Heat transfer between parallel plates: An approach based on the linearized Boltzmann equation for a binary mixture of rigid-sphere gases

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An analytical version of the discrete-ordinates method is used to develop a concise and particularly accurate solution of the heat-transfer problem for a binary gas mixture confined between two parallel plates. The formulation of the problem allows general (specular-diffuse) Maxwell boundary conditions for each of the two types of particles and is based on a form of the linearized Boltzmann equation that incorporates recently established analytical expressions for the relevant rigid-sphere kernels. Numerical results are reported for the density, the temperature, and the heat-flow profiles relative to each species in Ne-Ar and He-Xe mixtures. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2511039]

I. INTRODUCTION

The heat-transfer problem within the context of rarefied gas dynamics has been studied in terms of linear theory for a single-species gas based on the Bhatnagar-Gross-Krook (BGK) model (see, for example, the work by Thomas, Chang, and Siewert[1] and the references quoted therein). The problem has also been solved in terms of the linearized Boltzmann equation (LBE) for rigid-sphere interactions.[2,3] Recently, single-species studies of heat transfer between parallel plates have been extended to the case of binary mixtures of rigid spheres and the nonlinear Boltzmann equation.[4] Our own work[5] reports an essentially analytical solution of the heat-transfer problem as described by the McCormack kinetic model.[6] We do not discuss here many relevant works on this subject, but we refer instead to the books of Cercignani,[7,8] Williams,[9] and Ferziger and Kaper,[10] as well as review papers by Sharipov and Seleznev[11] and Williams,[12] for general background material.

II. A FORMULATION OF THE PROBLEM IN TERMS OF THE LINEARIZED BOLTZMANN EQUATION

We consider that a binary gas mixture is confined between two parallel plates that are kept at different temperatures. The two plates or walls reflect the gas particles both diffusely and specularly. The problem is thus to find the density profiles, the temperature profiles, and the heat-transfer profiles relevant to each of the two species of gas particles that are assumed to interact as rigid spheres. The particle velocity distributions are governed by the linearized Boltzmann equation.

Before starting our work that is specific to the heat-transfer problem, we briefly review our analytical formulation of the linearized Boltzmann equation for a binary mixture of rigid spheres. This formulation was started in Ref. 13 and was further developed in Refs. 14–17. In order to avoid too much duplication of previously reported aspects of our work, we consider that Refs. 13–17 can be consulted for some mathematical expressions that are not given in this work. To start, we write the coupled linearized Boltzmann equation (for variations only in the z direction) for the considered binary mixture as

$$c\mu\frac{\partial}{\partial z}\boldsymbol{H}(z,\boldsymbol{c}) + \varepsilon_0 V(c)\boldsymbol{H}(z,\boldsymbol{c})$$
$$= \varepsilon_0 \int e^{-c'^2}\boldsymbol{\mathcal{K}}(\boldsymbol{c}':\boldsymbol{c})\boldsymbol{H}(z,\boldsymbol{c}')\mathrm{d}^3 c', \tag{1}$$

where

$$\boldsymbol{H}(z,\boldsymbol{c}) = \begin{bmatrix} h_1(z,\boldsymbol{c}) \\ h_2(z,\boldsymbol{c}) \end{bmatrix}. \tag{2}$$

At this point, ε_0 is an arbitrary parameter that we (soon) will use to define a dimensionless spatial variable. Since Eq. (1) is written in terms of a dimensionless velocity variable $\boldsymbol{c}$, we note that the basic velocity distribution functions (for each of the two species of particles) are available from

$$f_\alpha(z,\boldsymbol{v}) = f_{\alpha,0}(v)[1 + h_\alpha(z,\lambda_\alpha^{1/2}\boldsymbol{v})], \quad \alpha = 1,2, \tag{3}$$

where $\lambda_\alpha = m_\alpha/(2kT_0)$, and where

$$f_{\alpha,0}(v) = n_\alpha(\lambda_\alpha/\pi)^{3/2}e^{-\lambda_\alpha v^2} \tag{4}$$

is the Maxwellian distribution for n_α particles of mass m_α in equilibrium at temperature T_0. Here, k is the Boltzmann constant. Continuing, we note that we use spherical coordinates $\{c, \theta, \phi\}$, with $\mu = \cos\theta$, to describe the dimensionless velocity vector, so that

$$\boldsymbol{H}(z,\boldsymbol{c}) \Leftrightarrow \boldsymbol{H}(z,c,\mu,\phi).$$

The basic elements of Eq. (1), viz., $V(c)$ and the scattering kernel $\boldsymbol{\mathcal{K}}(\boldsymbol{c}':\boldsymbol{c})$, are defined explicitly by Eqs. (23), (24), (64), and (65) of Ref. 15 in terms of (a) the ratio of the two particle masses m_1/m_2, (b) the ratio of the two particle diameters d_1/d_2, and (c) the ratio of particle densities n_1/n_2.

