IOP Publishing JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 1785–1789 doi:10.1088/1751-8113/40/8/007

A note on radiative transfer in a finite layer

C E Siewert

Mathematics Department, North Carolina State University, Raleigh, NC 27695-8205, USA

Received 19 September 2006, in final form 27 December 2006
Published 6 February 2007
Online at stacks.iop.org/JPhysA/40/1785

Abstract
An ADO (analytical discrete ordinates) solution is used to establish a concise and accurate result for a basic radiative transfer problem in a finite layer described by the grey equation of transfer with general anisotropic scattering. As a specific application, the solution is evaluated for the case of Fresnel boundary conditions to yield numerical results (of a high standard) for several specific cases.

PACS numbers: 42.25.Fx, 42.25.Gy, 44.40.+a

1. Introduction

We consider the problem to be defined by the grey equation of transfer [1]

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau, \mu') \, d\mu', \tag{1.1}$$

for $\mu \in [-1, 1]$ and $\tau \in (0, \tau_0)$, and the boundary conditions

$$I(0, \mu) = R_1(\mu) I(0, -\mu) + F_1(\mu), \qquad \mu \in (0, 1], \tag{1.2a}$$

and

$$I(\tau_0, -\mu) = R_2(\mu) I(\tau_0, \mu) + F_2(\mu), \qquad \mu \in (0, 1]. \tag{1.2b}$$

Here τ is the dimensionless spatial variable and the $\{\beta_l\}$ are the coefficients used to define an Lth-order Legendre expansion of the scattering law. We consider that $\varpi \in [0, 1)$ is given, along with L, τ_0 and the $\{\beta_l\}$. In addition, we assume that the functions $R_1(\mu)$ $R_2(\mu)$, $F_1(\mu)$ and $F_2(\mu)$ are given. We seek only specific [2] surface results, viz,

$$A = \int_0^1 T_1(\mu) I(0, -\mu) \mu \, d\mu + \int_0^1 \widehat{R}_1(\mu) \mu \, d\mu \tag{1.3a}$$

and

$$B = \int_0^1 T_2(\mu) I(\tau_0, \mu) \mu \, d\mu + \int_0^1 \widehat{R}_2(\mu) \mu \, d\mu, \tag{1.3b}$$

where $T_1(\mu)$, $T_2(\mu)$, $\widehat{R}_1(\mu)$ and $\widehat{R}_2(\mu)$ are considered given.

In regard to the boundary conditions listed as equations (1.2), we note that the functions $R_1(\mu)$ and $R_2(\mu)$ are used to account for internal reflections at the two bounding surfaces, and the functions $F_1(\mu)$ and $F_2(\mu)$ will be defined so as to incorporate the effects of transmission of externally incident radiation through the boundaries. Considering equations (1.3), we note that the functions $T_1(\mu)$ and $T_2(\mu)$ account for transmission from inside the layer through the surfaces, while the functions $\widehat{R}_1(\mu)$ and $\widehat{R}_2(\mu)$ will be defined so as to account for external reflections of the incoming radiation. While our formulation and solution of the problem is general in the sense that the input functions $F_1(\mu)$, $F_2(\mu)$, $R_1(\mu)$, $R_2(\mu)$, $T_1(\mu)$, $T_2(\mu)$, $\widehat{R}_1(\mu)$ and $\widehat{R}_2(\mu)$ are not specified, these functions are all explicitly used for the specific case of Fresnel boundary conditions for which our reported numerical calculation of the radiation currents exiting the scattering layer is given.

2. A solution

Here we use the ADO version [3] of the discrete-ordinates method used [4] to solve Chandrasekhar's basic problem in radiative transfer [1]. Since much of what we require here is already available [4], our presentation is brief. To begin we write

$$\boldsymbol{I}_{\pm}(\tau) = \sum_{j=1}^{N}[A_j \boldsymbol{\Phi}_{\pm}(\nu_j)\,\mathrm{e}^{-\tau/\nu_j} + B_j \boldsymbol{\Phi}_{\mp}(\nu_j)\,\mathrm{e}^{-(\tau_0-\tau)/\nu_j}]. \tag{2.1}$$

