcal{K}}(c', \mu' : c, \mu)\boldsymbol{\Pi}(c')\, \mathrm{d}c'\, \mathrm{d}c, \tag{3.5c}$$

where the $2 \times 2(K + 1)$ matrix $\boldsymbol{\Pi}(c)$ is defined as

$$\boldsymbol{\Pi}(c) = \big[\Pi_0(c)\boldsymbol{I} \;\; \Pi_1(c)\boldsymbol{I} \cdots \Pi_K(c)\boldsymbol{I}\big]. \tag{3.6}$$

In addition,

$$\boldsymbol{F} = \int_0^{\infty} \mathrm{e}^{-c^2} c^2 \boldsymbol{\Pi}^{\mathrm{T}}(c)\boldsymbol{\Pi}(c)\, \mathrm{d}c \tag{3.7a}$$

and

$$\boldsymbol{H} = \int_0^{\infty} \mathrm{e}^{-c^2} c^2 \boldsymbol{\Pi}^{\mathrm{T}}(c)(\boldsymbol{I} - \boldsymbol{\alpha})\boldsymbol{\Pi}(c)\, \mathrm{d}c. \tag{3.7b}$$

4. Elementary (ADO) solutions

Seeking separable solutions of Eq. (3.2), we substitute

$$\boldsymbol{G}(\tau, \mu) = \boldsymbol{\Phi}(\nu, \mu)\mathrm{e}^{-\tau/\nu} \tag{4.1}$$

into that equation to find

$$\boldsymbol{S}\boldsymbol{U}(\nu, \mu) - \int_0^{1} \boldsymbol{C}_+(\mu' : \mu)\boldsymbol{U}(\nu, \mu')\, \mathrm{d}\mu' = \frac{\mu}{\nu}\boldsymbol{A}\boldsymbol{V}(\nu, \mu) \tag{4.2a}$$

and

$$\boldsymbol{S}\boldsymbol{V}(\nu,\mu)-\int_0^1 \boldsymbol{C}_-(\mu':\mu)\boldsymbol{V}(\nu,\mu')\,\mathrm{d}\mu'=\frac{\mu}{\nu}\boldsymbol{A}\boldsymbol{U}(\nu,\mu), \tag{4.2b}$$

where

$$\boldsymbol{C}_\pm(\mu':\mu)=\boldsymbol{C}(\mu':\mu)\pm\boldsymbol{C}(-\mu':\mu). \tag{4.3}$$

Here

$$\boldsymbol{U}(\nu,\mu)=\boldsymbol{\Phi}(\nu,\mu)+\boldsymbol{\Phi}(\nu,-\mu) \tag{4.4a}$$

and

$$\boldsymbol{V}(\nu,\mu)=\boldsymbol{\Phi}(\nu,\mu)-\boldsymbol{\Phi}(\nu,-\mu). \tag{4.4b}$$

We can now substitute $\boldsymbol{U}(\nu,\mu)$ from Eq. (4.2b) into Eq. (4.2a) to find

$$\frac{1}{\mu^2}\left[\boldsymbol{\Sigma}^2\boldsymbol{V}(\nu,\mu)-\int_0^1 \boldsymbol{\mathcal{B}}(\mu':\mu)\boldsymbol{V}(\nu,\mu')\,\mathrm{d}\mu'\right]=\lambda\boldsymbol{V}(\nu,\mu), \tag{4.5}$$

where

$$\lambda=\frac{1}{\nu^2}, \tag{4.6}$$

$$\boldsymbol{\mathcal{B}}(\mu':\mu)=\frac{\mu}{\mu'}\boldsymbol{B}_+(\mu':\mu)\boldsymbol{\Sigma}+\boldsymbol{\Sigma}\boldsymbol{B}_-(\mu':\mu)-\int_0^1\frac{\mu}{\mu''}\boldsymbol{B}_+(\mu'':\mu)\boldsymbol{B}_-(\mu':\mu'')\,\mathrm{d}\mu'', \tag{4.7}$$

and

$$\boldsymbol{\Sigma}=\boldsymbol{A}^{-1}\boldsymbol{S}. \tag{4.8}$$

In Eq. (4.7), we have defined

$$\boldsymbol{B}_\pm(\mu':\mu)=\frac{1}{2}f(\mu',\mu)\sum_{n=1}^{\infty}(2n+1)\left[1\mp(-1)^n\right]P_n^1(\mu')P_n^1(\mu)\boldsymbol{B}_n, \tag{4.9}$$

where

$$\boldsymbol{B}_n=\boldsymbol{A}^{-1}\boldsymbol{C}_n. \tag{4.10}$$

Here the $2(K+1)\times 2(K+1)$ matrices $\boldsymbol{C}_n$ are given by

$$\boldsymbol{C}_n=\int_0^\infty\int_0^\infty \mathrm{e}^{-c^2}\mathrm{e}^{-c'^2}c^2c'^2\boldsymbol{\Pi}^{\mathrm{T}}(c)\boldsymbol{\mathcal{K}}_n(c',c)\boldsymbol{\Pi}(c')\,\mathrm{d}c'\,\mathrm{d}c. \tag{4.11}$$

Making use of Eq. (4.9), we rewrite Eq. (4.7) as

$$\boldsymbol{\mathcal{B}}(\mu':\mu)=\frac{\mu}{\mu'}\boldsymbol{B}_+(\mu':\mu)\boldsymbol{\Sigma}+\boldsymbol{\Sigma}\boldsymbol{B}_-(\mu':\mu)-\mu f(\mu',\mu)\sum_{m=0}^{\infty}\sum_{m'=m+1}^{\infty}\Delta_{m,m'}P_{2m'}^1(\mu')P_{2m+1}^1(\mu)\boldsymbol{B}_{2m+1}\boldsymbol{B}_{2m'}, \tag{4.12}$$

where the required

$$\Delta_{m,m'}=(4m+3)(4m'+1)\int_0^1 P_{2m'}^1(x)P_{2m+1}^1(x)\,\frac{\mathrm{d}x}{x} \tag{4.13}$$

can be shown to be given by

$$\Delta_{m,m'} = -\left[2\left(m'^2 + m'\right)^{1/2} P^1_{2m'+1}(0)\right]^{-1} (4m+3)(4m'+1) P^1_{2m+1}(0), \quad m \leqslant m'-1, \tag{4.14}$$

with

$$P^1_{2\alpha+1}(0) = -\frac{1}{2}\left[\frac{4\alpha^2-1}{\alpha(\alpha+1)}\right]^{1/2} P^1_{2\alpha-1}(0), \quad \alpha = 1, 2, \ldots, \quad \text{and} \quad P^1_1(0) = 2^{-1/2}. \tag{4.15a,b}$$

We now introduce a "half-range" quadrature scheme (with weights and nodes w_k and μ_k) and rewrite Eqs. (4.5) and (4.2b) evaluated at the quadrature points as

$$\frac{1}{\mu_i^2}\left[\boldsymbol{\Sigma}^2 \boldsymbol{V}(\nu_j, \mu_i) - \sum_{k=1}^{N} w_k \boldsymbol{\mathcal{B}}(\mu_k : \mu_i) \boldsymbol{V}(\nu_j, \mu_k)\right] = \lambda_j \boldsymbol{V}(\nu_j, \mu_i) \tag{4.16a}$$

and

$$\boldsymbol{U}(\nu_j, \mu_i) = \frac{\nu_j}{\mu_i}\left[\boldsymbol{\Sigma}\boldsymbol{V}(\nu_j, \mu_i) - \sum_{k=1}^{N} w_k \boldsymbol{B}_-(\mu_k : \mu_i) \boldsymbol{V}(\nu_j, \mu_k)\right], \tag{4.16b}$$

for $i = 1, 2, \ldots, N$. Equation (4.16a) defines our eigenvalue problem, to which we have added the subscript j to label the eigenvalues and eigenvectors. Once this eigenvalue problem is solved, we have the elementary solutions from

$$\boldsymbol{\Phi}(\nu_j, \mu_i) = \frac{1}{2}\left[\boldsymbol{U}(\nu_j, \mu_i) + \boldsymbol{V}(\nu_j, \mu_i)\right] \tag{4.17a}$$

and

$$\boldsymbol{\Phi}(\nu_j, -\mu_i) = \frac{1}{2}\left[\boldsymbol{U}(\nu_j, \mu_i) - \boldsymbol{V}(\nu_j, \mu_i)\right]. \tag{4.17b}$$

Note that the separation constants defined by

$$\nu_j = \pm\lambda_j^{-1/2} \tag{4.18}$$

occur in $\pm$ pairs. From this point, we take ν_j to be the positive root listed in Eq. (4.18). Once we have solved the eigenvalue problem defined by Eq. (4.16a), we can write our general (discrete ordinates) solution to Eq. (3.2) as

$$\boldsymbol{G}(\tau, \pm\mu_i) = \sum_{j=1}^{J}\left[A_j \boldsymbol{\Phi}(\nu_j, \pm\mu_i) \mathrm{e}^{-\tau/\nu_j} + B_j \boldsymbol{\Phi}(\nu_j, \mp\mu_i) \mathrm{e}^{\tau/\nu_j}\right], \tag{4.19}$$

for $i = 1, 2, \ldots, N$. Here $J = 2N(K+1)$, and the arbitrary constants $\{A_j\}$ and $\{B_j\}$ are to be determined from the conditions that define a specific problem.

5. The complete speed-dependent ADO solution

If we combine Eqs. (3.1), (3.4), (3.6) and (4.19) we can write our (approximate) solution as

$$\boldsymbol{\Psi}(\tau, c, \pm\mu_i) = \boldsymbol{\Pi}(c) \sum_{j=1}^{J}\left[A_j \boldsymbol{\Phi}(\nu_j, \pm\mu_i) \mathrm{e}^{-\tau/\nu_j} + B_j \boldsymbol{\Phi}(\nu_j, \mp\mu_i) \mathrm{e}^{\tau/\nu_j}\right], \tag{5.1}$$

for $i = 1, 2, \ldots, N$. While Eq. (5.1) is our general (discrete-ordinates) solution, we can make some improvements in that result. We have found that the eigenvalue problem defined by Eq. (4.16a) yields a separation constant, say ν_1, that approximates the expected unbounded separation constant. And so we ignore ν_1 in Eq. (5.1) and rewrite that equation as

$$\boldsymbol{\Psi}(\tau, c, \pm\mu_i) = \boldsymbol{\Psi}_*(\tau, c, \pm\mu_i) + \boldsymbol{\Pi}(c) \sum_{j=2}^{J}\left[A_j \boldsymbol{\Phi}(\nu_j, \pm\mu_i) \mathrm{e}^{-\tau/\nu_j} + B_j \boldsymbol{\Phi}(\nu_j, \mp\mu_i) \mathrm{e}^{\tau/\nu_j}\right], \tag{5.2}$$

for $i = 1, 2, \ldots, N$. We note that $\boldsymbol{\Psi}_*(\tau, c, \mu)$ is defined in terms of two of the exact elementary solutions we reported in a previous work [26], viz.,

$$\boldsymbol{\Psi}_*(\tau, c, \mu) = A_1 c\boldsymbol{\Phi} + B_1\big[c\tau\boldsymbol{\Phi} - \mu\boldsymbol{B}(c)\big], \tag{5.3}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 \\ a_{1,2} \end{bmatrix}, \tag{5.4}$$

and where $\boldsymbol{B}(c)$ is one of the generalized Chapman–Enskog (vector-valued) functions discussed in Ref. [26]. In our formulation of Kramers' problem there is no inhomogeneous driving term in the balance equation, and so we will use the arbitrary constants $\{A_j, B_j\}$ in Eqs. (5.2) and (5.3) to satisfy a condition as τ tends to infinity and a discrete-ordinates version of the relevant boundary condition. On the other hand, for the thermal-creep and diffusion-slip problems, when we introduce an inhomogeneous term in our balance equation, we will require that a particular solution be added to Eq. (5.2) before applying the appropriate conditions on the solutions.

6. The problems

Having developed our elementary solutions of Eq. (2.36), we are now ready to use them to solve the three specific problems basic to our current study.