In this work, we seek a solution of Eq. (1) that is valid for all $z \in (-z_0, z_0)$ and that satisfies Maxwell boundary conditions at the walls. If we denote the temperatures of the walls located at $z = -z_0$ and $z = z_0$ by T_{w1} and T_{w2}, respectively, we can follow a recent review paper by Williams[12] and linearize the boundary conditions about T_0 to express the relevant boundary conditions as

$$\boldsymbol{H}(-z_0, c, \mu, \phi) - (\boldsymbol{I} - \boldsymbol{\alpha})\boldsymbol{H}(-z_0, c, -\mu, \phi) - \boldsymbol{\alpha}\boldsymbol{\mathcal{I}}_-\{\boldsymbol{H}\}(-z_0)$$
$$= (c^2 - 2)\delta\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{5a}$$

and

$$\boldsymbol{H}(z_0, c, -\mu, \phi) - (\boldsymbol{I} - \boldsymbol{\beta})\boldsymbol{H}(z_0, c, \mu, \phi) - \boldsymbol{\beta}\boldsymbol{\mathcal{I}}_+\{\boldsymbol{H}\}(z_0)$$
$$= (2 - c^2)\delta\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \tag{5b}$$

for $\mu \in (0, 1]$, all c, and all ϕ. Here we have chosen T_0 to be the average of T_{w1} and T_{w2}, and so we have written

$$T_{w1} = T_0(1 + \delta) \tag{6a}$$

and

$$T_{w2} = T_0(1 - \delta), \tag{6b}$$

where δ is the parameter we use to specify the deviations of the wall temperatures relative to the reference temperature T_0. In writing Eqs. (5), we have used

$$\boldsymbol{\alpha} = \mathrm{diag}\{\alpha_1, \alpha_2\} \tag{7a}$$

and

$$\boldsymbol{\beta} = \mathrm{diag}\{\beta_1, \beta_2\} \tag{7b}$$

to compact our notation for the accommodation coefficients α_1, α_2 (for the wall located at $z = -z_0$) and β_1, β_2 (for the wall located at $z = z_0$). In addition,

$$\boldsymbol{\mathcal{I}}_{\mp}\{\boldsymbol{H}\}(z) = \frac{2}{\pi}\int_0^\infty \int_0^1 \int_0^{2\pi} e^{-c'^2}\boldsymbol{H}(z, c', \mp\mu', \phi')$$
$$\times \mu' c'^3 \mathrm{d}\phi' \mathrm{d}\mu' \mathrm{d}c' \tag{8}$$

is used to denote the diffuse terms in Eqs. (5).

In this work, we intend to compute, for $z \in [-z_0, z_0]$, the density, the temperature, and the heat-flow perturbations (see Appendix A of Ref. 16 for definitions of these and other macroscopic quantities of interest), i.e.,

$$N(z) = \frac{1}{\pi^{3/2}}\int e^{-c^2}\boldsymbol{H}(z, \boldsymbol{c})\mathrm{d}^3 c, \tag{9a}$$

$$T(z) = \frac{2}{3\pi^{3/2}}\int e^{-c^2}\boldsymbol{H}(z, \boldsymbol{c})(c^2 - 3/2)\mathrm{d}^3 c, \tag{9b}$$

and

$$Q(z) = \frac{1}{\pi^{3/2}}\int e^{-c^2}\boldsymbol{H}(z, \boldsymbol{c})(c^2 - 5/2)c\mu \mathrm{d}^3 c. \tag{9c}$$

Now, taking note of the Legendre expansion of $\mathcal{K}(\boldsymbol{c}' : \boldsymbol{c})$ that was reported in Ref. 15 and the boundary conditions listed as Eqs. (5), we find that here an expansion of $\boldsymbol{H}(z, \boldsymbol{c})$ in a Fou-

rier series of the azimuthal angle ϕ requires only the first term, and so, making use of the dimensionless spatial variable[15]

$$\tau = z\varepsilon_0, \tag{10}$$

with

$$\varepsilon_0 = (n_1 + n_2)\pi^{1/2}\left(\frac{n_1 d_1 + n_2 d_2}{n_1 + n_2}\right)^2, \tag{11}$$

we introduce

$$\boldsymbol{\Psi}(\tau, c, \mu) = \boldsymbol{H}(\tau/\varepsilon_0, \boldsymbol{c}), \tag{12}$$

