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CHANNEL FLOW OF A BINARY MIXTURE OF RIGID SPHERES DESCRIBED BY THE LINEARIZED BOLTZMANN EQUATION AND DRIVEN BY TEMPERATURE, PRESSURE, AND CONCENTRATION GRADIENTS*

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Abstract. An analytical version of the discrete-ordinates method (the ADO method) is used with recently established analytical expressions for the rigid-sphere scattering kernels in a study devoted to the flow of a binary gas mixture in a plane channel. In particular, concise and accurate solutions to basic flow problems in a plane channel driven by temperature, pressure, and concentration gradients and described by the linearized Boltzmann equation are established for the case of Maxwell boundary conditions for each of the two species. The velocity, heat-flow, and shear-stress profiles, as well as the mass- and heat-flow rates, are established for each species of particles, and numerical results are reported for two binary mixtures (Ne-Ar and He-Xe).

Key words. rarefied gas dynamics, binary mixtures, rigid spheres, channel flow, linearized Boltzmann equation

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1. Introduction.

While the classical problems of Poiseuille flow and thermal-creep flow in a plane channel in the general field of rarefied gas dynamics [24, 3, 5, 4] have been extensively studied for the case of a single-species gas (see, for example, [1, 25, 20, 22, 21, 14, 17] and the references therein), there are relatively few works (for example, [23, 16, 13]) devoted to these problems for gas mixtures. While [23] and [16] are based on the McCormack kinetic model [15], the work of Kosuge et al. [13] is carried out in terms of the linearized Boltzmann equation (LBE). It can be noted that the paper by Siewert and Valougeorgis [23] reports (in terms of the McCormack model) concise and accurate solutions to the problems of channel flow driven by pressure, temperature, and concentration gradients. While the approach used in [16], also based on the McCormack model, is purely numerical, that work does investigate flow in a two-dimensional channel. Most closely related to this work is [13], where purely numerical methods are used to establish some results for channel-flow problems based on the LBE.

In this work, we develop and evaluate concise and accurate solutions for flow problems in a plane-parallel channel driven by pressure, temperature, and concentration gradients. We make use of an analytical discrete-ordinates method (ADO method, [2]), and we use (in the LBE) explicit forms of the rigid-sphere collision kernels for binary gas mixtures [12, 6, 8]. The developed solutions depend (aside from some normalizations) only on the mass and diameter ratios and the relative equilibrium concentration of the two species of particles. We allow a free choice of the accommodation coefficients for each species at the confining surfaces of the channel. Our

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approach relies on a continuous treatment of both the space and speed variables that has proved to be particularly efficient and accurate for other classical problems for binary gas mixtures [7, 9, 10].

2. Basic formulation. The flow problems considered in this work are driven by a temperature gradient, a pressure gradient, or concentration gradients (or any linear combination of these effects), and so we base our linearizations of the particle distribution functions about local rather than absolute conditions, as was done in [9], for example. We use x to measure distance in the direction (parallel to the confining walls of the plane-parallel channel) of the mentioned gradients, and so we write the local Maxwellians (for the two species of particles identified by the subscripts $\alpha = 1$ and 2) as

$$f_{\alpha,0}(x,v) = n_\alpha(x)\left[\frac{m_\alpha}{2\pi kT(x)}\right]^{3/2}\exp\left\{-\frac{m_\alpha v^2}{2kT(x)}\right\}, \quad \alpha = 1, 2, \tag{2.1}$$

where v is the magnitude of the velocity $\boldsymbol{v}$. If we now express the considered linear variations in the number densities and the temperature as

$$n_\alpha(x) = n_\alpha(1 + R_\alpha x), \quad \alpha = 1, 2, \tag{2.2}$$

and

$$T(x) = T_0(1 + K_T x), \tag{2.3}$$

where R_α and K_T are considered to be given (small) constants, we can linearize (2.1) to obtain the approximations

$$f^*_{\alpha,0}(x,v) = f_{\alpha,0}(v)[1 + f_\alpha(v)x], \quad \alpha = 1, 2, \tag{2.4}$$

where

$$f_{\alpha,0}(v) = n_\alpha(\lambda_\alpha/\pi)^{3/2}\mathrm{e}^{-\lambda_\alpha v^2}, \quad \lambda_\alpha = m_\alpha/(2kT_0), \tag{2.5}$$

is the absolute Maxwellian distribution for n_α particles of mass m_α in equilibrium at temperature T_0. Here k is the Boltzmann constant, and the $f_\alpha(v)$ are to be determined. If we express the pressure distribution as

$$p(x) = p_0(1 + K_P x), \tag{2.6}$$

where $p_0 = nkT_0$, $n = n_1 + n_2$, and K_P is a given (small) constant, then using the perfect gas law

$$p(x) = n(x)kT(x), \tag{2.7}$$

where

$$n(x) = n_1(x) + n_2(x), \tag{2.8}$$

we find, after neglecting second-order effects,

$$c_1 R_1 + c_2 R_2 = K_P - K_T, \tag{2.9}$$

where $c_\alpha = n_\alpha/n$, $\alpha = 1, 2$. And so, making use of (2.9), we find that we can use

$$f_1(v) = [m_1 v^2/(2kT_0) - 5/2]K_T + K_P + c_2 K_C \tag{2.10a}$$

and

$$f_2(v) = [m_2 v^2/(2kT_0) - 5/2]K_T + K_P - c_1 K_C, \tag{2.10b}$$

with $K_C = R_1 - R_2$, to complete (2.4). Using the variable $z \in [-z_0, z_0]$ to measure the transverse or cross-channel direction, we now write the true velocity distributions as

$$f_\alpha(x, z, \boldsymbol{v}) = f_{\alpha,0}(v)\{1 + f_\alpha(v)x + h_\alpha(z, \lambda_\alpha^{1/2}\boldsymbol{v})\}, \tag{2.11}$$

where the perturbations $h_\alpha(z, \lambda_\alpha^{1/2}\boldsymbol{v})$ are to be determined from a form of the LBE used in [12, 6, 8, 7, 10] that has an added inhomogeneous driving term due to the x variation in (2.11).