6.1. The viscous-slip problem

For our formulation of the viscous-slip problem no driving term is added to Eq. (2.1). Instead the velocity profile

$$\boldsymbol{U}(\tau) = \frac{1}{\pi^{3/2}} \int_0^\infty \int_{-1}^1 \int_0^{2\pi} \mathrm{e}^{-c^2} \boldsymbol{H}(\tau/\varepsilon_0, \boldsymbol{c}) c^3 \big(1 - \mu^2\big)^{1/2} \cos\phi \, \mathrm{d}\phi \, \mathrm{d}\mu \, \mathrm{d}c \tag{6.1}$$

is considered to diverge linearly (but no faster) in τ as τ tends to infinity. And so, since the balance equation listed as Eq. (2.1) and the boundary condition listed as Eq. (2.47) are both homogeneous, and since the remaining condition is applied to $\boldsymbol{U}(\tau)$ as given by Eq. (6.1), there is no reason to include more than one term in a Fourier-series expansion of $\boldsymbol{H}(z, \boldsymbol{c})$. In this way, we justify the use of Eq. (2.32). Continuing, we neglect exponentially diverging (as τ tends to infinity) terms in Eq. (5.2) and write

$$\boldsymbol{\Psi}(\tau, c, \pm\mu_i) = A_1 c\boldsymbol{\Phi} + B_1\big[c\tau\boldsymbol{\Phi} \mp \mu_i \boldsymbol{B}(c)\big] + \boldsymbol{\Pi}(c) \sum_{j=2}^{J} A_j \boldsymbol{\Phi}(\nu_j, \pm\mu_i) \mathrm{e}^{-\tau/\nu_j}, \tag{6.2}$$

for $i = 1, 2, \ldots, N$. Once the arbitrary constants $\{A_j\}$ and B_1 are established, we can define the quantities of interest by using Eq. (6.2) in (discrete-ordinates versions of) Eqs. (2.33)–(2.35) to find

$$\boldsymbol{U}(\tau) = \frac{1}{2}(A_1 + B_1\tau)\boldsymbol{\Phi} + \sum_{j=2}^{J} A_j \boldsymbol{\mathcal{U}}_j \mathrm{e}^{-\tau/\nu_j}, \tag{6.3a}$$

$$\boldsymbol{P}(\tau) = -\frac{1}{2} B_1 \boldsymbol{\varepsilon}_p + \sum_{j=2}^{J} A_j \boldsymbol{\mathcal{P}}_j \mathrm{e}^{-\tau/\nu_j}, \tag{6.3b}$$

and

$$\boldsymbol{Q}(\tau) = \sum_{j=2}^{J} A_j \boldsymbol{\mathcal{Q}}_j \mathrm{e}^{-\tau/\nu_j}. \tag{6.3c}$$

Here $\boldsymbol{\Phi}$ is given by Eq. (5.4), and

$$\boldsymbol{\varepsilon}_p = \frac{16}{15\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2} \boldsymbol{B}(c) c^4 \, \mathrm{d}c. \tag{6.4}$$

We note that the components $\varepsilon_{p,1}$ and $\varepsilon_{p,2}$ of $\boldsymbol{\varepsilon}_p$ have been evaluated (for several data sets) in Ref. [26]. In writing Eqs. (6.3), we have used the definitions

$$\boldsymbol{\mathcal{U}}_j = \boldsymbol{\Pi}_1 \boldsymbol{X}_j, \tag{6.5a}$$

$$\boldsymbol{\mathcal{P}}_j = 2\boldsymbol{\Pi}_2 \boldsymbol{Y}_j, \tag{6.5b}$$

and

$$\boldsymbol{\mathcal{Q}}_j = \left[\boldsymbol{\Pi}_3 - \frac{5}{2}\boldsymbol{\Pi}_1\right]\boldsymbol{X}_j, \tag{6.5c}$$

where

$$\boldsymbol{X}_j = \frac{1}{\pi^{1/2}} \sum_{k=1}^{N} w_k \big(1-\mu_k^2\big)\big[\boldsymbol{\Phi}(\nu_j,\mu_k) + \boldsymbol{\Phi}(\nu_j,-\mu_k)\big], \tag{6.6a}$$

$$\boldsymbol{Y}_j = \frac{1}{\pi^{1/2}} \sum_{k=1}^{N} w_k \mu_k \big(1-\mu_k^2\big)\big[\boldsymbol{\Phi}(\nu_j,\mu_k) - \boldsymbol{\Phi}(\nu_j,-\mu_k)\big], \tag{6.6b}$$

and

$$\boldsymbol{\Pi}_n = \int_0^\infty \mathrm{e}^{-c^2} \boldsymbol{\Pi}(c) c^{n+2}\,\mathrm{d}c. \tag{6.7}$$

As the coefficient of the linear behavior (as τ tends to infinity) of $\boldsymbol{U}(\tau)$ is unspecified, we can normalize our solution by taking $B_1 = 2(m_1/m)^{1/2}$, where

$$m = c_1 m_1 + c_2 m_2. \tag{6.8}$$

We also let $A_1 = 2(m_1/m)^{1/2}\zeta_p$ and rewrite our results for the quantities of interest as

$$\boldsymbol{U}_p(\tau) = (\zeta_p + \tau)\boldsymbol{\Psi} + \sum_{j=2}^{J} A_j \boldsymbol{\mathcal{U}}_j \mathrm{e}^{-\tau/\nu_j}, \tag{6.9a}$$

$$\boldsymbol{P}_p(\tau) = -\left(\frac{m_1}{m}\right)^{1/2} \boldsymbol{\varepsilon}_p + \sum_{j=2}^{J} A_j \boldsymbol{\mathcal{P}}_j \mathrm{e}^{-\tau/\nu_j}, \tag{6.9b}$$

and

$$\boldsymbol{Q}_p(\tau) = \sum_{j=2}^{J} A_j \boldsymbol{\mathcal{Q}}_j \mathrm{e}^{-\tau/\nu_j}, \tag{6.9c}$$

where $(m_\alpha/m)^{1/2}$, $\alpha = 1, 2$, are the two components of the vector $\boldsymbol{\Psi}$ that appears in Eq. (6.9a). Using the definitions listed in Appendix A, we find that the average velocity

$$u_p(\tau) = v_0\left[c_1\left(\frac{m}{m_1}\right)^{1/2} U_1(\tau) + c_2\left(\frac{m}{m_2}\right)^{1/2} U_2(\tau)\right] \tag{6.10a}$$

and the bulk velocity

$$\hat{u}_p(\tau) = v_0\left[c_1\left(\frac{m_1}{m}\right)^{1/2} U_1(\tau) + c_2\left(\frac{m_2}{m}\right)^{1/2} U_2(\tau)\right], \tag{6.10b}$$

where $U_\alpha(\tau)$ are the two components of $\boldsymbol{U}_p(\tau)$, both satisfy the same asymptotic form, i.e.,

$$\lim_{\tau\to\infty} \frac{\mathrm{d}}{\mathrm{d}\tau} u_p(\tau) = v_0 \tag{6.11a}$$

and

$$\lim_{\tau\to\infty}\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{u}_p(\tau)=v_0, \tag{6.11b}$$

where

$$v_0=\left(\frac{2kT_0}{m}\right)^{1/2}. \tag{6.12}$$

Note that we have added the subscript p to Eqs. (6.9), (6.10) and (6.11) to denote our final forms for the viscous-slip problem.

In order to determine the required constants in our solution, we now write our discrete-ordinates version of the boundary condition listed as Eq. (2.49) in the form

$$\boldsymbol{\Psi}(0,c,\mu_i)-(\boldsymbol{I}-\boldsymbol{\alpha})\boldsymbol{\Psi}(0,c,-\mu_i)=\boldsymbol{0}, \tag{6.13}$$

for $i=1,2,\ldots,N$. We now substitute Eq. (6.2) into Eq. (6.13), multiply the resulting equation by

$$c^2\exp\bigl(-c^2\bigr)\Pi_l(c),$$

and integrate over all c to find, for $i=1,2,\ldots,N$ and $l=0,1,\ldots,K$, a system of linear algebraic equations which we can solve to find A_j, $j=1,2,\ldots,J$. In this way we have completed our solution, as given by Eq. (6.2), and our expressions for the quantities of interest, as given by Eqs. (6.9).

Finally, if we let

$$\boldsymbol{U}_{p,\mathrm{asy}}(\tau)=(\zeta_p+\tau)\boldsymbol{\Psi}, \tag{6.14}$$

with components $U_{1,\mathrm{asy}}(\tau)$ and $U_{2,\mathrm{asy}}(\tau)$, denote the asymptotic part of the velocity profile, then

$$\zeta_{p,\alpha}=\frac{U_{\alpha,\mathrm{asy}}(0)}{U'_{\alpha,\mathrm{asy}}(0)},\quad \alpha=1,2, \tag{6.15}$$

defines a viscous-slip coefficient for each species. We see from Eq. (6.14) that the two coefficients are the same, i.e.,

$$\zeta_{p,\alpha}=\zeta_p,\quad \alpha=1,2. \tag{6.16}$$

6.2. A common formulation of the thermal-creep and diffusion-slip problems

In regard to the problem of thermal creep (diffusion slip), the flow is caused by a constant temperature gradient (constant density gradients) in the direction x (parallel to the wall), and so it is helpful [29] to express the particle velocity-distribution functions in terms of linearized forms of the local Maxwellian distributions, rather than absolute Maxwellians, as was done in regard to Eq. (2.1). We write the local Maxwellians as

$$f_{\alpha,0}(x,v)=n_\alpha(x)\left[\frac{m_\alpha}{2\pi kT(x)}\right]^{3/2}\exp\left\{-\frac{m_\alpha v^2}{2kT(x)}\right\},\quad \alpha=1,2, \tag{6.17}$$

and if we express the considered linear variations in the number densities and the temperature (both of the wall and the gas mixture) as

$$n_\alpha(x)=n_\alpha(1+R_\alpha x),\quad \alpha=1,2, \tag{6.18}$$

and

$$T(x)=T_0(1+K_T x), \tag{6.19}$$

where R_α and K_T are considered to be given (small) constants, we can linearize Eq. (6.17) to obtain an approximation of the form

$$f^*_{\alpha,0}(x,v)=f_{\alpha,0}(v)\bigl[1+f_\alpha(v)x\bigr], \tag{6.20}$$

where $f_\alpha(v)$ is to be determined. In order to have the pressure constant, we find, after neglecting 2nd order terms, that we must have

$$c_1R_1+c_2R_2=-K_T, \tag{6.21}$$

and so, making use of Eq. (6.21), we find that we can use

$$f_1(v) = c_2 K_C + \left(\frac{m_1 v^2}{2kT_0} - \frac{5}{2}\right) K_T \tag{6.22}$$

and

$$f_2(v) = -c_1 K_C + \left(\frac{m_2 v^2}{2kT_0} - \frac{5}{2}\right) K_T, \tag{6.23}$$

with $K_C = R_1 - R_2$, to complete Eq. (6.20). Continuing our linearizations, we now write the velocity distribution functions we seek as

$$f_\alpha(x, z, \boldsymbol{v}) = f_{\alpha,0}(v)\big\{1 + f_\alpha(v)x + h_\alpha\big(z, \lambda_\alpha^{1/2}\boldsymbol{v}\big)\big\}, \quad \alpha = 1, 2. \tag{6.24}$$

As a result of Eq. (6.24), an inhomogeneous source

$$\boldsymbol{S}(\boldsymbol{c}) = c\big(1-\mu^2\big)^{1/2}\cos\phi\left\{\left(c^2 - \frac{5}{2}\right)K_T\begin{bmatrix}1\\1\end{bmatrix} + K_C\begin{bmatrix}c_2\\-c_1\end{bmatrix}\right\} \tag{6.25}$$

must be added to Eq. (2.1) to yield

$$\boldsymbol{S}(\boldsymbol{c}) + c\mu\frac{\partial}{\partial z}\boldsymbol{H}(z,\boldsymbol{c}) + \varepsilon_0\boldsymbol{V}(c)\boldsymbol{H}(z,\boldsymbol{c}) = \varepsilon_0\int \mathrm{e}^{-c'^2}\boldsymbol{\mathcal{K}}(\boldsymbol{c}':\boldsymbol{c})\boldsymbol{H}(z,\boldsymbol{c}')\,\mathrm{d}^3c'. \tag{6.26}$$

It is clear that we can solve the defined problem in two steps; for the first (generally referred to as the half-space thermal-creep problem) we take $K_C = 0$ and introduce the normalization $K_T = \varepsilon_0$, while for the second (often referred to as the diffusion-slip problem) we take $K_T = 0$ and use the normalization $K_C = \varepsilon_0$. Before considering the first of these two problems, we note that the inhomogeneous term in Eq. (6.26) has only one term in a Fourier-series representation, and so, since the boundary condition listed as Eq. (2.47) is homogeneous and since the only additional condition we impose is that $\boldsymbol{U}(\tau)$, as listed by Eq. (6.1), be bounded as τ tends to infinity, we are justified in using Eq. (2.32) also for the thermal-creep and diffusion-slip problems.

6.2.1. The thermal-creep problem

We use Eq. (2.32) to find from Eq. (6.26)

$$\boldsymbol{\Gamma}(c) + c\mu\frac{\partial}{\partial\tau}\boldsymbol{\Psi}(\tau,c,\mu) + \boldsymbol{V}(c)\boldsymbol{\Psi}(\tau,c,\mu) = \int_0^\infty\int_{-1}^1 \mathrm{e}^{-c'^2} f(\mu',\mu)\boldsymbol{\mathcal{K}}(c',\mu':c,\mu)\boldsymbol{\Psi}(\tau,c',\mu')c'^2\,\mathrm{d}\mu'\,\mathrm{d}c', \tag{6.27}$$

where

$$\boldsymbol{\Gamma}(c) = c\left(c^2 - \frac{5}{2}\right)\begin{bmatrix}1\\1\end{bmatrix}. \tag{6.28}$$

If we note how the two generalized Chapman–Enskog (vector-valued) functions $\boldsymbol{A}^{(1)}(c)$ and $\boldsymbol{A}^{(2)}(c)$ are defined in Ref. [26], then we can conclude that

$$\boldsymbol{\Psi}_{ps}(\tau,c,\mu) = -\boldsymbol{A}(c), \tag{6.29}$$

where

$$\boldsymbol{A}(c) = \boldsymbol{A}^{(1)}(c) + \boldsymbol{A}^{(2)}(c), \tag{6.30}$$

is a particular solution of Eq. (6.27).