so that Eqs. (9) can be written as

$$N(\tau) = \frac{2}{\pi^{1/2}}\int_0^\infty \int_{-1}^1 e^{-c^2}\boldsymbol{\Psi}(\tau, c, \mu)c^2 \mathrm{d}\mu \mathrm{d}c, \tag{13a}$$

$$T(\tau) = \frac{4}{3\pi^{1/2}}\int_0^\infty \int_{-1}^1 e^{-c^2}\boldsymbol{\Psi}(\tau, c, \mu)(c^2 - 3/2)c^2 \mathrm{d}\mu \mathrm{d}c \tag{13b}$$

and

$$Q(\tau) = \frac{2}{\pi^{1/2}}\int_0^\infty \int_{-1}^1 e^{-c^2}\boldsymbol{\Psi}(\tau, c, \mu)(c^2 - 5/2)c^3 \mu \mathrm{d}\mu \mathrm{d}c. \tag{13c}$$

It should be noted that to avoid excessive notation in writing Eqs. (13), we have followed the (common, but dubious) procedure of not always introducing new labels for dependent quantities (in this case N, T, and Q) when the independent variable is changed.

We can now use Eq. (12) in Eqs. (1) and (5) to deduce that $\boldsymbol{\Psi}(\tau, c, \mu)$ is to be determined from the balance equation

$$c\mu\frac{\partial}{\partial\tau}\boldsymbol{\Psi}(\tau, c, \mu) + V(c)\boldsymbol{\Psi}(\tau, c, \mu)$$
$$= \int_0^\infty \int_{-1}^1 e^{-c'^2}\boldsymbol{\mathcal{K}}(c', \mu' : c, \mu)\boldsymbol{\Psi}(\tau, c', \mu')c'^2 \mathrm{d}\mu' \mathrm{d}c' \tag{14}$$

and the boundary conditions

$$\boldsymbol{\Psi}(-a, c, \mu) - (\boldsymbol{I} - \boldsymbol{\alpha})\boldsymbol{\Psi}(-a, c, -\mu)$$
$$- 4\boldsymbol{\alpha}\int_0^\infty \int_0^1 e^{-c'^2}\boldsymbol{\Psi}(-a, c', -\mu')c'^3 \mu' \mathrm{d}\mu' \mathrm{d}c'$$
$$= (c^2 - 2)\delta\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{15a}$$

and

$$\boldsymbol{\Psi}(a, c, -\mu) - (\boldsymbol{I} - \boldsymbol{\beta})\boldsymbol{\Psi}(a, c, \mu)$$
$$- 4\boldsymbol{\beta}\int_0^\infty \int_0^1 e^{-c'^2}\boldsymbol{\Psi}(a, c', \mu')c'^3 \mu' \mathrm{d}\mu' \mathrm{d}c'$$
$$= (2 - c^2)\delta\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \tag{15b}$$

for $\mu \in (0,1]$ and all c, and where $a = z_0 \varepsilon_0$. In writing Eq. (14), we have used

$$\mathcal{K}(c',\mu':c,\mu) = \int_0^{2\pi} \mathcal{K}(c':c)\mathrm{d}\phi', \qquad (16)$$

which is expressed, as in Ref. 15, in the form

$$\mathcal{K}(c',\mu':c,\mu) = (1/2)\sum_{n=0}^{\infty}(2n+1)P_n(\mu')P_n(\mu)\mathcal{K}_n(c',c), \qquad (17)$$

where $P_n(\mu)$ denotes the nth order Legendre polynomial and $\mathcal{K}_n(c',c)$ is defined by Eqs. (73) and (74) of Ref. 15. We note that, for computational purposes, we use a truncated version (say, at $n=L$) of the expansion in Eq. (17).

III. THE COMPLETE SPEED-DEPENDENT ADO SOLUTION

In Ref. 16, a work devoted to the temperature-jump problem, we used the analytical discrete-ordinates (ADO) method,[18] supplemented with exact asymptotic solutions, and an expansion in terms of Legendre polynomials written in the form

$$\Psi(\tau,c,\mu) = \sum_{k=0}^{K}\Pi_k(c)G_k(\tau,\mu), \qquad (18)$$

with (for use in the computational work)

$$\Pi_k(c) = P_k(2\mathrm{e}^{-c}-1), \qquad (19)$$

to express a general solution of a discrete-ordinates version of Eq. (14), evaluated at $\{\pm\mu_i\}$, as

$$\begin{aligned} \Psi(\tau,c,\pm\mu_i) = {}&\Psi_*(\tau,c,\pm\mu_i)\\ &+ P(c)\sum_{j=4}^{J}[A_j\Phi(\nu_j,\pm\mu_i)\mathrm{e}^{-(a+\tau)/\nu_j}\\ &+ B_j\Phi(\nu_j,\mp\mu_i)\mathrm{e}^{-(a-\tau)/\nu_j}] \end{aligned} \qquad (20)$$

for $i=1,2,\ldots,N$ and $J=2N(K+1)$. Here, we use $\{\mu_i\}$ to denote the collection of N quadrature points,

$$P(c) = [P_0(2\mathrm{e}^{-c}-1)I \quad P_1(2\mathrm{e}^{-c}-1)I \quad \cdots \quad P_K(2\mathrm{e}^{-c}-1)I], \qquad (21)$$