And so we proceed with an inhomogeneous form of the LBE, for a binary mixture of rigid spheres, written as

$$\boldsymbol{S}(\boldsymbol{c}) + c\mu\frac{\partial}{\partial z}\boldsymbol{H}(z, \boldsymbol{c}) + \varepsilon_0 \boldsymbol{V}(c)\boldsymbol{H}(z, \boldsymbol{c}) = \varepsilon_0 \int \mathrm{e}^{-c'^2}\boldsymbol{\mathcal{K}}(\boldsymbol{c}' : \boldsymbol{c})\boldsymbol{H}(z, \boldsymbol{c}')\mathrm{d}^3c', \tag{2.12}$$

where ε_0 is, at this point, an arbitrary parameter that we will soon use to define a dimensionless spatial variable,

$$\boldsymbol{H}(z, \boldsymbol{c}) = \begin{bmatrix} h_1(z, \boldsymbol{c}) \\ h_2(z, \boldsymbol{c}) \end{bmatrix}, \tag{2.13}$$

and

$$\boldsymbol{S}(\boldsymbol{c}) = c(1-\mu^2)^{1/2}\cos\phi \left\{(c^2 - 5/2)K_T \begin{bmatrix} 1 \\ 1 \end{bmatrix} + K_P \begin{bmatrix} 1 \\ 1 \end{bmatrix} + K_C \begin{bmatrix} c_2 \\ -c_1 \end{bmatrix}\right\}. \tag{2.14}$$

Considering that the driving term in (2.12) is given by (2.14), we note that (i) the case of flow driven by a temperature gradient corresponds to $K_P = 0$, $K_C = 0$, and $K_T \neq 0$, (ii) the case of flow driven by a pressure gradient corresponds to $K_T = 0$, $K_C = 0$, and $K_P \neq 0$, and (iii) the case of flow driven by concentration gradients corresponds to $K_P = 0$, $K_T = 0$, and $K_C \neq 0$. Furthermore, we note that in writing (2.12), we have introduced the variable changes

$$h_\alpha(z, \boldsymbol{c}) = h_\alpha(z, \lambda_\alpha^{1/2}\boldsymbol{v}), \quad \alpha = 1, 2, \tag{2.15}$$

in order to work with the dimensionless velocity variable $\boldsymbol{c}$. Continuing, we note that we use spherical coordinates $\{c, \theta, \phi\}$, with $\mu = \cos\theta$, to describe the dimensionless velocity vector, so that

$$\boldsymbol{H}(z, \boldsymbol{c}) \Leftrightarrow \boldsymbol{H}(z, c, \mu, \phi).$$

In our notation, $c\mu$ is the component of the (dimensionless) velocity vector in the positive z direction, and

$$c_x = c(1-\mu^2)^{1/2}\cos\phi \tag{2.16}$$

is the component of velocity in the direction x (parallel to the confining surfaces) of the flow.

In regard to the homogeneous version of (2.12), we note that all of the defining elements have been developed in a recent series of papers [12, 6, 8]. We consider that these works [12, 6, 8] can be consulted if a complete understanding of all of the required elements is desired. And so at this point we simply quote from our previous work [12, 6, 8] and list without additional comments the required definitions. First,

$$\boldsymbol{V}(c) = (1/\varepsilon_0)\boldsymbol{\Sigma}(c) \tag{2.17}$$

and

$$\boldsymbol{\mathcal{K}}(\boldsymbol{c}' : \boldsymbol{c}) = (1/\varepsilon_0)\boldsymbol{K}(\boldsymbol{c}' : \boldsymbol{c}), \tag{2.18}$$

where

$$\boldsymbol{\Sigma}(c) = \begin{bmatrix} \varpi_1(c) & 0 \\ 0 & \varpi_2(c) \end{bmatrix}, \tag{2.19}$$

with

$$\varpi_\alpha(c) = \varpi_\alpha^{(1)}(c) + \varpi_\alpha^{(2)}(c) \tag{2.20}$$

and

$$\varpi_\alpha^{(\beta)}(c) = 4\pi^{1/2} n_\beta \sigma_{\alpha,\beta} a_{\beta,\alpha} \nu(a_{\alpha,\beta} c). \tag{2.21}$$

Here

$$\nu(c) = \frac{2c^2+1}{c}\int_0^c \mathrm{e}^{-x^2}\mathrm{d}x + \mathrm{e}^{-c^2} \tag{2.22}$$

and

$$a_{\alpha,\beta} = (m_\beta/m_\alpha)^{1/2}, \quad \alpha,\beta = 1,2. \tag{2.23}$$