Now since we wish $\boldsymbol{U}(\tau)$ to be bounded as τ tends to infinity, we write our discrete-ordinates solution to the half-space thermal-creep problem as

$$\boldsymbol{\Psi}(\tau,c,\pm\mu_i) = -\boldsymbol{A}(c) + A_1 c\boldsymbol{\Phi} + \boldsymbol{\Pi}(c)\sum_{j=2}^{J} A_j\boldsymbol{\Phi}(\nu_j,\pm\mu_i)\mathrm{e}^{-\tau/\nu_j}, \tag{6.31}$$

for $i = 1, 2, \ldots, N$. We now substitute Eqs. (6.31) into Eq. (6.13), multiply the resulting equation by

$$c^2 \exp\bigl(-c^2\bigr)\Pi_l(c),$$

and integrate over all c to find, for $i = 1, 2, \ldots, N$ and $l = 0, 1, \ldots, K$, a system of linear algebraic equations which we can solve to find A_j, $j = 1, 2, \ldots, J$. In this way we have completed Eq. (6.31). And so the quantities of interest here can be expressed as

$$\boldsymbol{U}_t(\tau) = \frac{1}{2}A_1\boldsymbol{\Phi} - \frac{4}{3\pi^{1/2}}\int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{A}(c)c^3\,\mathrm{d}c + \sum_{j=2}^{J} A_j\boldsymbol{\mathcal{U}}_j\mathrm{e}^{-\tau/\nu_j}, \tag{6.32a}$$

$$\boldsymbol{P}_t(\tau) = \sum_{j=2}^{J} A_j\boldsymbol{\mathcal{P}}_j\mathrm{e}^{-\tau/\nu_j}, \tag{6.32b}$$

and

$$\boldsymbol{Q}_t(\tau) = -\frac{4}{3\pi^{1/2}}\int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{A}(c)\left(c^2 - \frac{5}{2}\right)c^3\,\mathrm{d}c + \sum_{j=2}^{J} A_j\boldsymbol{\mathcal{Q}}_j\mathrm{e}^{-\tau/\nu_j}, \tag{6.32c}$$

where Eqs. (6.5)–(6.7) are to be used, and where we have added the subscript t for the thermal-creep problem. Taking note of Eq. (6.32a), we can define two (different) thermal-slip coefficients, viz.,

$$\zeta_{t,1} = [1 \quad 0]\boldsymbol{U}_t(\infty) \tag{6.33a}$$

and

$$\zeta_{t,2} = [0 \quad 1]\boldsymbol{U}_t(\infty), \tag{6.33b}$$

where

$$\boldsymbol{U}_t(\infty) = \frac{1}{2}A_1\boldsymbol{\Phi} - \frac{4}{3\pi^{1/2}}\int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{A}(c)c^3\,\mathrm{d}c. \tag{6.34}$$

A comment regarding the fact that we have defined two slip coefficients here is considered important. Following Ferziger and Kaper [3] and Appendix A of this work, we define the average flow velocity $u(\tau)$ and the bulk velocity $\hat{u}(\tau)$ as

$$u_t(\tau) = v_0\left[c_1\left(\frac{m}{m_1}\right)^{1/2}U_1(\tau) + c_2\left(\frac{m}{m_2}\right)^{1/2}U_2(\tau)\right] \tag{6.35a}$$

and

$$\hat{u}_t(\tau) = v_0\left[c_1\left(\frac{m_1}{m}\right)^{1/2}U_1(\tau) + c_2\left(\frac{m_2}{m}\right)^{1/2}U_2(\tau)\right], \tag{6.35b}$$

where $U_\alpha(\tau)$ are the components of $\boldsymbol{U}_t(\tau)$ and where Eqs. (6.8) and (6.12) are to be used. And so by reporting both $\zeta_{t,1}$ and $\zeta_{t,2}$ we make available both $u_t(\infty)$ and $\hat{u}_t(\infty)$.

6.2.2. The diffusion-slip problem

Using Eq. (2.32) again, we find from Eq. (6.26)

$$c\boldsymbol{N} + c\mu\frac{\partial}{\partial\tau}\boldsymbol{\Psi}(\tau, c, \mu) + \boldsymbol{V}(c)\boldsymbol{\Psi}(\tau, c, \mu) = \int_0^\infty\int_{-1}^{1} \mathrm{e}^{-c'^2} f(\mu', \mu)\boldsymbol{\mathcal{K}}(c', \mu' : c, \mu)\boldsymbol{\Psi}(\tau, c', \mu')c'^2\,\mathrm{d}\mu'\,\mathrm{d}c', \tag{6.36}$$

where

$$\boldsymbol{N} = \begin{bmatrix} c_2 \\ -c_1 \end{bmatrix}. \tag{6.37}$$

Since Eq. (6.36) has an inhomogeneous driving term, we must supplement our ADO solution of the homogeneous equation with a particular solution. We find we can express the particular solution required here as

$$\boldsymbol{\Psi}_{ps}(\tau, c, \mu) = c_2 \boldsymbol{A}^{(1)}(c) - c_1 \boldsymbol{A}^{(2)}(c). \tag{6.38}$$

Now since we wish $\boldsymbol{\Psi}(\tau, c, \mu)$ to be bounded as τ tends to infinity, we write our discrete-ordinates solution to the diffusion-slip problem as

$$\boldsymbol{\Psi}(\tau, c, \pm\mu_i) = c_2 \boldsymbol{A}^{(1)}(c) - c_1 \boldsymbol{A}^{(2)}(c) + A_1 c \boldsymbol{\Phi} + \boldsymbol{\Pi}(c) \sum_{j=2}^{J} A_j \boldsymbol{\Phi}(\nu_j, \pm\mu_i) \mathrm{e}^{-\tau/\nu_j}, \tag{6.39}$$

for $i = 1, 2, \ldots, N$. We now substitute Eqs. (6.39) into Eq. (6.13), multiply the resulting equation by

$$c^2 \exp\bigl(-c^2\bigr) \Pi_l(c),$$

and integrate over all c to find, for $i = 1, 2, \ldots, N$ and $l = 0, 1, \ldots, K$, a system of linear algebraic equations which we can solve to find A_j, $j = 1, 2, \ldots, J$. In this way we have completed Eq. (6.39). And so, using the subscript c for the diffusion-slip problem, we can express the quantities of interest here as

$$\boldsymbol{U}_c(\tau) = \frac{1}{2} A_1 \boldsymbol{\Phi} + \frac{4}{3\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2} \bigl[c_2 \boldsymbol{A}^{(1)}(c) - c_1 \boldsymbol{A}^{(2)}(c)\bigr] c^3 \, \mathrm{d}c + \sum_{j=2}^{J} A_j \boldsymbol{\mathcal{U}}_j \mathrm{e}^{-\tau/\nu_j}, \tag{6.40a}$$

$$\boldsymbol{P}_c(\tau) = \sum_{j=2}^{J} A_j \boldsymbol{\mathcal{P}}_j \mathrm{e}^{-\tau/\nu_j}, \tag{6.40b}$$

and

$$\boldsymbol{Q}_c(\tau) = \frac{4}{3\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2} \bigl[c_2 \boldsymbol{A}^{(1)}(c) - c_1 \boldsymbol{A}^{(2)}(c)\bigr] \left(c^2 - \frac{5}{2}\right) c^3 \, \mathrm{d}c + \sum_{j=2}^{J} A_j \boldsymbol{\mathcal{Q}}_j \mathrm{e}^{-\tau/\nu_j}, \tag{6.40c}$$

where Eqs. (6.5)–(6.7) are to be used. Taking note of Eq. (6.40a), we can define two (different) diffusion-slip coefficients, viz.,

$$\zeta_{c,1} = [1 \quad 0] \boldsymbol{U}_c(\infty) \tag{6.41a}$$

and

$$\zeta_{c,2} = [0 \quad 1] \boldsymbol{U}_c(\infty), \tag{6.41b}$$

where

$$\boldsymbol{U}_c(\infty) = \frac{1}{2} A_1 \boldsymbol{\Phi} + \frac{4}{3\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2} \bigl[c_2 \boldsymbol{A}^{(1)}(c) - c_1 \boldsymbol{A}^{(2)}(c)\bigr] c^3 \, \mathrm{d}c. \tag{6.42}$$

The fact that we have defined two slip coefficients here is motivated by the discussion that follows Eq. (6.34).

6.3. Relationships between the problems

In a somewhat recent work [30], Sharipov used physical arguments to provide, within the context of the S model for a single-species gas, a relationship between the heat flow from Kramers' problem and the thermal-slip coefficient. In a following work [31], Siewert used the defining equations relevant to the linearized Boltzmann equation and Maxwell or Cercignani–Lampis boundary conditions to generalize Sharipov's basic result. These two results [30,31] define, for the case of a single-species gas, a relationship between the viscous-slip problem and the half-space thermal-creep problem that can be used, for example, to help evaluate the quality of numerical results obtained for the two problems. As a generalization of the mentioned papers [30,31], Siewert and Valougeorgis [17] established a similar relationship between the viscous-slip problem and the thermal-creep problem for the case of a binary mixture described by the McCormack kinetic model [32]. Following these works [17,30,31], we can establish important relationships between the three half-space problems considered in this work.

6.3.1. A relationship between the viscous-slip and the thermal-creep problems

Since our derivation follows very closely the one given in Refs. [17,31], we can be brief here. If we add subscripts t for the thermal-creep problem and p for the viscous-slip problem, we find we can write

$$\boldsymbol{\Gamma}(c) - c\mu\frac{\partial}{\partial\tau}\boldsymbol{\Psi}_t(\tau, c, -\mu) + \boldsymbol{\mathcal{L}}\{\boldsymbol{\Psi}_t\}(\tau, c, -\mu) = \mathbf{0} \tag{6.43}$$

and

$$c\mu\frac{\partial}{\partial\tau}\boldsymbol{\Psi}_p(\tau, c, \mu) + \boldsymbol{\mathcal{L}}\{\boldsymbol{\Psi}_p\}(\tau, c, \mu) = \mathbf{0}, \tag{6.44}$$

where

$$\boldsymbol{\mathcal{L}}\{\boldsymbol{F}\}(\tau, c, \mu) = \boldsymbol{V}(c)\boldsymbol{F}(\tau, c, \mu) - \int_0^\infty\int_{-1}^1 \mathrm{e}^{-c'^2} f(\mu', \mu)\boldsymbol{\mathcal{K}}(c', \mu' : c, \mu)\boldsymbol{F}(\tau, c', \mu')c'^2\,\mathrm{d}\mu'\,\mathrm{d}c'. \tag{6.45}$$

We note from Ref. [26] that the kernel listed in Eq. (2.13) is such that

$$\boldsymbol{S}\boldsymbol{K}^{\mathrm{T}}(\boldsymbol{c} : \boldsymbol{c}') = \boldsymbol{K}(\boldsymbol{c}' : \boldsymbol{c})\boldsymbol{S}, \tag{6.46}$$

where

$$\boldsymbol{S} = \begin{bmatrix} c_2 & 0 \\ 0 & c_1 a_{1,2} \end{bmatrix}. \tag{6.47}$$

And so we can multiply Eq. (6.43) by

$$c^2\big(1-\mu^2\big)\mathrm{e}^{-c^2}\boldsymbol{\Psi}_p^{\mathrm{T}}(\tau, c, \mu)\boldsymbol{S}^{-1},$$

multiply Eq. (6.44) by

$$c^2\big(1-\mu^2\big)\mathrm{e}^{-c^2}\boldsymbol{\Psi}_t^{\mathrm{T}}(\tau, c, -\mu)\boldsymbol{S}^{-1},$$

integrate the resulting equations over all μ, over all c and over τ from 0 to τ_0, to find, after subtracting the two subsequently resulting equations, one from the other, and using Eq. (2.49),

$$\pi^{1/2}\int_0^{\tau_0}\boldsymbol{Q}_p^{\mathrm{T}}(\tau)\,\mathrm{d}\tau\begin{bmatrix} c_1 a_{1,2} \\ c_2 \end{bmatrix} = \int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}c^3\mu\big(1-\mu^2\big)\boldsymbol{\Psi}_p^{\mathrm{T}}(\tau_0, c, \mu)\boldsymbol{\Xi}\boldsymbol{\Psi}_t(\tau_0, c, -\mu)\,\mathrm{d}\mu\,\mathrm{d}c, \tag{6.48}$$

where

$$\boldsymbol{\Xi} = \begin{bmatrix} c_1 a_{1,2} & 0 \\ 0 & c_2 \end{bmatrix}, \tag{6.49}$$

and where Eq. (6.9c) is to be used. At this point we use Eqs. (6.2) and (6.31) in Eq. (6.48) and let τ_0 tend to infinity to find

$$[\,c_1 a_{1,2} \quad c_2\,]\int_0^\infty \boldsymbol{Q}_p(\tau)\,\mathrm{d}\tau = \beta - \boldsymbol{\varepsilon}_p^{\mathrm{T}}\boldsymbol{\Upsilon}\left[\boldsymbol{\zeta}_t + \frac{4}{3\pi^{1/2}}\int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{A}(c)c^3\,\mathrm{d}c\right], \tag{6.50}$$

where

$$\beta = \frac{8}{15\pi^{1/2}}\int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{B}^{\mathrm{T}}(c)\boldsymbol{\Upsilon}\boldsymbol{A}(c)c^3\,\mathrm{d}c \tag{6.51}$$

and

$$\boldsymbol{\Upsilon} = \begin{bmatrix} c_1(m_2/m)^{1/2} & 0 \\ 0 & c_2(m_1/m)^{1/2} \end{bmatrix}. \tag{6.52}$$

In addition, $\boldsymbol{\varepsilon}_p$ is as defined in Eq. (6.4), and

$$\boldsymbol{\zeta}_t = \begin{bmatrix} \zeta_{t,1} \\ \zeta_{t,2} \end{bmatrix}, \tag{6.53}$$

where $\zeta_{t,1}$ and $\zeta_{t,2}$ are as defined in Eqs. (6.33). While Eqs. (6.50) and (6.51) cannot be used to find both $\zeta_{t,1}$ and $\zeta_{t,2}$ from results derived from the viscous-slip problem, one of these quantities can be found in terms of the other. More importantly, however, Eqs. (6.50) and (6.51) can be used as a measure of the accuracy of numerical results found from solutions of the two problems: viscous slip and thermal creep. We note that β, as defined by Eq. (6.51), is independent of the accommodation coefficients α_1 and α_2.