I is the 2×2 identity matrix, and $\Psi_*(\tau,c,\mu)$ is as defined in terms of the elementary solutions we reported in Ref. 15, viz.,

$$\begin{aligned} \Psi_*(\tau,c,\mu) = {}&A_1 H_1 + A_2 H_2 + A_3 H_3(c) + B_1 H_4(c,\mu)\\ &+ B_2[\tau\Phi_1(c) - \mu A^{(1)}(c)]\\ &+ B_3[\tau\Phi_2(c) - \mu A^{(2)}(c)], \end{aligned} \qquad (22)$$

where

$$H_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad (23a)$$

$$H_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad (23b)$$

$$H_3(c) = c^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad (23c)$$

and

$$H_4(c,\mu) = c\mu \begin{bmatrix} 1 \\ (m_2/m_1)^{1/2} \end{bmatrix}. \qquad (23d)$$

In addition,

$$\Phi_1(c) = \begin{bmatrix} c_1(c^2-5/2)-c_2 \\ c_1(c^2-3/2) \end{bmatrix} \qquad (24a)$$

and

$$\Phi_2(c) = \begin{bmatrix} c_2(c^2-3/2) \\ c_2(c^2-5/2)-c_1 \end{bmatrix}, \qquad (24b)$$

where

$$c_\alpha = n_\alpha/(n_1+n_2), \quad \alpha = 1,2, \qquad (25)$$

and the $A^{(\alpha)}(c)$, $\alpha=1,2$, are two generalized Chapman-Enskog (vector-valued) functions (also defined in Ref. 15). While in Eqs. (20) and (22) A_j and B_j, for $j=1,2,\ldots,J$, are arbitrary constants that are to be determined (from the boundary conditions), the elementary solutions $\Phi(\nu_j,\pm\mu_i)$, the separation constants ν_j, and other elements of the solution are all defined in Refs. 15 and 16. Postponing briefly a description of the way we find the arbitrary constants in Eqs. (20) and (22), we use those equations in discrete-ordinates versions of Eqs. (13) to find

$$N(\tau) = N_*(\tau) + \sum_{j=4}^{J}[A_j\mathrm{e}^{-(a+\tau)/\nu_j} + B_j\mathrm{e}^{-(a-\tau)/\nu_j}]N_j, \qquad (26a)$$

$$T(\tau) = T_*(\tau) + \sum_{j=4}^{J}[A_j\mathrm{e}^{-(a+\tau)/\nu_j} + B_j\mathrm{e}^{-(a-\tau)/\nu_j}]T_j, \qquad (26b)$$

and

$$Q(\tau) = Q_* + \sum_{j=4}^{J}[A_j\mathrm{e}^{-(a+\tau)/\nu_j} - B_j\mathrm{e}^{-(a-\tau)/\nu_j}]Q_j, \qquad (26c)$$

where

$$N_*(\tau) = \begin{bmatrix} A_1 + (3/2)A_3 - B_2\tau \\ A_2 + (3/2)A_3 - B_3\tau \end{bmatrix}, \qquad (27a)$$

$$T_*(\tau) = \begin{bmatrix} A_3 + (c_1 B_2 + c_2 B_3)\tau \\ A_3 + (c_1 B_2 + c_2 B_3)\tau \end{bmatrix}, \qquad (27b)$$

and

$$Q_* = -\frac{4}{3\pi^{1/2}}\int_0^{\infty}\mathrm{e}^{-c^2}[B_2 A^{(1)}(c) + B_3 A^{(2)}(c)](c^2-5/2)c^3\mathrm{d}c. \qquad (27c)$$

We note that to complete Eqs. (26) and (27), we must find the constants $\{A_j,B_j\}$ and then use the definitions of the vectors N_j, T_j, and Q_j that are given in Ref. 16.