We use $\sigma_{\alpha,\beta}$ to denote the differential-scattering cross section, which (for the case of rigid-sphere scattering that is isotropic in the center-of-mass system) we write as [4]

$$\sigma_{\alpha,\beta} = \frac{1}{4}\left(\frac{d_\alpha + d_\beta}{2}\right)^2, \quad \alpha,\beta = 1,2, \tag{2.24}$$

where d_1 and d_2 are the atomic diameters of the two types of gas particles. We continue to follow [12, 6, 8] and write

$$\boldsymbol{K}(\boldsymbol{c}' : \boldsymbol{c}) = \begin{bmatrix} K_{1,1}(\boldsymbol{c}' : \boldsymbol{c}) & K_{1,2}(\boldsymbol{c}' : \boldsymbol{c}) \\ K_{2,1}(\boldsymbol{c}' : \boldsymbol{c}) & K_{2,2}(\boldsymbol{c}' : \boldsymbol{c}) \end{bmatrix}, \tag{2.25}$$

where

$$K_{1,1}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_1\sigma_{1,1}\pi^{1/2}\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) + n_2\sigma_{1,2}\pi^{1/2}\mathcal{F}_{1,2}(\boldsymbol{c}' : \boldsymbol{c}), \tag{2.26}$$

$$K_{1,2}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_2\sigma_{1,2}\pi^{1/2}\mathcal{G}_{1,2}(\boldsymbol{c}' : \boldsymbol{c}), \tag{2.27}$$

$$K_{2,1}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_1\sigma_{2,1}\pi^{1/2}\mathcal{G}_{2,1}(\boldsymbol{c}' : \boldsymbol{c}), \tag{2.28}$$

and

$$K_{2,2}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_2\sigma_{2,2}\pi^{1/2}\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) + n_1\sigma_{2,1}\pi^{1/2}\mathcal{F}_{2,1}(\boldsymbol{c}' : \boldsymbol{c}). \tag{2.29}$$

Here

$$\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{\pi}\left(\frac{2}{|\boldsymbol{c}'-\boldsymbol{c}|}\exp\left\{\frac{|\boldsymbol{c}'\times\boldsymbol{c}|^2}{|\boldsymbol{c}'-\boldsymbol{c}|^2}\right\} - |\boldsymbol{c}'-\boldsymbol{c}|\right) \tag{2.30}$$

is the basic kernel for a single-species gas used by Pekeris [18]. In addition,

$$\mathcal{F}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c}) = \mathcal{F}(a_{\alpha,\beta}; \boldsymbol{c}' : \boldsymbol{c}) \tag{2.31}$$

and

$$\mathcal{G}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c}) = \mathcal{G}(a_{\alpha,\beta}; \boldsymbol{c}' : \boldsymbol{c}), \tag{2.32}$$

where

$$\mathcal{F}(a; \boldsymbol{c}' : \boldsymbol{c}) = \frac{(a^2+1)^2}{a^3\pi|\boldsymbol{c}'-\boldsymbol{c}|}\exp\left\{a^2\frac{|\boldsymbol{c}'\times\boldsymbol{c}|^2}{|\boldsymbol{c}'-\boldsymbol{c}|^2} - \frac{(1-a^2)^2({c'}^2+c^2)}{4a^2} - \frac{(a^4-1)\boldsymbol{c}'\cdot\boldsymbol{c}}{2a^2}\right\} \tag{2.33}$$

and

$$\mathcal{G}(a; \boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{a\pi}\big|\boldsymbol{c}' - a\boldsymbol{c}\big|[J(a; \boldsymbol{c}' : \boldsymbol{c}) - 1], \tag{2.34}$$

with

$$J(a; \boldsymbol{c}' : \boldsymbol{c}) = \frac{(a+1/a)^2}{2\Delta(a; \boldsymbol{c}' : \boldsymbol{c})}\exp\left\{\frac{-2C(a; \boldsymbol{c}' : \boldsymbol{c})}{(a-1/a)^2}\right\}\sinh\left\{\frac{2\Delta(a; \boldsymbol{c}' : \boldsymbol{c})}{(a-1/a)^2}\right\}, \quad a \neq 1, \tag{2.35a}$$

or

$$J(a; \boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{|\boldsymbol{c}'-\boldsymbol{c}|^2}\exp\left\{\frac{|\boldsymbol{c}'\times\boldsymbol{c}|^2}{|\boldsymbol{c}'-\boldsymbol{c}|^2}\right\}, \quad a = 1. \tag{2.35b}$$

We have used the definitions [12, 6, 8]

$$\Delta(a; \boldsymbol{c}' : \boldsymbol{c}) = \left\{C^2(a; \boldsymbol{c}' : \boldsymbol{c}) + (a-1/a)^2|\boldsymbol{c}'\times\boldsymbol{c}|^2\right\}^{1/2} \tag{2.36}$$

and

$$C(a; \boldsymbol{c}' : \boldsymbol{c}) = {c'}^2 + c^2 - (a+1/a)\boldsymbol{c}'\cdot\boldsymbol{c}. \tag{2.37}$$

In this work, we intend to compute the velocity, the shear-stress, and the heat-flow profiles which we express as

$$\boldsymbol{U}(z) = \frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2}\boldsymbol{H}(z,\boldsymbol{c})c^3(1-\mu^2)^{1/2}\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.38}$$

$$\boldsymbol{P}(z) = \frac{2}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2}\boldsymbol{H}(z,\boldsymbol{c})c^4\mu(1-\mu^2)^{1/2}\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.39}$$

and

$$\boldsymbol{Q}(z) = \frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi} \mathrm{e}^{-c^2}\boldsymbol{H}(z,\boldsymbol{c})\left(c^2-\frac{5}{2}\right)c^3(1-\mu^2)^{1/2}\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.40}$$

where the components of $\boldsymbol{U}(z)$, $\boldsymbol{P}(z)$, and $\boldsymbol{Q}(z)$ are the functions $U_\alpha(z)$, $P_\alpha(z)$, and $Q_\alpha(z)$, for $\alpha = 1, 2$, that can be used, as mentioned in Appendix A of [9], to define the macroscopic quantities for a binary mixture.