6.3.2. A relationship between the viscous-slip and the diffusion-slip problems

The derivation reported in this section is very similar to that given in regard to the viscous-slip and thermal-creep problems, so we can (again) be very brief. If we add subscripts c for the diffusion-slip problem and p for the half-space viscous-slip problem, we find we can write

$$c\boldsymbol{N} - c\mu\frac{\partial}{\partial\tau}\boldsymbol{\Psi}_c(\tau, c, -\mu) + \boldsymbol{\mathcal{L}}\{\boldsymbol{\Psi}_c\}(\tau, c, -\mu) = \boldsymbol{0} \tag{6.54}$$

and

$$c\mu\frac{\partial}{\partial\tau}\boldsymbol{\Psi}_p(\tau, c, \mu) + \boldsymbol{\mathcal{L}}\{\boldsymbol{\Psi}_p\}(\tau, c, \mu) = \boldsymbol{0}, \tag{6.55}$$

where Eqs. (6.37) and (6.45) are to be used. We can multiply Eq. (6.54) by

$$c^2\big(1-\mu^2\big)\mathrm{e}^{-c^2}\boldsymbol{\Psi}_p^{\mathrm{T}}(\tau, c, \mu)\boldsymbol{S}^{-1},$$

multiply Eq. (6.55) by

$$c^2\big(1-\mu^2\big)\mathrm{e}^{-c^2}\boldsymbol{\Psi}_c^{\mathrm{T}}(\tau, c, -\mu)\boldsymbol{S}^{-1},$$

integrate the resulting equations over all μ, over all c and over τ from 0 to τ_0, to find, after subtracting the two subsequently resulting equations, one from the other, and using Eq. (2.49),

$$c_1c_2\pi^{1/2}\int_0^{\tau_0}\boldsymbol{U}_p^{\mathrm{T}}(\tau)\begin{bmatrix} a_{1,2} \\ -1 \end{bmatrix}\mathrm{d}\tau = \int_0^{\infty}\int_{-1}^{1}\mathrm{e}^{-c^2}c^3\mu\big(1-\mu^2\big)\boldsymbol{\Psi}_p^{\mathrm{T}}(\tau_0, c, \mu)\boldsymbol{\Xi}\boldsymbol{\Psi}_c(\tau_0, c, -\mu)\,\mathrm{d}\mu\,\mathrm{d}c, \tag{6.56}$$

where $\boldsymbol{\Xi}$ is given by Eq. (6.49), and where Eq. (6.9a) is to be used. At this point we can use Eqs. (6.2) and (6.39) in Eq. (6.56) and let τ_0 tend to infinity to find

$$c_1c_2\int_0^{\infty}[\,a_{1,2}\quad -1\,]\,\boldsymbol{U}_p(\tau)\,\mathrm{d}\tau = \gamma - \boldsymbol{\varepsilon}_p^{\mathrm{T}}\boldsymbol{\Upsilon}\left\{\boldsymbol{\zeta}_c + \frac{4}{3\pi^{1/2}}\int_0^{\infty}\mathrm{e}^{-c^2}\big[c_1\boldsymbol{A}^{(2)}(c) - c_2\boldsymbol{A}^{(1)}(c)\big]c^3\,\mathrm{d}c\right\}, \tag{6.57}$$

where

$$\gamma = \frac{8}{15\pi^{1/2}}\int_0^{\infty}\mathrm{e}^{-c^2}\boldsymbol{B}^{\mathrm{T}}(c)\boldsymbol{\Upsilon}\big[c_1\boldsymbol{A}^{(2)}(c) - c_2\boldsymbol{A}^{(1)}(c)\big]c^3\,\mathrm{d}c \tag{6.58}$$

and

$$\boldsymbol{\zeta}_c = \begin{bmatrix} \zeta_{c,1} \\ \zeta_{c,2} \end{bmatrix}, \tag{6.59}$$

where $\zeta_{c,1}$ and $\zeta_{c,2}$ are as defined in Eqs. (6.41). While Eqs. (6.57) and (6.58) cannot be used to find both $\zeta_{c,1}$ and $\zeta_{c,2}$ from results derived from the viscous-slip problem, one of these quantities can be found in terms of the other. More importantly, however, Eqs. (6.57) and (6.58) can be used as a measure of the accuracy of numerical results found from solutions of the two problems: viscous and diffusion slip.

6.3.3. Relationships between the thermal-creep and the diffusion-slip problems

Following the procedures used in Sections 6.3.1 and 6.3.2, we can deduce that

$$c_1c_2[a_{1,2} \quad -1]\boldsymbol{\zeta}_t = [c_1a_{1,2} \quad c_2]\boldsymbol{Q}_c(\infty) \tag{6.60a}$$

and

$$c_1c_2[a_{1,2} \quad -1]\int_0^\infty \big[\boldsymbol{U}_t(\tau) - \boldsymbol{\zeta}_t\big]\,\mathrm{d}\tau = [c_1a_{1,2} \quad c_2]\int_0^\infty \big[\boldsymbol{Q}_c(\tau) - \boldsymbol{Q}_c(\infty)\big]\,\mathrm{d}\tau, \tag{6.60b}$$

where Eqs. (6.33), (6.34), (6.40c), and (6.53) are to be used. While Eq. (6.60a) is a result we sought here, an identity involving the generalized Chapman–Enskog functions $\boldsymbol{A}^{(\alpha)}(c)$ can also be obtained when Eqs. (6.33), (6.34), (6.40c), and (6.53) are used with Eq. (6.60a). We find

$$\begin{aligned} &c_1c_2[a_{1,2} \quad -1]\int_0^\infty \mathrm{e}^{-c^2}\big[\boldsymbol{A}^{(1)}(c) + \boldsymbol{A}^{(2)}(c)\big]c^3\,\mathrm{d}c \\ &\quad = [c_1a_{1,2} \quad c_2]\int_0^\infty \mathrm{e}^{-c^2}\big[c_1\boldsymbol{A}^{(2)}(c) - c_2\boldsymbol{A}^{(1)}(c)\big]\left(c^2 - \frac{5}{2}\right)c^3\,\mathrm{d}c. \end{aligned} \tag{6.61}$$

As our solutions to the three considered problems are complete, we are ready to evaluate these solutions numerically for selected data cases.

7. Numerical results

To begin this section, we note that the computational implementation of our ADO solutions follows very closely that of our recent work on the temperature-jump problem [27]. Thus, for computational purposes, the kernel $\boldsymbol{\mathcal{K}}(c', \mu' : c, \mu)$ defined by Eq. (2.39) and the summation in Eq. (4.9) were both truncated at $n = L$ and, for consistency, the summations in Eq. (4.12) were truncated at $m = \lfloor L/2 \rfloor - 1$ and $m' = \lfloor L/2 \rfloor$, where $\lfloor x \rfloor$ denotes the floor (or integer part) of x. In addition, the Legendre components $\mathcal{K}_n(c', c)$ that are required in Eq. (4.11) were computed using a 200-point Gauss–Legendre quadrature set with the integration algorithms discussed in Appendix A of Ref. [26]. Along with the order M of the Gaussian quadrature used for integration over the speed variable in Eqs. (3.5a), (3.5b), (3.7), (4.11), (6.7), (6.51), and (6.58), the order K of the approximate representation of Eq. (3.1), with the choice

$$\Pi_k(c) = P_k\big(2\mathrm{e}^{-c} - 1\big),$$

the order N of the half-range Gaussian scheme introduced in Eqs. (4.16), and the number of spline functions K_s used to compute, without postprocessing [26], the generalized Chapman–Enskog vector functions $\boldsymbol{A}^{(1)}(c)$, $\boldsymbol{A}^{(2)}(c)$, and $\boldsymbol{B}(c)$, the kernel truncation parameter L defines the set of five approximation parameters

$$\{L, M, K, N, K_s\},$$

upon which our numerical results are based. We should also note that integrals involving only one of the generalized Chapman–Enskog vector functions were performed in this work as in Ref. [27], by applying a Gaussian quadrature of order four to each of the subintervals of integration defined by two consecutive knots.

In regard to computational linear algebra, we have used the sequence of EISPACK [33] routines BALANC, ELMHES, ELTRAN, HQR2, and BALBAK for solving the eigensystem defined by Eq. (4.16a) for $i = 1, 2, \ldots, N$ and subroutines DGECO and DGESL of the LINPACK package [34] for solving the systems of linear algebraic equations obtained for the superposition coefficients $\{A_j\}$ of the three problems, as discussed in Section 6. We note that we can write these three linear systems at once as

$$A_1\boldsymbol{\Pi}_1^{\mathrm{T}}\boldsymbol{\alpha}\boldsymbol{\Phi} + \sum_{j=2}^{J} A_j\big[\boldsymbol{F}\boldsymbol{\Phi}(\nu_j, \mu_i) - \boldsymbol{H}\boldsymbol{\Phi}(\nu_j, -\mu_i)\big] = \boldsymbol{R}(\mu_i), \tag{7.1}$$

for $i = 1, 2, \ldots, N$. Here, only the right-hand side vector $\boldsymbol{R}(\mu_i)$ of size $2(K+1)$ is problem-dependent. Using subscripts p, t, and c to identify the right-hand side vector respectively for the viscous-slip problem, for the thermal-creep problem, and for the diffusion-slip problem, we have

$$\boldsymbol{R}_p(\mu_i) = 2\left(\frac{m_1}{m}\right)^{1/2} \mu_i \int_0^\infty \mathrm{e}^{-c^2} c^2 \boldsymbol{\Pi}^{\mathrm{T}}(c)(2\boldsymbol{I} - \boldsymbol{\alpha})\boldsymbol{B}(c)\,\mathrm{d}c, \tag{7.2a}$$

$$\boldsymbol{R}_t(\mu_i) = \int_0^\infty \mathrm{e}^{-c^2} c^2 \boldsymbol{\Pi}^{\mathrm{T}}(c)\boldsymbol{\alpha}\boldsymbol{A}(c)\,\mathrm{d}c, \tag{7.2b}$$

and

$$\boldsymbol{R}_c(\mu_i) = \int_0^\infty \mathrm{e}^{-c^2} c^2 \boldsymbol{\Pi}^{\mathrm{T}}(c)\boldsymbol{\alpha}\big[c_1 \boldsymbol{A}^{(2)}(c) - c_2 \boldsymbol{A}^{(1)}(c)\big]\,\mathrm{d}c. \tag{7.2c}$$

It is thus clear that only one LU matrix factorization is sufficient for the three systems. We also note that, for some choices of the approximation parameters, a few eigenvalues of Eq. (4.16a) can appear in the calculation as complex conjugate pairs slightly off the real axis. This situation is resolved in our code without having to resort to programming in complex mode.

The numerical results reported in this work are based on two binary gas mixtures: Ne–Ar and He–Xe. For the Ne–Ar mixture, we use the basic data

$$m_2 = 39.948, \quad m_1 = 20.183, \quad \frac{d_2}{d_1} = 1.406,$$

while for the He–Xe mixture we use

$$m_2 = 131.30, \quad m_1 = 4.0026, \quad \frac{d_2}{d_1} = 2.226.$$

It should be noted that the values of the masses used here are taken from Ref. [17], while the diameter ratios are those reported in Ref. [14].

We report in Tables 1–5, for both of these mixtures, the viscous-slip, the thermal-slip, and the diffusion-slip coefficients computed with our method, for several values of c_1, the relative equilibrium concentration of the first species, and three different combinations of the accommodation coefficients for the first and second species at the wall. The numerical results in Tables 1–5 are thought to be correct to within ± 1 in the last digit and were obtained by increasing the values of the approximation parameters of our method in steps, until numerical convergence was observed. For this purpose, we have used $20 \leqslant L \leqslant 40$, $100 \leqslant M \leqslant 400$, $20 \leqslant K \leqslant 35$, $20 \leqslant N \leqslant 40$, and $80 \leqslant K_s - 2 \leqslant 1280$.

Table 1
The viscous-slip coefficient ζ_p

	Ne–Ar mixture			He–Xe mixture		
	$\alpha_1 = 0.3$ $\alpha_2 = 0.6$	$\alpha_1 = 0.4$ $\alpha_2 = 0.7$	$\alpha_1 = 1.0$ $\alpha_2 = 1.0$	$\alpha_1 = 0.3$ $\alpha_2 = 0.6$	$\alpha_1 = 0.4$ $\alpha_2 = 0.7$	$\alpha_1 = 1.0$ $\alpha_1 = 1.0$
c_1	ζ_p					
0.05	1.02260	8.20968(−1)	4.42845(−1)	9.93929(−1)	7.99122(−1)	4.39547(−1)
0.1	1.05230	8.43259(−1)	4.42380(−1)	9.94325(−1)	7.99261(−1)	4.35936(−1)
0.2	1.11723	8.91355(−1)	4.41544(−1)	9.98609(−1)	8.02167(−1)	4.29351(−1)
0.3	1.19095	9.44968(−1)	4.40852(−1)	1.00892	8.09579(−1)	4.23808(−1)
0.4	1.27573	1.00536	4.40329(−1)	1.02765	8.23240(−1)	4.19621(−1)
0.5	1.37465	1.07422	4.40008(−1)	1.05867	8.45920(−1)	4.17228(−1)
0.6	1.49207	1.15382	4.39931(−1)	1.10866	8.82313(−1)	4.17253(−1)
0.7	1.63430	1.24731	4.40150(−1)	1.19025	9.41073(−1)	4.20556(−1)
0.8	1.81086	1.35921	4.40729(−1)	1.33085	1.04023	4.28151(−1)
0.9	2.03681	1.49621	4.41755(−1)	1.60486	1.22546	4.39975(−1)
0.95	2.17561	1.57714	4.42468(−1)	1.86222	1.38965	4.45247(−1)