To find the required constants $\{A_j, B_j\}$, we substitute Eq. (20) into discrete-ordinates versions of Eqs. (15), multiply the resulting equations by

$$c^2 \exp\{-c^2\} P^T(c),$$

where the superscript T is used to denote the transpose operation, and integrate over all c to define a system of $2J$ linear algebraic equations for the $2J$ unspecified constants. However, there is an issue of importance. Since solutions H_1 and H_2 each satisfy homogeneous versions of Eqs. (15), the constants A_1 and A_2 cannot be determined from the established linear system. It follows that the boundary conditions listed as Eqs. (15) are not sufficient to define a unique solution to the considered heat-transfer problem. We follow our previous papers[3,5,19] and impose the additional (vector) condition

$$\int_{-a}^{a} N(\tau) \, d\tau = 0. \tag{28}$$

And so we solve the constructed $2J+2$ system of linear equations (of rank $2J$) to find the $2J$ constants $\{A_j, B_j\}$ needed to complete our solution for $N(\tau)$, $T(\tau)$, and $Q(\tau)$ as listed in Eqs. (26) and (27).

IV. NUMERICAL RESULTS

In order to demonstrate that our ADO solution for the considered heat-transfer problem can yield accurate results with a relatively modest computational effort, we report detailed numerical results for two test cases. As in the related work[5] that is based on the McCormack model, the test cases are defined for a Ne-Ar mixture in the first case and for a He-Xe mixture in the second case. We note that only the mass ratio m_1/m_2, the diameter ratio d_1/d_2, and the density ratio n_1/n_2 are needed to define the LBE for rigid-sphere interactions, and so we use the basic data,

$$m_2 = 39.948, \quad m_1 = 20.183, \quad d_2/d_1 = 1.406, \quad n_1/n_2 = 2/3$$

for the Ne-Ar mixture, and

$$m_2 = 131.30, \quad m_1 = 4.0026, \quad d_2/d_1 = 2.226, \quad n_1/n_2 = 2/3$$

for the He-Xe mixture. It should be noted that the values of the masses used here are taken from Ref. 20, while the diameter ratios are those reported by Sharipov and Kalempa.[21] As we wish to compare our results found here with our previous work[5] that was based on the McCormack model, we find it convenient to define our half-width a in terms of the half-width $a_M = 1.5$ that was used in Ref. 5. In Ref. 16, we expressed the relationship between the dimensionless spatial variable τ_M used in our work[5] with the McCormack model and τ, the dimensionless spatial variable used in this work, as

$$\xi_M = \tau/\tau_M, \tag{29}$$

where ξ_M can be computed from

$$\xi_M = \frac{c_2[Y_1 + X_{2,1}^{(4)}] + c_1[Y_2 + X_{1,2}^{(4)}]}{Y_1 Y_2 - X_{1,2}^{(4)} X_{2,1}^{(4)}}, \tag{30}$$

where

$$Y_1 = X_{1,1}^{(3)} + X_{1,2}^{(3)} - X_{1,1}^{(4)}, \tag{31}$$

$$Y_2 = X_{2,2}^{(3)} + X_{2,1}^{(3)} - X_{2,2}^{(4)}, \tag{32}$$

$$X_{\alpha,\beta}^{(3)} = \left(\frac{10}{3} + \frac{2m_\beta}{m_\alpha}\right) F_{\alpha,\beta}, \tag{33}$$

and

$$X_{\alpha,\beta}^{(4)} = \tfrac{4}{3} F_{\alpha,\beta}, \tag{34}$$

with

$$F_{\alpha,\beta} = \frac{2c_\beta m_\alpha}{5m_\beta} \left(\frac{m_\beta}{m_\alpha + m_\beta}\right)^{3/2} \left(\frac{c_1 m_1 + c_2 m_2}{m_\alpha}\right)^{1/2} \times \left(\frac{d_\alpha + d_\beta}{c_1 d_1 + c_2 d_2}\right)^2. \tag{35}$$

And so, for Tables I and II, we use the half-width $a = 1.5\xi_M$. It can be observed that by tabulating the tempera-

TABLE I. The density, temperature, and heat-flow profiles for the Ne-Ar mixture: $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, $n_1/n_2 = 2/3$, $\delta = 1$, $a_M = 1.5 \Rightarrow a = 0.644\,978\,249\dots$