As in [9], it is clear (for the specific flow problems considered here) that an expansion of $\boldsymbol{H}(z, \boldsymbol{c})$ in a Fourier series (in the angle ϕ) requires only one term—that is, one proportional to $\cos\phi$. And so we follow [8] and introduce the dimensionless spatial variable

$$\tau = z\varepsilon_0, \tag{2.41}$$

where

$$\varepsilon_0 = (n_1 + n_2)\pi^{1/2}\left(\frac{n_1 d_1 + n_2 d_2}{n_1 + n_2}\right)^2, \tag{2.42}$$

and write

$$\boldsymbol{H}(\tau/\varepsilon_0, \boldsymbol{c}) = \boldsymbol{\Psi}(\tau, c, \mu)(1 - \mu^2)^{1/2}\cos\phi, \tag{2.43}$$

where $\boldsymbol{\Psi}(\tau, c, \mu)$ is the (vector-valued) function to be determined. We now let $z = \tau/\varepsilon_0$ in (2.38)–(2.40) and consider that

$$\boldsymbol{U}(\tau) = \frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\boldsymbol{\Psi}(\tau, c, \mu)c^3(1 - \mu^2)\mathrm{d}\mu\mathrm{d}c, \tag{2.44}$$

$$\boldsymbol{P}(\tau) = \frac{2}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\boldsymbol{\Psi}(\tau, c, \mu)c^4(1 - \mu^2)\mu\mathrm{d}\mu\mathrm{d}c, \tag{2.45}$$

and

$$\boldsymbol{Q}(\tau) = \frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c^2}\boldsymbol{\Psi}(\tau, c, \mu)\left(c^2 - \frac{5}{2}\right)c^3(1 - \mu^2)\mathrm{d}\mu\mathrm{d}c \tag{2.46}$$

are the quantities to be computed. It should be noted that to avoid excessive notation, we have, in writing (2.44)–(2.46), followed the (often-used) procedure of not always introducing new labels for dependent quantities (in this case $\boldsymbol{U}$, $\boldsymbol{P}$, and $\boldsymbol{Q}$) when the independent variable is changed.

We can now use (2.43) in (2.12), multiply the resulting equation by $\cos\phi$, integrate over all ϕ, and use the Legendre expansion of the scattering kernel $\boldsymbol{\mathcal{K}}(\boldsymbol{c}' : \boldsymbol{c})$ that was introduced in a previous work—see equations (26) and (65) of [8]—to find

$$\begin{aligned}\boldsymbol{\Upsilon}(c) + c\mu\frac{\partial}{\partial\tau}\boldsymbol{\Psi}(\tau, c, \mu) + \boldsymbol{V}(c)\boldsymbol{\Psi}(\tau, c, \mu)\\ = \int_0^\infty\int_{-1}^1 \mathrm{e}^{-c'^2} f(\mu', \mu)\boldsymbol{\mathcal{K}}(c', \mu' : c, \mu)\boldsymbol{\Psi}(\tau, c', \mu')c'^2\mathrm{d}\mu'\mathrm{d}c',\end{aligned} \tag{2.47}$$

where

$$f(\mu', \mu) = \left(\frac{1 - \mu'^2}{1 - \mu^2}\right)^{1/2}. \tag{2.48}$$

In addition,

$$\boldsymbol{\mathcal{K}}(c', \mu' : c, \mu)\cos\phi' = \int_0^{2\pi}\boldsymbol{\mathcal{K}}(\boldsymbol{c}' : \boldsymbol{c})\cos\phi\mathrm{d}\phi, \tag{2.49}$$

which we can express, in the notation of [8], as

$$\boldsymbol{\mathcal{K}}(c', \mu' : c, \mu) = (1/2)\sum_{n=1}^{\infty}(2n+1)P_n^1(\mu')P_n^1(\mu)\boldsymbol{\mathcal{K}}_n(c', c), \tag{2.50}$$

where $P_n^1(x)$ is used to denote one of the normalized associated Legendre functions. More explicitly,

$$P_l^m(\mu) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2}(1-\mu^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}\mu^m}P_l(\mu), \tag{2.51}$$

where $P_l(\mu)$ is the Legendre polynomial. In addition,

$$\boldsymbol{\mathcal{K}}_n(c', c) = \begin{bmatrix} \mathcal{K}_n^{(1,1)}(c', c) & \mathcal{K}_n^{(1,2)}(c', c) \\ \mathcal{K}_n^{(2,1)}(c', c) & \mathcal{K}_n^{(2,2)}(c', c) \end{bmatrix}, \tag{2.52}$$

with

$$\mathcal{K}_n^{(1,1)}(c', c) = p_1\mathcal{P}^{(n)}(c', c) + (g_2/4)\mathcal{F}^{(n)}(a_{1,2}; c', c), \tag{2.53a}$$

$$\mathcal{K}_n^{(1,2)}(c', c) = g_2\mathcal{G}^{(n)}(a_{1,2}; c', c), \tag{2.53b}$$

$$\mathcal{K}_n^{(2,1)}(c', c) = g_1\mathcal{G}^{(n)}(a_{2,1}; c', c), \tag{2.53c}$$

and

$$\mathcal{K}_n^{(2,2)}(c', c) = p_2\mathcal{P}^{(n)}(c', c) + (g_1/4)\mathcal{F}^{(n)}(a_{2,1}; c', c). \tag{2.53d}$$