Table 2
The thermal-slip coefficient $\zeta_{t,1}$

	Ne–Ar mixture			He–Xe mixture		
	$\alpha_1 = 0.3$ $\alpha_2 = 0.6$	$\alpha_1 = 0.4$ $\alpha_2 = 0.7$	$\alpha_1 = 1.0$ $\alpha_2 = 1.0$	$\alpha_1 = 0.3$ $\alpha_2 = 0.6$	$\alpha_1 = 0.4$ $\alpha_2 = 0.7$	$\alpha_1 = 1.0$ $\alpha_1 = 1.0$
c_1	$\zeta_{t,1}$					
0.05	2.36503(−1)	2.40184(−1)	2.51134(−1)	2.29346(−1)	2.30227(−1)	2.32747(−1)
0.1	2.33394(−1)	2.37086(−1)	2.48521(−1)	2.24463(−1)	2.25325(−1)	2.27804(−1)
0.2	2.27397(−1)	2.31125(−1)	2.43646(−1)	2.14949(−1)	2.15781(−1)	2.18246(−1)
0.3	2.21707(−1)	2.25493(−1)	2.39268(−1)	2.05841(−1)	2.06658(−1)	2.09222(−1)
0.4	2.16341(−1)	2.20211(−1)	2.35429(−1)	1.97261(−1)	1.98084(−1)	2.00911(−1)
0.5	2.11319(−1)	2.15306(−1)	2.32186(−1)	1.89399(−1)	1.90263(−1)	1.93593(−1)
0.6	2.06666(−1)	2.10811(−1)	2.29613(−1)	1.82568(−1)	1.83527(−1)	1.87745(−1)
0.7	2.02411(−1)	2.06764(−1)	2.27806(−1)	1.77322(−1)	1.78475(−1)	1.84249(−1)
0.8	1.98581(−1)	2.03216(−1)	2.26895(−1)	1.74747(−1)	1.76298(−1)	1.84940(−1)
0.9	1.95199(−1)	2.00225(−1)	2.27058(−1)	1.77312(−1)	1.79794(−1)	1.94399(−1)
0.95	1.93679(−1)	1.98958(−1)	2.27617(−1)	1.82340(−1)	1.85833(−1)	2.06200(−1)

Table 3
The thermal-slip coefficient $\zeta_{t,2}$

	Ne–Ar mixture			He–Xe mixture		
	$\alpha_1 = 0.3$ $\alpha_2 = 0.6$	$\alpha_1 = 0.4$ $\alpha_2 = 0.7$	$\alpha_1 = 1.0$ $\alpha_2 = 1.0$	$\alpha_1 = 0.3$ $\alpha_2 = 0.6$	$\alpha_1 = 0.4$ $\alpha_2 = 0.7$	$\alpha_1 = 1.0$ $\alpha_1 = 1.0$
c_1	$\zeta_{t,2}$					
0.05	2.05171(−1)	2.10350(−1)	2.25754(−1)	2.02735(−1)	2.07781(−1)	2.22215(−1)
0.1	2.01813(−1)	2.07007(−1)	2.23095(−1)	1.96908(−1)	2.01842(−1)	2.16045(−1)
0.2	1.95327(−1)	2.00571(−1)	2.18187(−1)	1.85401(−1)	1.90161(−1)	2.04280(−1)
0.3	1.89153(−1)	1.94479(−1)	2.13859(−1)	1.74135(−1)	1.78810(−1)	1.93500(−1)
0.4	1.83297(−1)	1.88743(−1)	2.10152(−1)	1.63192(−1)	1.67907(−1)	1.84097(−1)
0.5	1.77768(−1)	1.83378(−1)	2.07125(−1)	1.52700(−1)	1.57647(−1)	1.76720(−1)
0.6	1.72571(−1)	1.78402(−1)	2.04854(−1)	1.42879(−1)	1.48370(−1)	1.72529(−1)
0.7	1.67710(−1)	1.73835(−1)	2.03437(−1)	1.34139(−1)	1.40739(−1)	1.73813(−1)
0.8	1.63177(−1)	1.69699(−1)	2.03011(−1)	1.27358(−1)	1.36239(−1)	1.85737(−1)
0.9	1.58940(−1)	1.66011(−1)	2.03762(−1)	1.24802(−1)	1.39017(−1)	2.22664(−1)
0.95	1.56909(−1)	1.64337(−1)	2.04656(−1)	1.27051(−1)	1.47061(−1)	2.63709(−1)

Table 4
The diffusion-slip coefficient $\zeta_{c,1}$

	Ne–Ar mixture			He–Xe mixture		
	$\alpha_1 = 0.3$ $\alpha_2 = 0.6$	$\alpha_1 = 0.4$ $\alpha_2 = 0.7$	$\alpha_1 = 1.0$ $\alpha_2 = 1.0$	$\alpha_1 = 0.3$ $\alpha_2 = 0.6$	$\alpha_1 = 0.4$ $\alpha_2 = 0.7$	$\alpha_1 = 1.0$ $\alpha_1 = 1.0$
c_1	$-\zeta_{c,1}$					
0.05	3.11893(−1)	3.10993(−1)	3.05877(−1)	3.81849(−1)	3.81546(−1)	3.79748(−1)
0.1	2.96294(−1)	2.94534(−1)	2.84735(−1)	3.57949(−1)	3.57355(−1)	3.53860(−1)
0.2	2.65338(−1)	2.61991(−1)	2.44161(−1)	3.12228(−1)	3.11091(−1)	3.04507(−1)
0.3	2.34609(−1)	2.29875(−1)	2.05858(−1)	2.69231(−1)	2.67601(−1)	2.58339(−1)
0.4	2.03956(−1)	1.98076(−1)	1.69817(−1)	2.28876(−1)	2.26802(−1)	2.15287(−1)
0.5	1.73171(−1)	1.66448(−1)	1.36026(−1)	1.91032(−1)	1.88559(−1)	1.75243(−1)
0.6	1.41962(−1)	1.34790(−1)	1.04467(−1)	1.55467(−1)	1.52641(−1)	1.38032(−1)
0.7	1.09901(−1)	1.02823(−1)	7.51194(−2)	1.21725(−1)	1.18602(−1)	1.03337(−1)
0.8	7.63442(−2)	7.01476(−2)	4.79530(−2)	8.87572(−2)	8.54351(−2)	7.05108(−2)
0.9	4.02773(−2)	3.61750(−2)	2.29300(−2)	5.34294(−2)	5.02635(−2)	3.79460(−2)
0.95	2.08124(−2)	1.84371(−2)	1.12072(−2)	3.16125(−2)	2.90980(−2)	2.04286(−2)

Table 5
The diffusion-slip coefficient $\zeta_{c,2}$

	Ne–Ar mixture			He–Xe mixture		
	$\alpha_1 = 0.3$ $\alpha_2 = 0.6$	$\alpha_1 = 0.4$ $\alpha_2 = 0.7$	$\alpha_1 = 1.0$ $\alpha_2 = 1.0$	$\alpha_1 = 0.3$ $\alpha_2 = 0.6$	$\alpha_1 = 0.4$ $\alpha_2 = 0.7$	$\alpha_1 = 1.0$ $\alpha_1 = 1.0$
c_1	$\zeta_{c,2}$					
0.05	8.45039(−3)	9.71652(−3)	1.69144(−2)	1.03574(−2)	1.20941(−2)	2.23910(−2)
0.1	1.69611(−2)	1.94367(−2)	3.32223(−2)	2.04741(−2)	2.38738(−2)	4.38908(−2)
0.2	3.42280(−2)	3.89369(−2)	6.40213(−2)	4.00843(−2)	4.65921(−2)	8.43059(−2)
0.3	5.19561(−2)	5.86160(−2)	9.24050(−2)	5.90896(−2)	6.84221(−2)	1.21473(−1)
0.4	7.03554(−2)	7.86269(−2)	1.18384(−1)	7.78968(−2)	8.97793(−2)	1.55731(−1)
0.5	8.97160(−2)	9.91742(−2)	1.41975(−1)	9.71832(−2)	1.11349(−1)	1.87616(−1)
0.6	1.10447(−1)	1.20537(−1)	1.63197(−1)	1.18170(−1)	1.34353(−1)	2.18029(−1)
0.7	1.33146(−1)	1.43103(−1)	1.82078(−1)	1.43307(−1)	1.61193(−1)	2.48625(−1)
0.8	1.58710(−1)	1.67428(−1)	1.98653(−1)	1.78339(−1)	1.97366(−1)	2.82844(−1)
0.9	1.88560(−1)	1.94332(−1)	2.12966(−1)	2.40666(−1)	2.58798(−1)	3.29346(−1)
0.95	2.05778(−1)	2.09120(−1)	2.19292(−1)	3.00413(−1)	3.14814(−1)	3.64468(−1)

Table 6
Viscous-slip problem: velocity, heat-flow and shear-stress profiles for the He–Xe mixture with $c_1 = 0.3$, $\alpha_1 = 0.3$, and $\alpha_2 = 0.6$

τ	$U_1(\tau)$	$U_2(\tau)$	$Q_1(\tau)$	$Q_2(\tau)$	$-P_1(\tau)$	$-P_2(\tau)$
0.0	2.3116(−1)	9.4111(−1)	7.4923(−3)	8.8403(−2)	5.2476(−2)	5.2017(−1)
0.1	2.5003(−1)	1.1897	6.0126(−3)	4.9819(−2)	6.7090(−2)	5.1391(−1)
0.2	2.6766(−1)	1.3496	4.9948(−3)	3.5255(−2)	7.7586(−2)	5.0941(−1)
0.3	2.8570(−1)	1.4919	4.2000(−3)	2.6009(−2)	8.5723(−2)	5.0593(−1)
0.4	3.0418(−1)	1.6260	3.5581(−3)	1.9574(−2)	9.2216(−2)	5.0314(−1)
0.5	3.2304(−1)	1.7555	3.0306(−3)	1.4904(−2)	9.7484(−2)	5.0088(−1)
0.6	3.4220(−1)	1.8820	2.5918(−3)	1.1432(−2)	1.0181(−1)	4.9903(−1)
0.7	3.6163(−1)	2.0065	2.2238(−3)	8.8118(−3)	1.0539(−1)	4.9750(−1)
0.8	3.8126(−1)	2.1297	1.9133(−3)	6.8132(−3)	1.0837(−1)	4.9622(−1)
0.9	4.0107(−1)	2.2519	1.6499(−3)	5.2773(−3)	1.1086(−1)	4.9515(−1)
1.0	4.2102(−1)	2.3733	1.4257(−3)	4.0907(−3)	1.1296(−1)	4.9425(−1)
2.0	6.2483(−1)	3.5709	3.5536(−4)	2.6841(−4)	1.2249(−1)	4.9017(−1)

Table 7
Thermal-creep problem: velocity, heat-flow and shear-stress profiles for the He–Xe mixture with $c_1 = 0.3$, $\alpha_1 = 0.3$, and $\alpha_2 = 0.6$

τ	$U_1(\tau)$	$U_2(\tau)$	$-Q_1(\tau)$	$-Q_2(\tau)$	$-P_1(\tau)$	$P_2(\tau)$
0.0	1.6685(−1)	8.6058(−2)	7.0316(−1)	4.4891(−1)	2.9330(−3)	1.2570(−3)
0.1	1.7926(−1)	1.2053(−1)	7.5050(−1)	5.6611(−1)	3.3504(−3)	1.4359(−3)
0.2	1.8553(−1)	1.3514(−1)	7.7256(−1)	6.1220(−1)	3.3008(−3)	1.4146(−3)
0.3	1.8984(−1)	1.4463(−1)	7.8699(−1)	6.4050(−1)	3.0878(−3)	1.3234(−3)
0.4	1.9303(−1)	1.5136(−1)	7.9724(−1)	6.5965(−1)	2.8113(−3)	1.2048(−3)
0.5	1.9546(−1)	1.5632(−1)	8.0485(−1)	6.7325(−1)	2.5174(−3)	1.0789(−3)
0.6	1.9737(−1)	1.6007(−1)	8.1066(−1)	6.8322(−1)	2.2292(−3)	9.5536(−4)
0.7	1.9888(−1)	1.6296(−1)	8.1518(−1)	6.9068(−1)	1.9584(−3)	8.3931(−4)
0.8	2.0010(−1)	1.6520(−1)	8.1874(−1)	6.9636(−1)	1.7104(−3)	7.3301(−4)
0.9	2.0109(−1)	1.6696(−1)	8.2159(−1)	7.0072(−1)	1.4870(−3)	6.3728(−4)
1.0	2.0189(−1)	1.6835(−1)	8.2389(−1)	7.0412(−1)	1.2881(−3)	5.5205(−4)
2.0	2.0514(−1)	1.7337(−1)	8.3299(−1)	7.1584(−1)	2.7563(−4)	1.1813(−4)

In addition to the slip coefficients reported in Tables 1–5, we report in Tables 6–8 the velocity, heat-flow, and shear-stress profiles for the considered problems, for one of the studied He–Xe cases. These results are also thought to be correct to within ± 1 in the last reported digit and were obtained using the same range of values for the approximation parameters as above, except the upper limit of L, which was extended to 100 for these calculations. The reason for this extension is the fact that we have found that the entries that correspond to $Q_2(0)$ in Tables 6 and 8 require more

Table 8
Diffusion-slip problem: velocity, heat-flow and shear-stress profiles for the He–Xe mixture with $c_1 = 0.3$, $\alpha_1 = 0.3$, and $\alpha_2 = 0.6$

τ	$-U_1(\tau)$	$U_2(\tau)$	$Q_1(\tau)$	$Q_2(\tau)$	$P_1(\tau)$	$-P_2(\tau)$
0.0	2.2768(−1)	3.4232(−2)	1.0290(−1)	1.8802(−4)	4.0753(−2)	1.7465(−2)
0.1	2.4279(−1)	4.2934(−2)	1.1026(−1)	−1.4997(−3)	3.1748(−2)	1.3606(−2)
0.2	2.4980(−1)	4.6755(−2)	1.1361(−1)	−1.7065(−3)	2.5420(−2)	1.0894(−2)
0.3	2.5436(−1)	4.9303(−2)	1.1577(−1)	−1.6374(−3)	2.0639(−2)	8.8455(−3)
0.4	2.5758(−1)	5.1161(−2)	1.1728(−1)	−1.4677(−3)	1.6918(−2)	7.2506(−3)
0.5	2.5997(−1)	5.2579(−2)	1.1839(−1)	−1.2626(−3)	1.3966(−2)	5.9856(−3)
0.6	2.6179(−1)	5.3692(−2)	1.1923(−1)	−1.0514(−3)	1.1595(−2)	4.9691(−3)
0.7	2.6319(−1)	5.4582(−2)	1.1987(−1)	−8.4819(−4)	9.6702(−3)	4.1444(−3)
0.8	2.6430(−1)	5.5303(−2)	1.2037(−1)	−6.5945(−4)	8.0965(−3)	3.4699(−3)
0.9	2.6518(−1)	5.5894(−2)	1.2077(−1)	−4.8785(−4)	6.8017(−3)	2.9150(−3)
1.0	2.6589(−1)	5.6382(−2)	1.2108(−1)	−3.3396(−4)	5.7306(−3)	2.4560(−3)
2.0	2.6865(−1)	5.8505(−2)	1.2228(−1)	4.7335(−4)	1.1502(−3)	4.9293(−4)

terms than the others in the scattering kernel expansion of Eq. (2.39) to be determined with the desired five-figure accuracy.