η	$N_1(-a+2\eta a)$	$N_2(-a+2\eta a)$	$-T_1(-a+2\eta a)$	$-T_2(-a+2\eta a)$	$Q_1(-a+2\eta a)$	$Q_2(-a+2\eta a)$
0.0	−1.6801(−1)	−3.6136(−1)	1.7754(−1)	−5.4437(−3)	1.5790(−1)	3.0481(−1)
0.1	−1.2351(−1)	−2.3950(−1)	2.3943(−1)	1.3845(−1)	1.7993(−1)	2.8414(−1)
0.2	−9.1550(−2)	−1.6723(−1)	2.8324(−1)	2.1561(−1)	1.9411(−1)	2.7085(−1)
0.3	−6.1223(−2)	−1.0555(−1)	3.2422(−1)	2.7809(−1)	2.0384(−1)	2.6172(−1)
0.4	−3.1289(−2)	−4.8759(−2)	3.6424(−1)	3.3382(−1)	2.1044(−1)	2.5553(−1)
0.5	−1.3257(−3)	5.5732(−3)	4.0403(−1)	3.8625(−1)	2.1460(−1)	2.5163(−1)
0.6	2.8918(−2)	5.8953(−2)	4.4409(−1)	4.3747(−1)	2.1668(−1)	2.4968(−1)
0.7	5.9720(−2)	1.1270(−1)	4.8493(−1)	4.8935(−1)	2.1681(−1)	2.4956(−1)
0.8	9.1603(−2)	1.6851(−1)	5.2743(−1)	5.4426(−1)	2.1490(−1)	2.5135(−1)
0.9	1.2599(−1)	2.2995(−1)	5.7376(−1)	6.0713(−1)	2.1052(−1)	2.5545(−1)
1.0	1.7374(−1)	3.2019(−1)	6.3923(−1)	7.0843(−1)	2.0237(−1)	2.6310(−1)

TABLE II. The density, temperature, and heat-flow profiles for the He-Xe mixture: $\alpha_1=0.2$, $\alpha_2=0.4$, $\beta_1=0.6$, $\beta_2=0.8$, $n_1/n_2=2/3$, $\delta=1$, $a_M=1.5 \Rightarrow a=0.530\ 099\ 262\ldots$.

η	$N_1(-a+2\eta a)$	$N_2(-a+2\eta a)$	$-T_1(-a+2\eta a)$	$-T_2(-a+2\eta a)$	$Q_1(-a+2\eta a)$	$Q_2(-a+2\eta a)$
0.0	−1.0728(−1)	−3.7765(−1)	2.9276(−1)	6.7202(−3)	1.6823(−1)	3.1227(−1)
0.1	−7.7050(−2)	−2.4863(−1)	3.4526(−1)	1.5038(−1)	1.7123(−1)	3.0083(−1)
0.2	−5.5818(−2)	−1.7442(−1)	3.8228(−1)	2.2641(−1)	1.7353(−1)	2.9207(−1)
0.3	−3.6531(−2)	−1.1021(−1)	4.1587(−1)	2.8897(−1)	1.7538(−1)	2.8499(−1)
0.4	−1.8049(−2)	−5.0652(−2)	4.4801(−1)	3.4517(−1)	1.7688(−1)	2.7925(−1)
0.5	1.2253(−4)	6.4315(−3)	4.7958(−1)	3.9803(−1)	1.7809(−1)	2.7465(−1)
0.6	1.8299(−2)	6.2356(−2)	5.1115(−1)	4.4940(−1)	1.7902(−1)	2.7111(−1)
0.7	3.6780(−2)	1.1826(−1)	5.4325(−1)	5.0088(−1)	1.7968(−1)	2.6856(−1)
0.8	5.5995(−2)	1.7564(−1)	5.7666(−1)	5.5454(−1)	1.8009(−1)	2.6702(−1)
0.9	7.6891(−2)	2.3772(−1)	6.1303(−1)	6.1463(−1)	1.8020(−1)	2.6659(−1)
1.0	1.0511(−1)	3.2931(−1)	6.6202(−1)	7.1084(−1)	1.7991(−1)	2.6768(−1)

ture and density profiles in fractions of the half-width, we are able to compare our results directly with the results[5] of the McCormack model once the expression

$$a = a_M \xi_M \quad (36)$$

is used to relate the two relevant half-widths. We believe that the results in Tables I and II are good to plus/minus one digit in the last place listed. In comparing our results with the results of Tables I and II of Ref. 5, we found that the largest differences displayed by the McCormack-model results with respect to the LBE results occur for the density profiles: up to 15% for Table I and 51% for Table II, without taking into account the difference in $N_1(0)$. With regard to the temperature profiles, the maximum differences, without taking into account the differences in $T_1(-a)$ and $T_2(-a)$, are 12% for Table I and 20% for Table II, while for the heat-flow profiles the maximum differences are <4% for both Tables I and II.