We also can write

$$\boldsymbol{V}(c) = \begin{bmatrix} v_1(c) & 0 \\ 0 & v_2(c) \end{bmatrix}, \tag{2.54}$$

where now

$$v_1(c) = p_1\nu(c) + g_2 a_{2,1}\nu(a_{1,2}c) \tag{2.55a}$$

and

$$v_2(c) = p_2\nu(c) + g_1 a_{1,2}\nu(a_{2,1}c). \tag{2.55b}$$

In writing (2.53) and (2.55), we have used

$$p_\alpha = c_\alpha\left(\frac{nd_\alpha}{n_1d_1 + n_2d_2}\right)^2, \quad \alpha = 1, 2, \tag{2.56a}$$

and

$$g_\alpha = c_\alpha\left(\frac{nd_{\mathrm{avg}}}{n_1d_1 + n_2d_2}\right)^2, \quad \alpha = 1, 2, \tag{2.56b}$$

where

$$d_{\mathrm{avg}} = (d_1 + d_2)/2. \tag{2.57}$$

In order to avoid too much repetition, we do not list here our expressions for the Legendre moments

$$\mathcal{P}^{(n)}(c', c), \quad \mathcal{F}^{(n)}(a; c', c), \quad \text{and} \quad \mathcal{G}^{(n)}(a; c', c),$$

since they are explicitly given in Appendix A of [8]. To complete (2.47), we note that the inhomogeneous driving term is

$$\boldsymbol{\Upsilon}(c) = (c/\varepsilon_0) \begin{bmatrix} (c^2 - 5/2)K_T + K_P + c_2 K_C \\ (c^2 - 5/2)K_T + K_P - c_1 K_C \end{bmatrix}. \tag{2.58}$$

At the walls located at $\tau = -a$ and $\tau = a$, we use a combination of specular and diffuse reflection, and so, in regard to (2.12), we write the boundary conditions as

$$\boldsymbol{H}(-a, c, \mu, \phi) - (\boldsymbol{I} - \boldsymbol{\alpha})\boldsymbol{H}(-a, c, -\mu, \phi) - \boldsymbol{\alpha}\boldsymbol{\mathcal{I}}_-\{\boldsymbol{H}\}(-a) = \boldsymbol{0} \tag{2.59a}$$

and

$$\boldsymbol{H}(a, c, -\mu, \phi) - (\boldsymbol{I} - \boldsymbol{\beta})\boldsymbol{H}(a, c, \mu, \phi) - \boldsymbol{\beta}\boldsymbol{\mathcal{I}}_+\{\boldsymbol{H}\}(a) = \boldsymbol{0} \tag{2.59b}$$

for $\mu \in (0, 1]$ and all c and all ϕ. Here

$$\boldsymbol{\mathcal{I}}_{\mp}\{\boldsymbol{H}\}(z) = \frac{2}{\pi} \int_0^{\infty} \int_0^1 \int_0^{2\pi} \mathrm{e}^{-c'^2} \boldsymbol{H}(z, c', \mp\mu', \phi')\mu' c'^3 \mathrm{d}\phi' \mathrm{d}\mu' \mathrm{d}c', \tag{2.60}$$

$$\boldsymbol{\alpha} = \mathrm{diag}\{\alpha_1, \alpha_2\}, \tag{2.61a}$$

and

$$\boldsymbol{\beta} = \mathrm{diag}\{\beta_1, \beta_2\}, \tag{2.61b}$$

where α_1, α_2, β_1, and β_2 are the accommodation coefficients to be used for the two species of gas particles at the confining surfaces. Taking note of (2.43), we find from (2.59) the boundary conditions subject to which we must solve (2.47), that is,

$$\boldsymbol{\Psi}(-a, c, \mu) - (\boldsymbol{I} - \boldsymbol{\alpha})\boldsymbol{\Psi}(-a, c, -\mu) = \boldsymbol{0} \tag{2.62a}$$

and

$$\boldsymbol{\Psi}(a, c, -\mu) - (\boldsymbol{I} - \boldsymbol{\beta})\boldsymbol{\Psi}(a, c, \mu) = \boldsymbol{0}, \tag{2.62b}$$

for $\mu \in (0, 1]$ and all c. We use $\boldsymbol{I}$ to denote the 2×2 identity matrix.