We note that a useful identity related to the x–z component of the shear-stress profile for the gas mixture,

$$p(\tau) = p_0\big[c_1 P_1(\tau) + c_2 P_2(\tau)\big], \tag{7.3}$$

where $P_\alpha(\tau)$, $\alpha = 1, 2$, is the shear-stress profile for species α, as given by Eq. (2.28) or Eq. (2.34), and

$$p_0 = nkT_0 \tag{7.4}$$

is the equilibrium pressure, can be derived and used to check the accuracy of the numerical results obtained for the shear-stress profiles. Multiplying Eq. (2.36), (6.27), or (6.36) by

$$[c_1 \quad c_2]\mathrm{e}^{-c^2}\big(1-\mu^2\big)c^3$$

and integrating over all μ and all c, we find, after using Eq. (6.46) and (from Ref. [26])

$$\left\{c\boldsymbol{V}(c) - \int_0^\infty \mathrm{e}^{-c'^2}\boldsymbol{\mathcal{K}}_1(c', c)c'^3\,\mathrm{d}c'\right\}\begin{bmatrix}1\\ a_{1,2}\end{bmatrix} = \boldsymbol{0}, \tag{7.5}$$

that

$$P_0 = [c_1 \quad c_2]\boldsymbol{P}(\tau) \tag{7.6}$$

is a constant. And so we deduce from Eq. (6.9b) that

$$P_0 = -[c_1 \quad c_2]\left(\frac{m_1}{m}\right)^{1/2}\boldsymbol{\varepsilon}_p \tag{7.7}$$

for the viscous-slip problem. Similarly, we conclude from Eqs. (6.32b) and (6.40b) that P_0 is zero for the thermal-creep and the diffusion-slip problems.

Typically, we have found that our solutions (for all three of the considered problems) obtained with $L = 50$, $M = 100$, $K = 20$, $N = 20$, and $K_s = 82$, yield at least five figures of accuracy for the slip coefficients and four figures for the profiles, except possibly at positions located in the neighborhoods of the zeros of these quantities (when zeros exist), and require less than a minute of CPU time on an AMD Athlon 64 3200+ machine running at 2 GHz. We report in Table 9 results of our evaluation, for the considered mixtures, of the constants β and γ defined by Eqs. (6.51) and (6.58).

To increase our level of confidence in the correctness of our computational implementation, we have considered additional test cases for which numerical results are available in the literature. While we have found the work reported in Refs. [21,22] to be very important, we consider it unfortunate that the numerical results reported in those works are limited to the special case of pure diffuse reflection at the bounding surface, to the special case of equal-diameter particles, and to the special case of unit mass for the lighter of the two particle species. In order to make some

Table 9
The constants β and γ

	Ne–Ar mixture		He–Xe mixture	
c_1	β	γ	β	γ
0.05	1.221262(−1)	9.666591(−4)	1.199337(−1)	2.807231(−3)
0.1	1.226397(−1)	1.802326(−3)	1.184587(−1)	5.161078(−3)
0.2	1.240895(−1)	3.099505(−3)	1.166436(−1)	8.615523(−3)
0.3	1.261612(−1)	3.929240(−3)	1.166506(−1)	1.056734(−2)
0.4	1.289443(−1)	4.329468(−3)	1.190599(−1)	1.120242(−2)
0.5	1.325572(−1)	4.338643(−3)	1.248834(−1)	1.068357(−2)
0.6	1.371586(−1)	3.996221(−3)	1.360234(−1)	9.143439(−3)
0.7	1.429646(−1)	3.343468(−3)	1.564641(−1)	6.677131(−3)
0.8	1.502735(−1)	2.424839(−3)	1.960454(−1)	3.359170(−3)
0.9	1.595074(−1)	1.290379(−3)	2.874894(−1)	−4.928570(−4)
0.95	1.650274(−1)	6.603696(−4)	3.942041(−1)	−1.914827(−3)

comparisons with the numerical results reported in Refs. [21,22], we first must take note of the differing choices of mean-free path used by the Kyoto group [21,22] and that used in this work. If we let τ_K denote the dimensionless spatial variable used in Refs. [21,22], then we find that

$$\xi_K = \frac{\tau}{\tau_K} \tag{7.8}$$

can be computed from

$$\xi_K = 2^{1/2}\left(\frac{c_1 d_1 + c_2 d_2}{2 d_1}\right)^2. \tag{7.9}$$

Thus, upon dividing our converged numerical results for the viscous-slip coefficient by ξ_K, we found agreement (within ± 1 in the fifth figure) with the five-figure results reported in Table II of Ref. [21]. We note that since the species-dependent viscous-slip coefficients we defined by Eq. (6.15) are the same for each species and since the normalization used in Ref. [21] for this problem was also used here, the comparison with the coefficient b of Ref. [21] was easily made, viz.

$$b = \frac{\zeta_p}{\xi_K}. \tag{7.10}$$

For the thermal-creep and diffusion-slip problems, we found that the coefficients b_{I} and b_{II} listed respectively in Tables III and IV of Ref. [22] are given, in our notation, by

$$b_{\mathrm{I}} = \frac{c_1 (m_1/m)^{1/2} \zeta_{t,1} + c_2 (m_2/m)^{1/2} \zeta_{t,2}}{\xi_K (c_1 m_1 + c_2 m_2)^{1/2}} \tag{7.11}$$

and

$$b_{\mathrm{II}} = \frac{c_1 (m_1/m)^{1/2} \zeta_{c,1} + c_2 (m_1/m)^{1/2} \zeta_{c,2}}{c_1 c_2 \xi_K (c_1 m_1 + c_2 m_2)^{1/2}}, \tag{7.12}$$

where $\zeta_{t,\alpha}$ and $\zeta_{c,\alpha}$, for $\alpha = 1, 2$, are the species-dependent slip coefficients defined by Eqs. (6.33) and (6.41). Some comments about Eqs. (7.11) and (7.12) can be made. First of all the factor ξ_K appears in Eqs. (7.11) and (7.12) in order to take into account the different mean-free paths (and thus different temperature and/or concentration gradients) used by the Kyoto group and here. The factor $1/(c_1 c_2)$ appears in Eq. (7.12) because of the differing (here and in Ref. [22]) ways the driving term for the diffusion-slip problem is normalized. If we use

$$\hat{u}_*(z) = \hat{u}(z)/v_0 \tag{7.13}$$

to denote our reduced bulk velocity, and subscripts t and c to distinguish between thermal-creep and diffusion-slip quantities, then Eqs. (7.11) and (7.12) can be expressed as

$$b_{\mathrm{I}} = \frac{\hat{u}_{*,t}(\infty)}{\xi_K (c_1 m_1 + c_2 m_2)^{1/2}} \tag{7.14}$$

and

$$b_{\rm II} = \frac{\hat{u}_{*,c}(\infty)}{c_1 c_2 \xi_K (c_1 m_1 + c_2 m_2)^{1/2}}. \tag{7.15}$$

We note that, while all of our slip coefficients and our reduced bulk velocity depend only on the ratios

$$\frac{n_1}{n_2}, \quad \frac{m_1}{m_2}, \quad \frac{d_1}{d_2},$$

the coefficients $b_{\rm I}$ and $b_{\rm II}$ depend also on the individual masses m_1 and m_2. We can report that we have used Eqs. (7.11) and (7.12) to confirm (to within ± 1 in the last digit shown) all of the (four-digit) entries listed in Tables III and IV of Ref. [22].

In order to have independent checks on our velocity and heat-flow profiles, we have used our solutions to recompute the Knudsen-layer functions tabulated in Refs. [21] and [22]. To be specific: in regard to Tables IV and V of Ref. [21], we found a maximum difference of 1 unit in the last reported digit, with respect to Tables V and VI of Ref. [22] we found maximum differences of 4 and 3 units, respectively, in the last reported digits, and for Tables IX and X of Ref. [22] we found a maximum difference of 5 units in the last reported digit. We found that the Knudsen-layer functions $U^\alpha(\tau_K)$ and $H^\alpha(\tau_K)$ of Refs. [21] and [22], where τ_K is the dimensionless spatial variable used (and denoted as x_1) in these works, can be computed from our solutions by

$$U^\alpha(\tau_K) = f^\alpha \sum_{j=2}^{J} A_j \mathcal{U}_{j,\alpha} {\rm e}^{-\xi_K \tau_K/\nu_j} \tag{7.16a}$$

and

$$H^\alpha(\tau_K)/c_\alpha = f^\alpha \sum_{j=2}^{J} A_j \mathcal{Q}_{j,\alpha} {\rm e}^{-\xi_K \tau_K/\nu_j}. \tag{7.16b}$$

Here $\alpha = 1, 2$ (A, B in the notation of Refs. [21] and [22]), the problem-independent quantities $\{\mathcal{U}_{j,\alpha}\}$ and $\{\mathcal{Q}_{j,\alpha}\}$ are the components of the vectors $\{\boldsymbol{\mathcal{U}}_j\}$ and $\{\boldsymbol{\mathcal{Q}}_j\}$ defined by Eqs. (6.5a) and (6.5c), and the coefficients $\{A_j\}$ are found for each problem by solving the linear system defined by Eq. (7.1) and the corresponding right-hand side vector, i.e. Eq. (7.2a) or Eq. (7.2b) or Eq. (7.2c). Making use again of the subscripts p, t, and c to distinguish between problems, we write the conversion factor f^α used in Eqs. (7.16) as

$$f_p^\alpha = \left(\frac{m}{m_\alpha}\right)^{1/2} \xi_K^{-1}, \tag{7.17a}$$

$$f_t^\alpha = \left(\frac{m_\alpha}{m_1}\right)^{1/2} \xi_K^{-1}, \tag{7.17b}$$

and

$$f_c^\alpha = \left(\frac{m_\alpha}{m_1}\right)^{1/2} (c_1 c_2 \xi_K)^{-1}, \tag{7.17c}$$

respectively for the viscous-slip, the thermal-creep, and the diffusion-slip problems.

We have also compared our numerical results with those based on the McCormack model and reported in Ref. [17] for the viscous-slip and thermal-creep problems. For this purpose, we must first take note of the differing definitions of the mean-free path used in this work and in Ref. [17]. Recalling that we use τ, as defined by Eqs. (2.30) and (2.31), to denote our dimensionless spatial variable and using τ_M for that of Ref. [17], we find that the relationship between these two quantities,

$$\xi_M = \frac{\tau}{\tau_M}, \tag{7.18}$$

can be computed from

$$\xi_M = \frac{c_2[\Upsilon_1 + X_{2,1}^{(4)}] + c_1[\Upsilon_2 + X_{1,2}^{(4)}]}{\Upsilon_1 \Upsilon_2 - X_{1,2}^{(4)} X_{2,1}^{(4)}}, \tag{7.19}$$

Table 10
A comparison: slip coefficients (relative to τ_M) for the He–Xe mixture with $\alpha_1 = 1.0$ and $\alpha_2 = 1.0$

	McCormack			LBE		
c_1	ζ_p	$\zeta_{t,1}$	$\zeta_{t,2}$	ζ_p	$\zeta_{t,1}$	$\zeta_{t,2}$
0.0	1.018	4.423(−1)	5.874(−1)	1.003	5.380(−1)	5.171(−1)
0.1	1.057	4.546(−1)	6.049(−1)	1.043	5.450(−1)	5.168(−1)
0.2	1.099	4.682(−1)	6.256(−1)	1.087	5.523(−1)	5.170(−1)
0.3	1.145	4.836(−1)	6.507(−1)	1.135	5.601(−1)	5.180(−1)
0.4	1.197	5.011(−1)	6.820(−1)	1.187	5.685(−1)	5.209(−1)
0.5	1.253	5.215(−1)	7.226(−1)	1.245	5.777(−1)	5.273(−1)
0.6	1.313	5.458(−1)	7.776(−1)	1.306	5.878(−1)	5.402(−1)
0.7	1.375	5.755(−1)	8.563(−1)	1.368	5.994(−1)	5.654(−1)
0.8	1.424	6.127(−1)	9.764(−1)	1.417	6.122(−1)	6.148(−1)
0.9	1.413	6.546(−1)	1.168	1.404	6.203(−1)	7.104(−1)
1.0	1.018	5.874(−1)	1.286	1.003	5.171(−1)	7.751(−1)

Table 11
A comparison: velocity and heat-flow profiles for the viscous-slip problem ($\tau \Rightarrow \tau_M$) for the He–Xe mixture with $c_1 = 0.3$, $\alpha_1 = 1.0$, and $\alpha_2 = 1.0$