To have an additional comparison of our numerical results, we have also used our code to compute the normalized heat flow reported by Kosuge, Aoki, and Takata[4] for the problem of a binary mixture of rigid-sphere gases confined between two diffusely reflecting parallel plates with different temperatures. These authors employed an iterative finite-difference technique to solve the two coupled nonlinear

Boltzmann equations that describe the problem and reported numerical results in tabular form for a normalized heat flow defined as

$$q_1^* = [2p_0(2kT_{w1}/m_1)^{1/2}]^{-1}$$
$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [m_1 f_1(x, \mathbf{v}) + m_2 f_2(x, \mathbf{v})] v^2 v_x \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z,$$
$$(37)$$

where, except for the pressure $p_0 = k(n_1 + n_2)T_{w1}$, all symbols have been defined in our work. We have found that q_1^* can be expressed in terms of the *constant*

$$q_0 = [c_1 \quad c_2(m_1/m_2)^{1/2}]\mathbf{Q}(\tau), \quad (38)$$

where $\mathbf{Q}(\tau)$ is given by Eq. (26c), in the following way:

$$q_1^* = (1 + \delta)^{-3/2}q_0, \quad (39)$$

and so we report in Table III our numerical results for q_1^*, along with those based on the nonlinear Boltzmann equation (NLBE) for rigid-sphere interactions reported by Kosuge, Aoki, and Takata[4] and those listed in Ref. 5 for the McCormack model. Since our mean free path is defined in a way

TABLE III. Comparison results for a normalized heat flux ($-q_1^*$): $\alpha_1=1.0$, $\alpha_2=1.0$, $\beta_1=1.0$, $\beta_2=1.0$, $\delta=-1/3$.

n_1/n_2	Kn	$m_1/m_2=2$ and $d_1/d_2=1$			$m_1/m_2=4$ and $d_1/d_2=2$		
		NLBE[a]	McCormack model[b]	This work	NLBE[a]	McCormack model[b]	This work
1.0(1)	1.0(−1)	0.184	0.181	0.187	0.207	0.202	0.210
1.0(1)	1.0	0.509	0.519	0.529	0.547	0.557	0.568
1.0(1)	1.0(1)	0.656	0.683	0.684	0.693	0.721	0.723
1.0	1.0(−1)	0.209	0.205	0.212	0.370	0.358	0.376
1.0	1.0	0.589	0.599	0.610	0.814	0.830	0.846
1.0	1.0(1)	0.763	0.794	0.795	0.966	1.006	1.008
1.0(−1)	1.0(−1)	0.245	0.241	0.249	0.659	0.653	0.677
1.0(−1)	1.0	0.677	0.689	0.702	1.124	1.159	1.170
1.0(−1)	1.0(1)	0.871	0.906	0.908	1.244	1.298	1.299

[a]Reference 4.
[b]Reference 5.

TABLE IV. Refined (LBE) results for the single-gas cases listed in Tables 1 and 2 of Ref. 3.

η	$T(-a+2\eta a)$		$N(-a+2\eta a)$	
	Table 1	Table 2	Table 1	Table 2
0.0	6.7605(−1)	8.2332(−1)	−2.5367(−1)	−5.6833(−1)
0.1	5.9100(−1)	6.6684(−1)	−1.8265(−1)	−4.2825(−1)
0.2	5.3593(−1)	5.5223(−1)	−1.3342(−1)	−3.1770(−1)
0.3	4.8692(−1)	4.4421(−1)	−8.8300(−2)	−2.1140(−1)
0.4	4.4033(−1)	3.3853(−1)	−4.4787(−2)	−1.0661(−1)
0.5	3.9450(−1)	2.3353(−1)	−1.7755(−3)	−2.2547(−3)
0.6	3.4824(−1)	1.2816(−1)	4.1504(−2)	1.0237(−1)
0.7	3.0025(−1)	2.1165(−2)	8.5893(−2)	2.0807(−1)
0.8	2.4843(−1)	−8.9846(−2)	1.3278(−1)	3.1637(−1)
0.9	1.8785(−1)	−2.1157(−1)	1.8556(−1)	4.3158(−1)
1.0	8.4286(−2)	−4.0266(−1)	2.6931(−1)	5.9527(−1)

different from that of Kosuge, Aoki, and Takata,[4] the equivalent half-width in our formulation is computed from

$$a = (c_1 + c_2 d_2/d_1)^2/[2(2\pi)^{1/2}\text{Kn}], \quad (40)$$

where Kn is the Knudsen number used by Kosuge, Aoki, and Takata.[4] We note also that to compute our entries in Table III, we have put all our accommodation coefficients equal to unity, and we have used $\delta=-1/3$. As the approach based on the LBE is good only for small deviations from the equilibrium state, it is anticipated that better agreement between the NLBE and the LBE results would be observed for absolute values of δ smaller than 1/3.