3. Solutions. Following our previous work as reported in [9, 11], we express our solution (evaluated at the N pairs of discrete ordinates $\pm\mu_i$) of (2.47) in the form

$$\boldsymbol{\Psi}(\tau, c, \pm\mu_i) = \boldsymbol{\Psi}_{ps}(\tau, c, \pm\mu_i) + \boldsymbol{\Psi}_*(\tau, c, \pm\mu_i) + \boldsymbol{\Psi}_{app}(\tau, c, \pm\mu_i) \tag{3.1}$$

for $i = 1, 2, \ldots, N$. We note that $\boldsymbol{\Psi}_*(\tau, c, \mu)$ is defined in terms of two of the exact elementary solutions we reported in a previous work [8], that is,

$$\boldsymbol{\Psi}_*(\tau, c, \mu) = A_1 c\boldsymbol{\Phi} + B_1\big[c\tau\boldsymbol{\Phi} - \mu\boldsymbol{B}(c)\big], \tag{3.2}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 \\ a_{1,2} \end{bmatrix}, \tag{3.3}$$

and where $\boldsymbol{B}(c)$ is one of the generalized Chapman–Enskog (vector-valued) functions discussed in [8]. In addition,

(3.4)
$$\boldsymbol{\Psi}_{app}(\tau, c, \pm\mu_i) = \boldsymbol{\Pi}(c) \sum_{j=2}^{J} \big[A_j \boldsymbol{\Phi}(\nu_j, \pm\mu_i) \mathrm{e}^{-(a+\tau)/\nu_j} + B_j \boldsymbol{\Phi}(\nu_j, \mp\mu_i) \mathrm{e}^{-(a-\tau)/\nu_j}\big].$$

For our computations, we use the $2 \times 2(K+1)$ matrix

$$\boldsymbol{\Pi}(c) = \Big[P_0(2\mathrm{e}^{-c} - 1)\boldsymbol{I} \;\; P_1(2\mathrm{e}^{-c} - 1)\boldsymbol{I} \cdots P_K(2\mathrm{e}^{-c} - 1)\boldsymbol{I}\Big], \tag{3.5}$$

where $K+1$ is the number of basis functions used to represent the speed dependence of the approximate part of our solution. We note that [9] can be consulted if a complete understanding of the eigenvalue spectrum $\{\nu_j\}$ and the elementary solutions $\{\boldsymbol{\Phi}(\nu_j, \pm\mu_i)\}$ is desired. Since (2.47) has the inhomogeneous driving term $\boldsymbol{\Upsilon}(c)$, we have included in (3.1) the particular solution

$$\boldsymbol{\Psi}_{ps}(\tau, c, \mu) = \boldsymbol{\Psi}_P(\tau, c, \mu) + \boldsymbol{\Psi}_T(\tau, c, \mu) + \boldsymbol{\Psi}_C(\tau, c, \mu), \tag{3.6}$$

the elements of which were developed and reported in [11]. We repeat from [11]:

(3.7)
$$\boldsymbol{\Psi}_P(\tau, c, \mu) = [1/(\varepsilon_0\varepsilon_p)]\{c\tau^2\boldsymbol{\Phi} - 2\mu\tau\boldsymbol{B}(c) + (1/5)\boldsymbol{D}(c) + [(5\mu^2 - 1)/5]\boldsymbol{E}(c)\}K_P,$$

$$\boldsymbol{\Psi}_T(\tau, c, \mu) = -(1/\varepsilon_0)[\boldsymbol{A}^{(1)}(c) + \boldsymbol{A}^{(2)}(c)]K_T, \tag{3.8}$$

and

$$\boldsymbol{\Psi}_C(\tau, c, \mu) = (1/\varepsilon_0)[c_2\boldsymbol{A}^{(1)}(c) - c_1\boldsymbol{A}^{(2)}(c)]K_C. \tag{3.9}$$

In [8] and [11], we have defined and computed, for selected cases, the generalized Chapman–Enskog and Burnett (vector-valued) functions $\boldsymbol{A}^{(1)}(c)$, $\boldsymbol{A}^{(2)}(c)$, $\boldsymbol{B}(c)$, $\boldsymbol{D}(c)$, and $\boldsymbol{E}(c)$ that appear in (3.7)–(3.9). In addition, the constant ε_p is expressed in [11] as

$$\varepsilon_p = \big[c_1 \quad c_2\big]\boldsymbol{\varepsilon}_p, \tag{3.10}$$

where

$$\boldsymbol{\varepsilon}_p = \frac{16}{15\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{B}(c)c^4\mathrm{d}c. \tag{3.11}$$

We note that the components $\varepsilon_{p,1}$ and $\varepsilon_{p,2}$ of $\boldsymbol{\varepsilon}_p$ have been evaluated (for several data sets) in [8].

Finally, to complete our discussion of (3.1), we note that the arbitrary constants $\{A_j, B_j\}$ are to be determined from boundary conditions to be applied at $\tau = \pm a$. For this purpose, we substitute (3.1) into discrete-ordinates versions of (2.62), multiply the resulting equations by

$$c^2 \exp\{-c^2\}\boldsymbol{\Pi}^T(c),$$

where the superscript T is used to denote the transpose operation, and integrate over all c to define a system of $2J$ linear algebraic equations for the $2J$ unspecified constants. We note that only the right-hand-side vector of such system is problem-dependent.