	McCormack				LBE			
τ	$U_1(\tau)$	$U_2(\tau)$	$Q_1(\tau)$	$Q_2(\tau)$	$U_1(\tau)$	$U_2(\tau)$	$Q_1(\tau)$	$Q_2(\tau)$
0.0	9.134(−1)	4.466	2.174(−1)	1.043	9.974(−1)	4.545	5.860(−2)	7.431(−1)
0.1	1.072	5.530	1.785(−1)	8.181(−1)	1.150	5.740	5.239(−2)	5.763(−1)
0.2	1.201	6.351	1.575(−1)	7.005(−1)	1.274	6.581	4.786(−2)	4.942(−1)
0.3	1.321	7.102	1.418(−1)	6.146(−1)	1.390	7.333	4.405(−2)	4.334(−1)
0.4	1.436	7.812	1.292(−1)	5.466(−1)	1.501	8.039	4.071(−2)	3.845(−1)
0.5	1.549	8.497	1.187(−1)	4.906(−1)	1.610	8.715	3.775(−2)	3.436(−1)
0.6	1.659	9.163	1.097(−1)	4.432(−1)	1.717	9.371	3.509(−2)	3.086(−1)
0.7	1.767	9.815	1.018(−1)	4.025(−1)	1.822	1.001(1)	3.268(−2)	2.782(−1)
0.8	1.874	1.046(1)	9.479(−2)	3.671(−1)	1.926	1.064(1)	3.049(−2)	2.516(−1)
0.9	1.980	1.109(1)	8.855(−2)	3.359(−1)	2.030	1.126(1)	2.848(−2)	2.281(−1)
1.0	2.086	1.171(1)	8.293(−2)	3.083(−1)	2.133	1.188(1)	2.663(−2)	2.071(−1)
2.0	3.117	1.774(1)	4.686(−2)	1.448(−1)	3.142	1.782(1)	1.424(−2)	8.423(−2)

where

$$\Upsilon_1 = X^{(3)}_{1,1} + X^{(3)}_{1,2} - X^{(4)}_{1,1}, \tag{7.20a}$$

$$\Upsilon_2 = X^{(3)}_{2,2} + X^{(3)}_{2,1} - X^{(4)}_{2,2}, \tag{7.20b}$$

$$X^{(3)}_{\alpha,\beta} = \left(\frac{10}{3} + \frac{2m_\beta}{m_\alpha}\right) F_{\alpha,\beta}, \tag{7.21a}$$

and

$$X^{(4)}_{\alpha,\beta} = \frac{4}{3} F_{\alpha,\beta}, \tag{7.21b}$$

with

$$F_{\alpha,\beta} = \frac{2c_\beta m_\alpha}{5m_\beta} \left(\frac{m_\beta}{m_\alpha + m_\beta}\right)^{3/2} \left(\frac{c_1 m_1 + c_2 m_2}{m_\alpha}\right)^{1/2} \left(\frac{d_\alpha + d_\beta}{c_1 d_1 + c_2 d_2}\right)^2. \tag{7.22}$$

And so, to compare our results for the viscous-slip and thermal-slip coefficients with the equivalent results of Ref. [17], we find that we must divide our results by the ξ_M factor defined by Eq. (7.19). Having done this conversion for all of the Ne–Ar and He–Xe cases studied in Ref. [17], we found that the relative deviations of the McCormack results for the viscous-slip coefficient with respect to our results are generally small and reach a maximum value of 1.5% for both the Ne–Ar and He–Xe mixtures. For the thermal-slip coefficients, the maximum deviations are much larger: 24% for the Ne–Ar mixture and 66% for the He–Xe mixture. As an example of the kind of results we obtained,

Table 12
A comparison: velocity and heat-flow profiles for the thermal-creep problem ($\tau \Rightarrow \tau_M$) for the He–Xe mixture with $c_1 = 0.3$, $\alpha_1 = 1.0$, and $\alpha_2 = 1.0$

	McCormack				LBE			
τ	$U_1(\tau)$	$U_2(\tau)$	$-Q_1(\tau)$	$-Q_2(\tau)$	$U_1(\tau)$	$U_2(\tau)$	$-Q_1(\tau)$	$-Q_2(\tau)$
0.0	1.647(−1)	1.533(−1)	9.407(−1)	8.484(−1)	2.538(−1)	1.459(−1)	1.086	7.987(−1)
0.1	2.268(−1)	2.246(−1)	1.137	1.070	3.070(−1)	2.196(−1)	1.302	1.062
0.2	2.626(−1)	2.681(−1)	1.248	1.195	3.392(−1)	2.601(−1)	1.429	1.200
0.3	2.897(−1)	3.023(−1)	1.330	1.287	3.640(−1)	2.909(−1)	1.526	1.301
0.4	3.114(−1)	3.308(−1)	1.396	1.360	3.844(−1)	3.158(−1)	1.604	1.381
0.5	3.295(−1)	3.554(−1)	1.450	1.420	4.016(−1)	3.367(−1)	1.669	1.446
0.6	3.448(−1)	3.770(−1)	1.495	1.470	4.164(−1)	3.546(−1)	1.724	1.500
0.7	3.580(−1)	3.961(−1)	1.534	1.513	4.293(−1)	3.702(−1)	1.772	1.546
0.8	3.695(−1)	4.133(−1)	1.568	1.550	4.407(−1)	3.839(−1)	1.814	1.585
0.9	3.796(−1)	4.289(−1)	1.598	1.583	4.508(−1)	3.960(−1)	1.851	1.620
1.0	3.886(−1)	4.430(−1)	1.624	1.611	4.598(−1)	4.067(−1)	1.884	1.650
2.0	4.407(−1)	5.356(−1)	1.776	1.772	5.143(−1)	4.704(−1)	2.077	1.815

we report in Table 10 a comparison between the McCormack and LBE slip coefficients for various choices of the relative He equilibrium concentration in the He–Xe mixture and complete accommodation for both species at the half-space boundary. Finally, we believe it important to note that, while we have found reasonable agreement (relative differences $<10\%$) between the McCormack and LBE velocity profiles for the viscous-slip problem, we have also found that the agreement between the heat-flow profiles for this problem is poor, with relative differences that usually exceed 40% and can even be larger than 100% when the masses of the species are very different, as is the case of the He–Xe mixture. With regard to the thermal-creep problem, we have found that the relative differences between the McCormack and LBE profiles (for both the velocity and heat flow) vary, in general, from a few percent up to 40%. These observations are illustrated in Tables 11 and 12 for a typical (but different from those of Ref. [17]) He–Xe case.

8. Concluding remarks

We have reported in this work what we believe to be concise and accurate solutions for the viscous-slip, the thermal-creep, and the diffusion-slip problems, as described by the (vector) linearized Boltzmann equation for a binary mixture of rigid spheres.

In addition to the comparisons with numerical results of other works for binary mixtures that are reported in Section 7, we have also performed comparisons with the single-gas LBE results of Ref. [35], using three different ways of achieving the single-gas limit in our formulation:

$$\text{(i) } c_1 = 0, \quad \text{(ii) } c_2 = 0, \quad \text{or} \quad \text{(iii) } m_1 = m_2,\ d_1 = d_2, \text{ and } \alpha_1 = \alpha_2.$$

We note that to convert our results to the same spatial units used in Ref. [35] we made use of the factors

$$\xi_{S,p} = 0.449027806\ldots \quad \text{and} \quad \xi_{S,t} = 0.679630049\ldots,$$

which (for a single-species case) are the ratios between our dimensionless spatial variable, as defined by Eqs. (2.30) and (2.31), and those of Ref. [35]. Doing this, we found good but not perfect agreement with the six-figure results for the slip coefficients and the five-figure results for the velocity and heat-flow profiles that are tabulated in Ref. [35]. While we found at most a difference of one unit in the sixth digit listed for the slip coefficients, we did find a maximum difference of 8 units in the fifth digit listed in Ref. [35] for the velocity and heat-flow profiles. We have confirmed that the (very slight) loss of accuracy in Tables 1–5 of Ref. [35] was due to using $L = 8$ in those computations. To make available our current results (based on $L = 40$), we list in Tables 13–17 improved versions of Tables 1–5 of Ref. [35]. To be clear, we note that in Ref. [35] the mean-free path for the viscous-slip problem was defined in terms of viscosity (hence the conversion factor $\xi_{S,p}$ was used), while for the thermal-creep problem a mean-free path based on thermal conductivity (and the factor $\xi_{S,t}$) was used. To be consistent, we have made use these same [35] definitions in our Tables 13–17. We also note that the constant β defined by Eq. (6.51) of this work can be expressed, in the single-gas limit, as

Table 13
Single-species gas [35]: the slip coefficients

α	ζ_P	ζ_T
0.1	1.70478(1)	2.65765(−1)
0.2	8.17247	2.74450(−1)
0.3	5.20563	2.82899(−1)
0.4	3.71609	2.91124(−1)
0.5	2.81761	2.99133(−1)
0.6	2.21478	3.06938(−1)
0.7	1.78098	3.14546(−1)
0.8	1.45291	3.21968(−1)
0.9	1.19540	3.29210(−1)
1.0	9.87328(−1)	3.36280(−1)

Table 14
Single-species gas [35]: velocity profile $u_P(\tau)$ for Kramers' problem

τ	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$
0.0	1.6472(1)	4.7033	2.3851	1.4151	8.9297(−1)	7.1555(−1)
0.1	1.6771(1)	4.9753	2.6322	1.6386	1.0943	9.0630(−1)
0.2	1.6956(1)	5.1494	2.7960	1.7925	1.2389	1.0463
0.3	1.7111(1)	5.2981	2.9380	1.9281	1.3684	1.1729
0.4	1.7252(1)	5.4335	3.0685	2.0541	1.4899	1.2922
0.5	1.7383(1)	5.5606	3.1920	2.1740	1.6064	1.4071
0.6	1.7507(1)	5.6821	3.3105	2.2898	1.7195	1.5189
0.7	1.7627(1)	5.7994	3.4255	2.4025	1.8301	1.6284
0.8	1.7743(1)	5.9137	3.5379	2.5130	1.9388	1.7363
0.9	1.7857(1)	6.0255	3.6482	2.6218	2.0461	1.8429
1.0	1.7968(1)	6.1356	3.7568	2.7292	2.1523	1.9484
2.0	1.9023(1)	7.1839	4.7987	3.7649	3.1820	2.9752

Table 15
Single-species gas [35]: heat-flow profile $q_P(\tau)$ for Kramers' problem

τ	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$
0.0	2.3951(−1)	2.0553(−1)	1.7413(−1)	1.4506(−1)	1.1811(−1)	1.0537(−1)
0.1	1.9023(−1)	1.6449(−1)	1.4042(−1)	1.1786(−1)	9.6679(−2)	8.6566(−2)
0.2	1.6365(−1)	1.4191(−1)	1.2148(−1)	1.0225(−1)	8.4100(−2)	7.5405(−2)
0.3	1.4360(−1)	1.2475(−1)	1.0699(−1)	9.0208(−2)	7.4326(−2)	6.6698(−2)
0.4	1.2736(−1)	1.1079(−1)	9.5138(−2)	8.0319(−2)	6.6262(−2)	5.9498(−2)
0.5	1.1373(−1)	9.9038(−2)	8.5133(−2)	7.1945(−2)	5.9410(−2)	5.3371(−2)
0.6	1.0206(−1)	8.8950(−2)	7.6525(−2)	6.4722(−2)	5.3488(−2)	4.8069(−2)
0.7	9.1925(−2)	8.0175(−2)	6.9023(−2)	5.8416(−2)	4.8308(−2)	4.3428(−2)
0.8	8.3041(−2)	7.2470(−2)	6.2426(−2)	5.2863(−2)	4.3740(−2)	3.9332(−2)
0.9	7.5195(−2)	6.5656(−2)	5.6585(−2)	4.7940(−2)	3.9686(−2)	3.5695(−2)
1.0	6.8225(−2)	5.9597(−2)	5.1386(−2)	4.3554(−2)	3.6070(−2)	3.2449(−2)
2.0	2.7542(−2)	2.4123(−2)	2.0852(−2)	1.7718(−2)	1.4709(−2)	1.3248(−2)

$$\beta = \left(\frac{m_2}{m}\right)^{1/2} \xi_{S,p}\xi_{S,t}\beta_S, \quad \text{single-gas case,} \tag{8.1}$$

where

$$\beta_S = 0.398935128\ldots \tag{8.2}$$

was reported in Ref. [31].

Finally, we note that measurements of viscous-slip and thermal-slip coefficients for binary-gas mixtures have been made available by Loyalka and co-workers [36,37].

Table 16
Single-species gas [35]: velocity profile $u_T(\tau)$ for the thermal-creep problem

τ	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$
0.0	2.3877(−1)	2.0478(−1)	1.7338(−1)	1.4433(−1)	1.1742(−1)	1.0471(−1)
0.1	2.4634(−1)	2.2627(−1)	2.0732(−1)	1.8939(−1)	1.7238(−1)	1.6421(−1)
0.2	2.5011(−1)	2.3715(−1)	2.2478(−1)	2.1296(−1)	2.0163(−1)	1.9614(−1)
0.3	2.5279(−1)	2.4491(−1)	2.3731(−1)	2.2996(−1)	2.2283(−1)	2.1935(−1)
0.4	2.5484(−1)	2.5089(−1)	2.4699(−1)	2.4312(−1)	2.3931(−1)	2.3741(−1)
0.5	2.5648(−1)	2.5567(−1)	2.5474(−1)	2.5369(−1)	2.5256(−1)	2.5196(−1)
0.6	2.5781(−1)	2.5957(−1)	2.6108(−1)	2.6236(−1)	2.6344(−1)	2.6391(−1)
0.7	2.5892(−1)	2.6282(−1)	2.6636(−1)	2.6957(−1)	2.7250(−1)	2.7387(−1)
0.8	2.5985(−1)	2.6554(−1)	2.7079(−1)	2.7564(−1)	2.8013(−1)	2.8225(−1)
0.9	2.6064(−1)	2.6784(−1)	2.7454(−1)	2.8078(−1)	2.8660(−1)	2.8937(−1)
1.0	2.6131(−1)	2.6981(−1)	2.7774(−1)	2.8516(−1)	2.9212(−1)	2.9544(−1)
2.0	2.6456(−1)	2.7935(−1)	2.9333(−1)	3.0657(−1)	3.1912(−1)	3.2516(−1)

Table 17
Single-species gas [35]: heat-flow profile $-q_T(\tau)$ for the thermal-creep problem

τ	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$
0.0	1.1662	1.0071	8.5848(−1)	7.1948(−1)	5.8923(−1)	5.2716(−1)
0.1	1.1930	1.0838	9.8034(−1)	8.8227(−1)	7.8910(−1)	7.4423(−1)
0.2	1.2056	1.1202	1.0390	9.6162(−1)	8.8780(−1)	8.5212(−1)
0.3	1.2141	1.1450	1.0791	1.0162	9.5599(−1)	9.2682(−1)
0.4	1.2205	1.1635	1.1090	1.0569	1.0070	9.8276(−1)
0.5	1.2254	1.1778	1.1322	1.0886	1.0467	1.0264
0.6	1.2293	1.1891	1.1507	1.1139	1.0784	1.0612
0.7	1.2324	1.1984	1.1657	1.1344	1.1042	1.0896
0.8	1.2350	1.2059	1.1781	1.1513	1.1255	1.1129
0.9	1.2371	1.2122	1.1883	1.1653	1.1432	1.1324
1.0	1.2389	1.2175	1.1969	1.1771	1.1580	1.1487
2.0	1.2472	1.2418	1.2365	1.2315	1.2266	1.2242

Acknowledgements

The authors would like to thank A.M. Yacout for his help with the computer systems used to perform the calculations reported in this work.