In our two recent works[16,17] concerning the temperature-jump problem and three half-space flow problems (viscous slip, thermal creep, and diffusion slip), we reported detailed descriptions of computational aspects related to implementations of our solutions. To be brief, we do not repeat this type of discussion here, especially since no new numerical complications were encountered in this work.

V. CONCLUDING REMARKS

We have reported in this work what we believe to be a concise, accurate, and essentially analytical solution for the problem of heat transfer between two parallel plates, as described by the (vector) linearized Boltzmann equation for a binary mixture of rigid spheres.

In addition to the comparisons with numerical results of other works for binary mixtures that are reported in Sec. IV, we have also performed a comparison with the single-gas results of Ref. 3, using three different ways of achieving the single-gas limit in our formulation,

(i) $c_1 = 0$,   (ii) $c_2 = 0$,

or

(iii) $m_1 = m_2$, $d_1 = d_2$, $\alpha_1 = \alpha_2$,   and $\beta_1 = \beta_2$.

Since the mean free path used in this work and that used in Ref. 3 are different, we have used

$$a = a_S \xi_{S,t}, \quad (41)$$

where

$$\xi_{S,t} = 0.679\,630\,049\ldots \quad (42)$$

and where a_S is the half-width used in Ref. 3, to compute the half-width in our current notation. While we found at most a difference of 14 units (for one small entry) in the last two of the five digits listed for the temperature and density profiles in Tables 1 and 2 of Ref. 3, we found only a maximum difference of 3 units in the sixth digit listed in Table 3 of Ref. 3. We have confirmed that the (very slight) loss of accuracy in Tables 1–3 of Ref. 3 was due to using $L=8$ in those computations. To make available our current results (based on $L=100$), we list in Tables IV and V improved versions of Tables 1–3 of Ref. 3. It can be noted that Table 1 of Ref. 3 is based on $a_S=1$ and accommodation coefficients $\{0.7, 0.3\}$, while Table 2 of Ref. 3 is based on $a_S=2.5$ and accommodation coefficients $\{1.0, 0.5\}$. The definition of the normalized

TABLE V. Refined results (normalized heat flux q) for the single-gas cases listed in Table 3 of Ref. 3.

α_1	α_2	$a=0.1$	$a=0.5$	$a=1.0$	$a=1.5$	$a=2.0$	$a=2.5$
0.7	0.1	9.85340(−1)	9.44283(−1)	9.04245(−1)	8.69276(−1)	8.37449(−1)	8.08047(−1)
0.7	0.3	9.61125(−1)	8.61659(−1)	7.75503(−1)	7.08002(−1)	6.52111(−1)	6.04642(−1)
0.7	0.5	9.42050(−1)	8.03656(−1)	6.93254(−1)	6.12618(−1)	5.49560(−1)	4.98494(−1)
0.7	0.7	9.26733(−1)	7.60978(−1)	6.36429(−1)	5.49802(−1)	4.84615(−1)	4.33436(−1)
0.7	0.9	9.14237(−1)	7.28470(−1)	5.95003(−1)	5.05448(−1)	4.39903(−1)	3.89560(−1)
0.7	1.0	9.08834(−1)	7.15014(−1)	5.78283(−1)	4.87862(−1)	4.22420(−1)	3.72597(−1)
0.9	0.1	9.85020(−1)	9.43316(−1)	9.02501(−1)	8.66748(−1)	8.34195(−1)	8.04152(−1)
0.9	0.3	9.58141(−1)	8.52559(−1)	7.61853(−1)	6.91377(−1)	6.33547(−1)	5.84854(−1)
0.9	0.5	9.34747(−1)	7.83174(−1)	6.65323(−1)	5.81139(−1)	5.16540(−1)	4.65054(−1)
0.9	0.7	9.14237(−1)	7.28470(−1)	5.95003(−1)	5.05448(−1)	4.39903(−1)	3.89560(−1)
0.9	0.9	8.96137(−1)	6.84284(−1)	5.41522(−1)	4.50277(−1)	3.85821(−1)	3.37631(−1)
0.9	1.0	8.87872(−1)	6.65246(−1)	5.19319(−1)	4.27947(−1)	3.64341(−1)	3.17305(−1)
1.0	1.0	8.78056(−1)	6.43427(−1)	4.94555(−1)	4.03496(−1)	3.41131(−1)	2.95558(−1)

heat flux q reported in Table V is given in Ref. 3, and finally it should be noted that Table V is defined in terms of the notation of Ref. 3, i.e., $a \Rightarrow a_S$, and the relevant accommodation coefficients are $\{\alpha_1, \alpha_2\}$.

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