Here $\boldsymbol{I}_{\pm}(\tau)$ are vectors the components of which are the intensities evaluated at $\pm\mu_k$, where the $\{\mu_k\}$ are the N quadrature points used to evaluate integrals defined on the interval [0, 1]. In addition, the separation constants ν_j and the elementary vectors $\boldsymbol{\Phi}_{\pm}(\nu_j)$ are as defined previously [4]. Making use of the boundary conditions listed as equations (1.2), we find that the constants $\{A_j, B_j\}$ required to complete equation (2.1) are defined as the solutions to the linear system listed as

$$\sum_{j=1}^{N}\{A_j[\boldsymbol{\Phi}_+(\nu_j) - \boldsymbol{R}_1\boldsymbol{\Phi}_-(\nu_j)] + B_j[\boldsymbol{\Phi}_-(\nu_j) - \boldsymbol{R}_1\boldsymbol{\Phi}_+(\nu_j)]\,\mathrm{e}^{-\tau_0/\nu_j}\} = \boldsymbol{F}_1 \tag{2.2a}$$

and

$$\sum_{j=1}^{N}\{A_j[\boldsymbol{\Phi}_-(\nu_j) - \boldsymbol{R}_2\boldsymbol{\Phi}_+(\nu_j)]\,\mathrm{e}^{-\tau_0/\nu_j} + B_j[\boldsymbol{\Phi}_+(\nu_j) - \boldsymbol{R}_2\boldsymbol{\Phi}_-(\nu_j)]\} = \boldsymbol{F}_2, \tag{2.2b}$$

where

$$\boldsymbol{R}_\alpha = \mathrm{diag}\{R_\alpha(\mu_1), R_\alpha(\mu_2), \ldots, R_\alpha(\mu_N)\} \tag{2.3}$$

and

$$\boldsymbol{F}_\alpha = [F_\alpha(\mu_1)\ F_\alpha(\mu_2)\ \cdots\ F_\alpha(\mu_N)]^T. \tag{2.4}$$

The superscript T is used to denote the transpose operation. Once we have solved equations (2.2) to establish the constants $\{A_j, B_j\}$, we can compute the desired results from

$$A = \sum_{j=1}^{N}\left[A_j X_j^{(1)} + B_j Y_j^{(1)}\mathrm{e}^{-\tau_0/\nu_j}\right] + \int_0^1 \widehat{R}_1(\mu)\mu\,\mathrm{d}\mu \tag{2.5a}$$

and

$$B = \sum_{j=1}^{N}\left[A_j Y_j^{(2)}\mathrm{e}^{-\tau_0/\nu_j} + B_j X_j^{(2)}\right] + \int_0^1 \widehat{R}_2(\mu)\mu\,\mathrm{d}\mu, \tag{2.5b}$$

where

$$X_j^{(\alpha)} = [w_1\mu_1 T_\alpha(\mu_1)\ w_2\mu_2 T_\alpha(\mu_2)\ \cdots\ w_N\mu_N T_\alpha(\mu_N)]\mathbf{\Phi}_-(\nu_j) \tag{2.6a}$$

and

$$Y_j^{(\alpha)} = [w_1\mu_1 T_\alpha(\mu_1)\ w_2\mu_2 T_\alpha(\mu_2)\ \cdots\ w_N\mu_N T_\alpha(\mu_N)]\mathbf{\Phi}_+(\nu_j), \tag{2.6b}$$

for $\alpha = 1, 2$. Here the $\{w_k\}$ are the weights that go with the nodes $\{\mu_k\}$ to define the N-point quadrature scheme.

3. A calculation

We consider the problem of Fresnel boundary conditions solved recently by Williams [2], and so to report some numerical results we consider that the slab is surrounded by non-participating media, that n is the index of refraction, and that

$$\tau_0 = 1.0, \quad \varpi = 0.9, \quad F_2(\mu) = 0, \quad R_1(\mu) = R(n, \mu), \quad R_2(\mu) = R(n, \mu),$$

along with the definitions

$$R(n, \mu) = 1 + [G(n, \mu) - 1]H[\mu - \mu_c(n)], \tag{3.1}$$

$$G(n, \mu) = \frac{1}{2}\left\{\left[\frac{\mu - nf(n, \mu)}{\mu + nf(n, \mu)}\right]^2 + \left[\frac{n\mu - f(n, \mu)}{n\mu + f(n, \mu)}\right]^2\right\}, \tag{3.2}$$