Appendix A. Macroscopic quantities for mixtures

In Appendix A of Ref. [27], we did a review of the way we followed Ferziger and Kaper [3] to define relevant macroscopic quantities for each species and for a binary mixture, as they applied to the temperature-jump problem where the temperature, density, velocity, and heat flow (in the transverse direction z) varied with z. Here, we continue that work [27] and define the velocity and heat flow (as functions of z) in direction of the flow x. We also list here the basic definition of the shear stress defined by the x and z directions.

A.1. Single-species quantities

For flow problems driven in the x direction that have variations along only the (transverse) spatial variable z, the average velocity in the x direction $u_\alpha(z)$ and the heat flux (also called heat flow) in the x direction $q_\alpha(z)$ for a species (with particle mass m_α and equilibrium number density n_α) are expressed as

$$u_\alpha(z) = v_{\alpha,0} U_\alpha(z) \tag{A.1a}$$

and

$$q_\alpha(z) = p_{\alpha,0} v_{\alpha,0} Q_\alpha(z), \tag{A.1b}$$

where $p_{\alpha,0} = n_\alpha k T_0$, $v_{\alpha,0} = (2kT_0/m_\alpha)^{1/2}$,

$$U_\alpha(z) = \frac{1}{\pi^{3/2}} \int \mathrm{e}^{-c^2} h_\alpha(z, \boldsymbol{c}) c \big(1-\mu^2\big)^{1/2} \cos\phi \, \mathrm{d}^3 c \tag{A.2a}$$

and

$$Q_\alpha(z) = \frac{1}{\pi^{3/2}} \int \mathrm{e}^{-c^2} h_\alpha(z, \boldsymbol{c}) \left(c^2 - \frac{5}{2}\right) c \big(1-\mu^2\big)^{1/2} \cos\phi \, \mathrm{d}^3 c. \tag{A.2b}$$

Since we are reporting shear-stress profiles in this work, we also consider

$$p_{ab}^{(\alpha)}(x, z) = m_\alpha \int f_{\alpha,0}(v) \big[1 + f_\alpha(v) x + h_\alpha\big(z, \lambda_\alpha^{1/2} \boldsymbol{v}\big)\big] w_a w_b \, \mathrm{d}^3 v, \tag{A.3}$$

where a and b can be x, y or z, $w_x = v_x - u_\alpha(z)$, $w_y = v_y$ and $w_z = v_z$. Note that while $f_\alpha(v)$ is given by Eqs. (6.22) and (6.23) for the thermal-creep and diffusion-slip problems, the term $f_\alpha(v)x$ should not, for our formulation of the viscous-slip problem, be included in Eq. (A.3). Some elementary work leads to

$$p_{ab}^{(\alpha)}(x, z) = p_{\alpha,0} \big\{ P_{ab}^{(\alpha)}(z) + \delta_{a,b} \big[1 + (K_T + R_\alpha) x\big] \big\}, \quad a, b = x, y, z, \tag{A.4}$$

where the x-dependent part of Eq. (A.4) is to be ignored for the viscous-slip problem, and where

$$P_{ab}^{(\alpha)}(z) = \frac{2}{\pi^{3/2}} \int \mathrm{e}^{-c^2} h_\alpha(z, \boldsymbol{c}) c_a c_b \, \mathrm{d}^3 c. \tag{A.5}$$

We can use Eq. (2.32) in Eq. (A.5) to show, for the considered problems of viscous slip, thermal creep, and diffusion slip that

$$P_{ab}^{(\alpha)}(z) = 0, \quad ab \neq xz. \tag{A.6}$$

In this work, we are concerned with the shear stress (defined by the x and z directions) which we write as

$$p_{xz}^{(\alpha)}(z) = p_{\alpha,0} P_\alpha(z), \tag{A.7}$$

where

$$P_\alpha(z) = \frac{2}{\pi^{3/2}} \int \mathrm{e}^{-c^2} h_\alpha(z, \boldsymbol{c}) c^2 \mu \big(1-\mu^2\big)^{1/2} \cos\phi \, \mathrm{d}^3 c. \tag{A.8}$$

A.2. Mixture quantities

In Ref. [27] we listed (for mixtures) expressions, as defined by Ferziger and Kaper [3], for the number density $n(z)$, the mass density $\rho(z)$, the average velocity $u(z)$, the bulk velocity $\hat{u}(z)$, the temperature $t(z)$ and the heat flux $q(z)$, as they applied to the temperature-jump problem. Continuing, we follow similar logic and list the definitions relevant here for the considered flow problems, viz.

$$u(z) = v_0 \left[c_1 \left(\frac{m}{m_1}\right)^{1/2} U_1(z) + c_2 \left(\frac{m}{m_2}\right)^{1/2} U_2(z) \right], \tag{A.9a}$$

$$\hat{u}(z) = v_0 \left[c_1 \left(\frac{m_1}{m}\right)^{1/2} U_1(z) + c_2 \left(\frac{m_2}{m}\right)^{1/2} U_2(z) \right], \tag{A.9b}$$

$$q(z) = p_0 v_0 \left[c_1 \left(\frac{m}{m_1}\right)^{1/2} Q_1(z) + c_2 \left(\frac{m}{m_2}\right)^{1/2} Q_2(z) \right], \tag{A.9c}$$

and (the x–z component of the shear stress)

$$p(z) = p_0 \big[c_1 P_1(z) + c_2 P_2(z) \big], \tag{A.9d}$$

where $p_0 = nkT_0$, $v_0 = (2kT_0/m)^{1/2}$, $m = c_1 m_1 + c_2 m_2$, and $c_\alpha = n_\alpha/(n_1 + n_2)$, $\alpha = 1, 2$.

References

[1] M.M.R. Williams, Mathematical Methods in Particle Transport Theory, Butterworth, London, 1971.
[2] C. Cercignani, The Boltzmann Equation and its Applications, Springer, New York, 1988.
[3] J.H. Ferziger, H.G. Kaper, Mathematical Theory of Transport Processes in Gases, North-Holland, Amsterdam, 1972.
[4] S. Chapman, T.G. Cowling, The Mathematical Theory of Non-Uniform Gases, Cambridge University Press, Cambridge, 1952.
[5] C.E. Siewert, Viscous-slip, thermal-slip, and temperature-jump coefficients as defined by the linearized Boltzmann equation and the Cercignani–Lampis boundary condition, Phys. Fluids 15 (2003) 1696–1701.
[6] L.B. Barichello, M. Camargo, P. Rodrigues, C.E. Siewert, Unified solutions to classical flow problems based on the BGK model, J. Appl. Math. Phys. (ZAMP) 52 (2001) 517–534.
[7] C.E. Siewert, F. Sharipov, Model equations in rarefied gas dynamics: Viscous-slip and thermal-slip coefficients, Phys. Fluids 14 (2002) 4123–4129.
[8] M.M.R. Williams, A review of the rarefied gas dynamics theory associated with some classical problems in flow and heat transfer, J. Appl. Math. Phys. (ZAMP) 52 (2001) 500–516.
[9] S.K. Loyalka, K.A. Hickey, The Kramers problem: Velocity slip and defect for a hard sphere gas with arbitrary accommodation, J. Appl. Math. Phys. (ZAMP) 41 (1990) 245–253.
[10] T. Ohwada, Y. Sone, K. Aoki, Numerical analysis of the shear and thermal creep flows of a rarefied gas over a plane wall on the basis of the linearized Boltzmann equation for hard-sphere molecules, Phys. Fluids A 1 (1989) 1588–1599.
[11] S.K. Loyalka, Temperature jump and thermal creep slip: Rigid sphere gas, Phys. Fluids A 1 (1989) 403–408.
[12] H.A. Kramers, J. Kistemaker, On the slip of a diffusing gas mixture along a wall, Physica 10 (1943) 669–713.
[13] Y. Onishi, On the behavior of a slightly rarefied gas mixture over plane boundaries, J. Appl. Math. Phys. (ZAMP) 37 (1986) 573–596.
[14] F. Sharipov, D. Kalempa, Velocity slip and temperature jump coefficients for gaseous mixtures. I. Viscous slip coefficient, Phys. Fluids 15 (2003) 1800–1806.
[15] F. Sharipov, D. Kalempa, Velocity slip and temperature jump coefficients for gaseous mixtures. II. Thermal slip coefficient, Phys. Fluids 16 (2004) 759–764.
[16] F. Sharipov, D. Kalempa, Velocity slip and temperature jump coefficients for gaseous mixtures. III. Diffusion slip coefficient, Phys. Fluids 16 (2004) 3779–3785.
[17] C.E. Siewert, D. Valougeorgis, Concise and accurate solutions to half-space binary-gas flow problems defined by the McCormack model and specular-diffuse wall conditions, Eur. J. Mech. B Fluids 23 (2004) 709–726.
[18] S.K. Loyalka, Velocity slip coefficient and the diffusion slip velocity for a multicomponent gas mixture, Phys. Fluids 14 (1971) 2599–2604.
[19] I.N. Ivchenko, S.K. Loyalka, R.V. Tompson, Slip coefficients for binary gas mixtures, J. Vac. Sci. Technol. A 15 (1997) 2375–2381.
[20] I.N. Ivchenko, S.K. Loyalka, R.V. Tompson, Boundary slip phenomena in a binary mixture, J. Appl. Math. Phys. (ZAMP) 53 (2002) 58–72.
[21] S. Yasuda, S. Takata, K. Aoki, Numerical analysis of the shear flow of a binary mixture of hard-sphere gases over a plane wall, Phys. Fluids 16 (2004) 1989–2003.
[22] S. Takata, S. Yasuda, S. Kosuge, K. Aoki, Numerical analysis of thermal-slip and diffusion-slip flows of a binary mixture of hard-sphere molecular gases, Phys. Fluids 15 (2003) 3745–3766.
[23] R.D.M. Garcia, C.E. Siewert, M.M.R. Williams, A formulation of the linearized Boltzmann equations for a binary mixture of rigid spheres, Eur. J. Mech. B Fluids 24 (2005) 614–620.
[24] L.B. Barichello, C.E. Siewert, A discrete-ordinates solution for a non-grey model with complete frequency redistribution, J. Quant. Spectros. Radiat. Transfer 62 (1999) 665–675.
[25] R.D.M. Garcia, C.E. Siewert, Some exact results basic to the linearized Boltzmann equations for a binary mixture of rigid spheres, J. Appl. Math. Phys. (ZAMP) 57 (2006) 999–1010.
[26] R.D.M. Garcia, C.E. Siewert, Some solutions (linear in the spatial variables) and generalized Chapman–Enskog functions basic to the linearized Boltzmann equations for a binary mixture of rigid spheres, J. Appl. Math. Phys. (ZAMP) (2007), doi:10.1007/s00033-006-0074-5.
[27] R.D.M. Garcia, C.E. Siewert, The temperature-jump problem based on the linearized Boltzmann equation for a binary mixture of rigid spheres, Eur. J. Mech. B Fluids 26 (2007) 132–153.
[28] C.L. Pekeris, Solution of the Boltzmann–Hilbert integral equation, Proc. Natl. Acad. Sci. 41 (1955) 661–669.
[29] C.E. Siewert, D. Valougeorgis, The McCormack model: channel flow of a binary gas mixture driven by temperature, pressure and density gradients, Eur. J. Mech. B Fluids 23 (2004) 645–664.
[30] F. Sharipov, Heat transfer in the Knudsen layer, Phys. Rev. E 69 (2004) 061201.
[31] C.E. Siewert, On computing the thermal-slip coefficient from Kramers' problem, Phys. Fluids 16 (2004) 2132–2135.
[32] F.J. McCormack, Construction of linearized kinetic models for gaseous mixtures and molecular gases, Phys. Fluids 16 (1973) 2095–2105.
[33] B.T. Smith, J.M. Boyle, J.J. Dongarra, B.S. Garbow, Y. Ikebe, V.C. Klema, C.B. Moler, Matrix Eigensystem Routines – EISPACK Guide, Springer-Verlag, Berlin, 1976.
[34] J.J. Dongarra, J.R. Bunch, C.B. Moler, G.W. Stewart, LINPACK Users' Guide, SIAM, Philadelphia, 1979.
[35] C.E. Siewert, The linearized Boltzmann equation: Concise and accurate solutions to basic flow problems, J. Appl. Math. Phys. (ZAMP) 54 (2003) 273–303.
[36] J.A. Bentz, R.V. Tompson, S.K. Loyalka, Viscosity and velocity slip coefficients for gas mixtures: Measurements with a spinning rotor gauge, J. Vac. Sci. Technol. A 17 (1999) 235–241.
[37] C.M. Huang, R.V. Tompson, T.K. Ghosh, I.N. Ivchenko, S.K. Loyalka, Measurements of thermal creep in binary gas mixtures, Phys. Fluids 11 (1999) 1662–1672.