4. Quantities of interest. Considering that we have solved the system of linear algebraic equations to establish the arbitrary constants $\{A_j, B_j\}$, we can use (3.1) to find our final expressions for the quantities of interest here, that is, the velocity, heat-flow, and shear-stress profiles. And so, using (3.1) in discrete-ordinates versions of (2.44)–(2.46), we find

(4.1a)
$$\boldsymbol{U}(\tau) = \boldsymbol{U}_{ps}(\tau) + (1/2)(A_1 + B_1\tau)\boldsymbol{\Phi} + \sum_{j=2}^{J} \big[A_j \mathrm{e}^{-(a+\tau)/\nu_j} + B_j \mathrm{e}^{-(a-\tau)/\nu_j}\big]\boldsymbol{\mathcal{U}}_j,$$

(4.1b)
$$\boldsymbol{Q}(\tau) = \boldsymbol{Q}_{ps}(\tau) + \sum_{j=2}^{J} \big[A_j \mathrm{e}^{-(a+\tau)/\nu_j} + B_j \mathrm{e}^{-(a-\tau)/\nu_j}\big]\boldsymbol{\mathcal{Q}}_j,$$

and

(4.1c)
$$\boldsymbol{P}(\tau) = \boldsymbol{P}_{ps}(\tau) - (1/2)B_1\boldsymbol{\varepsilon}_p + \sum_{j=2}^{J} \big[A_j \mathrm{e}^{-(a+\tau)/\nu_j} - B_j \mathrm{e}^{-(a-\tau)/\nu_j}\big]\boldsymbol{\mathcal{P}}_j.$$

In writing (4.1), we have used the definitions

(4.2a)
$$\boldsymbol{\mathcal{U}}_j = \boldsymbol{\Pi}_1 \boldsymbol{X}_j,$$

(4.2b)
$$\boldsymbol{\mathcal{P}}_j = 2\boldsymbol{\Pi}_2 \boldsymbol{Y}_j,$$

and

(4.2c)
$$\boldsymbol{\mathcal{Q}}_j = [\boldsymbol{\Pi}_3 - (5/2)\boldsymbol{\Pi}_1]\boldsymbol{X}_j,$$

where

(4.3a)
$$\boldsymbol{X}_j = \frac{1}{\pi^{1/2}} \sum_{k=1}^{N} w_k (1-\mu_k^2)[\boldsymbol{\Phi}(\nu_j, \mu_k) + \boldsymbol{\Phi}(\nu_j, -\mu_k)],$$

(4.3b)
$$\boldsymbol{Y}_j = \frac{1}{\pi^{1/2}} \sum_{k=1}^{N} w_k \mu_k (1-\mu_k^2)[\boldsymbol{\Phi}(\nu_j, \mu_k) - \boldsymbol{\Phi}(\nu_j, -\mu_k)],$$

and

(4.4)
$$\boldsymbol{\Pi}_n = \int_0^\infty \mathrm{e}^{-c^2} \boldsymbol{\Pi}(c) c^{n+2} \mathrm{d}c.$$

In (4.3), we use the weights $\{w_k\}$, along with the nodes $\{\mu_k\}$, to complete the definition of our N-point, half-range quadrature scheme. Finally, to complete (4.1), we must compute

(4.5)
$$\boldsymbol{U}_{ps}(\tau) = \frac{1}{\pi^{1/2}} \int_0^\infty \int_{-1}^{1} \mathrm{e}^{-c^2} \boldsymbol{\Psi}_{ps}(\tau, c, \mu) c^3 (1-\mu^2) \mathrm{d}\mu \mathrm{d}c,$$

(4.6)
$$\boldsymbol{Q}_{ps}(\tau) = \frac{1}{\pi^{1/2}} \int_0^\infty \int_{-1}^{1} \mathrm{e}^{-c^2} \boldsymbol{\Psi}_{ps}(\tau, c, \mu) \left(c^2 - \frac{5}{2}\right) c^3 (1-\mu^2) \mathrm{d}\mu \mathrm{d}c,$$

and

$$\boldsymbol{P}_{ps}(\tau) = \frac{2}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 \mathrm{e}^{-c^2} \boldsymbol{\Psi}_{ps}(\tau, c, \mu) c^4 (1-\mu^2) \mu \mathrm{d}\mu \mathrm{d}c. \tag{4.7}$$

Using (3.6)–(3.9), we find

$$\boldsymbol{U}_{ps}(\tau) = (1/\varepsilon_0)\{(1/\varepsilon_p)[(1/2)\tau^2 \boldsymbol{\Phi} + \boldsymbol{D}_U] K_p - [\boldsymbol{A}_U^{(1)} + \boldsymbol{A}_U^{(2)}] K_T + [c_2 \boldsymbol{A}_U^{(1)} - c_1 \boldsymbol{A}_U^{(2)}] K_C\}, \tag{4.8}$$

$$\boldsymbol{Q}_{ps}(\tau) = (1/\varepsilon_0)\{(1/\varepsilon_p) \boldsymbol{D}_Q K_p - [\boldsymbol{A}_Q^{(1)} + \boldsymbol{A}_Q^{(2)}] K_T + [c_2 \boldsymbol{A}_Q^{(1)} - c_1 \boldsymbol{A}_Q^{(2)}] K_C\}, \tag{4.9}$$

and

$$\boldsymbol{P}_{ps}(\tau) = -[\tau/(\varepsilon_0 \varepsilon_p)] \boldsymbol{\varepsilon}_p K_p, \tag{4.10}$$

where

$$\boldsymbol{D}_U = \frac{4}{15\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2} \boldsymbol{D}(c) c^3 \mathrm{d}c, \tag{4.11a}$$

$$\boldsymbol{A}_U^{(\alpha)} = \frac{4}{3\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2} \boldsymbol{A}^{(\alpha)}(c) c^3 \mathrm{d}c, \quad \alpha = 1, 2, \tag{4.11b}$$

$$\boldsymbol{D}_Q = \frac{4}{15\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2} \boldsymbol{D}(c) \left(c^2 - \frac{5}{2}\right) c^3 \mathrm{d}c, \tag{4.11c}$$

and

$$\boldsymbol{A}_Q^{(\alpha)} = \frac{4}{3\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2} \boldsymbol{A}^{(\alpha)}(c) \left(c^2 - \frac{5}{2}\right) c^3 \mathrm{d}c, \quad \alpha = 1, 2. \tag{4.11d}$$