$$f(n, \mu) = [1 - n^2(1 - \mu^2)]^{1/2}, \tag{3.3}$$

and

$$\mu_c(n) = (1 - 1/n^2)^{1/2}. \tag{3.4}$$

We have used $H(x)$ for the Heaviside step function. As noted by Williams [2], when the slab is illuminated (say on the surface at $\tau = 0$) by an incident distribution given by $\psi(\mu)$, then we must use

$$F_1(\mu) = F(n, \mu) = n^2[1 - R(n, \mu)]\psi\{[1 - n^2(1 - \mu^2)]^{1/2}\}, \tag{3.5}$$

in the boundary condition listed as equation (1.2a). For this specific computation, we consider isotropic incidence and write

$$\psi(\mu) = 2, \tag{3.6}$$

and so, we obtain

$$F(n, \mu) = 2n^2[1 - R(n, \mu)]. \tag{3.7}$$

Continuing to follow Williams [2], we compute the total reflectance and transmittance for the slab

$$A(n) = \int_0^1 [1 - R(n, \mu)]I(0, -\mu)\mu\, \mathrm{d}\mu + 2\int_0^1 G(1/n, \mu)\mu\, \mathrm{d}\mu \tag{3.8}$$

and

$$B(n) = \int_0^1 [1 - R(n, \mu)]I(\tau_0, \mu)\mu\, \mathrm{d}\mu. \tag{3.9}$$

Since

$$\int_0^1 \psi(\mu)\mu\, \mathrm{d}\mu = 1, \tag{3.10}$$

Table 1. The reflection and transmission functions $A(n)$ and $B(n)$ for the case of isotropic incidence with the Lth-order binomial scattering law, $\tau_0 = 1.0$, and $\varpi = 0.9$.

	$A(n)$			$B(n)$		
n	$L = 0$	$L = 10$	$L = 100$	$L = 0$	$L = 10$	$L = 100$
1.0	3.527 120(−1)	1.056 963(−1)	2.078 116(−2)	4.747 459(−1)	7.210 146(−1)	8.087 338(−1)
1.2	2.951 590(−1)	1.231 536(−1)	7.508 137(−2)	4.864 328(−1)	6.724 406(−1)	7.662 932(−1)
4/3	2.845 970(−1)	1.518 604(−1)	1.092 776(−1)	4.685 624(−1)	6.247 627(−1)	7.322 807(−1)
1.4	2.818 441(−1)	1.647 357(−1)	1.252 187(−1)	4.579 480(−1)	6.027 303(−1)	7.155 156(−1)
1.6	2.794 295(−1)	1.988 698(−1)	1.696 890(−1)	4.241 081(−1)	5.425 879(−1)	6.666 752(−1)
1.8	2.826 753(−1)	2.278 706(−1)	2.098 356(−1)	3.905 958(−1)	4.900 438(−1)	6.206 368(−1)
2.0	2.895 234(−1)	2.530 848(−1)	2.460 557(−1)	3.592 396(−1)	4.439 980(−1)	5.778 185(−1)

the functions $A(n)$ and $B(n)$ are the normalized currents (or partial fluxes) leaving the slab. The function $A(n)$ has two components: one due to external reflection (of the incoming distribution) at the surface and the other due to scattering within the slab. In this example, the function $B(n)$ is the result of the radiation passing through the slab. To complete the description of our specific calculation, we use the binomial form [5]

$$p(\cos\Theta) = \frac{L+1}{2^L}(1+\cos\Theta)^L \tag{3.11}$$

to describe the scattering process. We note that Θ is the scattering angle and that (here) L is a nonnegative integer. It is known that equation (3.11) can be rewritten in terms of Legendre polynomials as

$$p(\cos\Theta) = \sum_{l=0}^{L} \beta_l P_l(\cos\Theta), \tag{3.12}$$

where, as reported by McCormick and Sanchez [6], the β coefficients can be computed from the recursion formula

$$\beta_l = \left(\frac{2l+1}{2l-1}\right)\left(\frac{L+1-l}{L+1+l}\right)\beta_{l-1}, \tag{3.13}$$

for $l = 1, 2, \ldots, L$, with $\beta_0 = 1$. Our numerical results for $A(n)$ and $B(n)$, thought to be good to all figures listed, are shown (for three values of L) in the accompanying table 1. We note that our results for $L = 0$ are consistent with Williams' calculation [2].

While this computation required only a simple modification of the work given, say, in [4], there was one especially interesting aspect of this calculation: since the derivative of the function $R(n, \mu)$ is discontinuous at $\mu = \mu_c(n)$, the standard Gauss–Legendre quadrature scheme mapped onto the interval [0, 1] had to be modified. We simply used a composite quadrature scheme with a breakpoint at $\mu = \mu_c(n)$. So, as in [7], the final quadrature scheme used depended on the boundary condition (and computed quantities) as well as the equation of transfer.

4. A closing comment

The author would like to point out that this work is considered a brief note, and so only a modest attempt has been made to define the physical aspects of the problem—works by, say, Aronson [8, 9], Elias and coworkers [10–12], and Jin and Stamnes [13], as well as the classical text by Born and Wolf [14], could be consulted for more insight into physical and applicational aspects of Fresnel scattering. Having said that, we believe the reported numerical results that include the effects of anisotropic scattering can be considered a contribution to the field. Importantly, from the computational point-of-view, the fact that the quadrature scheme used to define the final solution incorporated aspects of the boundary data is considered especially noteworthy. As our solution to the single-layer problem for the case of Fresnel boundary conditions yielded what we believe to be numerical results of a very high standard, we will next be looking at the case of the two-layer and then the multi-layer version of this problem. In this context, we should mention here the important works by Liou and Wu [15, 16], and Caron, Andraud and Lafait [17].

Acknowledgments

The author would like to thank M M R Williams for pointing out the interest in this problem and for some subsequent discussions. Useful comments from R Aronson, R D M Garcia, and N J McCormick are noted with thanks, and thanks are also extended to M Elias and A M Yacout for some helpful communications regarding the numerical results reported herein.

References

[1] Chandrasekhar S 1950 *Radiative Transfer* (London: Oxford University Press)
[2] Williams M M R 2006 On an integral equation arising in the transport of radiation through a slab involving internal reflection: II. Generalisation and numerical results for the Fresnel case *Eur. Phys. J.* B **53** 245–55
[3] Barichello L B and Siewert C E 1999 A discrete-ordinates solution for a non-grey model with complete frequency redistribution *J. Quant. Spectrosc. Radiat. Transfer* **62** 665–75
[4] Siewert C E 2000 A concise and accurate solution to Chandrasekhar's basic problem in radiative transfer *J. Quant. Spectrosc. Radiat. Transfer* **64** 109–30
[5] Kaper H G, Shultis J K and Veninga J G 1970 Numerical evaluation of the slab albedo problem solution in one-speed anisotropic transport theory *J. Comput. Phys.* **6** 288–313
[6] McCormick N J and Sanchez R 1981 Inverse problem transport calculations for anisotropic scattering coefficients *J. Math. Phys.* **22** 199–208
[7] Barichello L B and Siewert C E 1999 A discrete-ordinates solution for a polarization model with complete frequency redistribution *Astrophys. J.* **513** 370–82
[8] Aronson R 1995 Boundary conditions for diffusion of light *J. Opt. Soc. Am.* A **12** 2532–9
[9] Aronson R 1997 Radiative transfer implies a modified reciprocity relation *J. Opt. Soc. Am.* A **14** 486–90
[10] da Silva A, Elias M, Andraud C and Lafait J 2003 Comparison of the auxiliary function method and the discrete-ordinate method for solving the radiative transfer equation for light scattering *J. Opt. Soc. Am.* A **20** 2321–9
[11] Elias M and Elias G 2004 Radiative transfer in inhomogeneous stratified scattering media with use of the auxiliary function method *J. Opt. Soc. Am.* A **21** 580–9
[12] Simonot L, Elias M and Charron E 2004 Special visual effect of art glazes explained by the radiative transfer equation *Appl. Opt.* **43** 2580–7
[13] Jin Z and Stamnes K 1994 Radiative transfer in nonuniformly refracting layered media: atmosphere–ocean system *Appl. Opt.* **33** 431–42
[14] Born M and Wolf E 1964 *Principles of Optics* (New York: Pergamon)
[15] Liou B-T and Wu C-Y 1996 Composite discrete-ordinates solutions for radiative transfer in a two-layer medium with Fresnel interfaces *Numer. Heat Transfer* A **30** 739–51
[16] Liou B-T and Wu C-Y 1996 Radiative transfer in a multi-layer medium with Fresnel interfaces *Heat Mass Transfer* **32** 103–7
[17] Caron J, Andraud C and Lafait J 2004 Radiative transfer calculations in multilayer systems with smooth or rough interfaces *J. Mod. Opt.* **51** 575–95