Since the expressions listed as (4.1a), (4.1b), (4.8), and (4.9) are analytical and continuous in the space variable, we can immediately find results for the normalized mass- and heat-flow rates

$$\boldsymbol{U} = \frac{1}{2a^2} \int_{-a}^{a} \boldsymbol{U}(\tau) \mathrm{d}\tau \tag{4.12}$$

and

$$\boldsymbol{Q} = \frac{1}{2a^2} \int_{-a}^{a} \boldsymbol{Q}(\tau) \mathrm{d}\tau, \tag{4.13}$$

where the factor $1/(2a^2)$ is included in order to be consistent with definitions adopted in other works and to facilitate comparisons with numerical results reported in these works. We find

$$\boldsymbol{U} = \frac{1}{2a^2} \left[\boldsymbol{U}_{ps} + a A_1 \boldsymbol{\Phi} + \sum_{j=2}^{J} \nu_j (A_j + B_j)(1 - \mathrm{e}^{-2a/\nu_j}) \boldsymbol{\mathcal{U}}_j \right] \tag{4.14}$$

and

$$\boldsymbol{Q} = \frac{1}{2a^2} \left[\boldsymbol{Q}_{ps} + \sum_{j=2}^{J} \nu_j (A_j + B_j)(1 - \mathrm{e}^{-2a/\nu_j}) \boldsymbol{\mathcal{Q}}_j \right], \tag{4.15}$$

Table 1

Pressure-driven flow: species-specific velocity, heat-flow, and shear-stress profiles for the Ne-Ar mixture with $2a = 0.1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, *and* $n_1/n_2 = 2/3$.

η	$-U_1(-a+2\eta a)$	$-U_2(-a+2\eta a)$	$Q_1(-a+2\eta a)$	$Q_2(-a+2\eta a)$	$P_1(-a+2\eta a)$	$P_2(-a+2\eta a)$
0.0	2.0012(−1)	1.6301(−1)	6.6293(−2)	4.8754(−2)	1.7779(−2)	3.2742(−2)
0.1	2.0308(−1)	1.7049(−1)	6.7557(−2)	5.1904(−2)	9.6303(−3)	2.1508(−2)
0.2	2.0430(−1)	1.7393(−1)	6.8083(−2)	5.3332(−2)	1.4419(−3)	1.0300(−2)
0.3	2.0455(−1)	1.7547(−1)	6.8208(−2)	5.3981(−2)	−6.7713(−3)	−8.9089(−4)
0.4	2.0395(−1)	1.7549(−1)	6.7987(−2)	5.4019(−2)	−1.5002(−2)	−1.2071(−2)
0.5	2.0254(−1)	1.7409(−1)	6.7431(−2)	5.3493(−2)	−2.3243(−2)	−2.3243(−2)
0.6	2.0028(−1)	1.7128(−1)	6.6527(−2)	5.2399(−2)	−3.1491(−2)	−3.4411(−2)
0.7	1.9708(−1)	1.6691(−1)	6.5233(−2)	5.0678(−2)	−3.9742(−2)	−4.5577(−2)
0.8	1.9274(−1)	1.6068(−1)	6.3464(−2)	4.8191(−2)	−4.7990(−2)	−5.6745(−2)
0.9	1.8682(−1)	1.5183(−1)	6.1023(−2)	4.4605(−2)	−5.6229(−2)	−6.7919(−2)
1.0	1.7702(−1)	1.3641(−1)	5.6912(−2)	3.8171(−2)	−6.4447(−2)	−7.9107(−2)

where

$$\text{(4.16)}\quad \boldsymbol{U}_{ps} = (2a/\varepsilon_0)\{(1/\varepsilon_p)[(1/6)a^2\boldsymbol{\Phi} + \boldsymbol{D}_U]K_p - [\boldsymbol{A}_U^{(1)} + \boldsymbol{A}_U^{(2)}]K_T + [c_2\boldsymbol{A}_U^{(1)} - c_1\boldsymbol{A}_U^{(2)}]K_C\}$$

and

$$\text{(4.17)}\quad \boldsymbol{Q}_{ps} = (2a/\varepsilon_0)\{(1/\varepsilon_p)\boldsymbol{D}_Q K_p - [\boldsymbol{A}_Q^{(1)} + \boldsymbol{A}_Q^{(2)}]K_T + [c_2\boldsymbol{A}_Q^{(1)} - c_1\boldsymbol{A}_Q^{(2)}]K_C\}.$$

As our solutions are now complete, we are ready for some numerical results.

5. Numerical results. The sample cases for which we report numerical results in this work are defined in terms of two binary mixtures: Ne-Ar and He-Xe. We note that only the mass ratio m_1/m_2, the diameter ratio d_1/d_2, and the density ratio n_1/n_2 are needed to define the LBE for rigid-sphere interactions, and so we use the basic data:

$$m_2 = 39.948, \quad m_1 = 20.183, \quad d_2/d_1 = 1.406, \quad n_1/n_2 = 2/3$$

for the Ne-Ar mixture and

$$m_2 = 131.30, \quad m_1 = 4.0026, \quad d_2/d_1 = 2.226, \quad n_1/n_2 = 2/3$$

for the He-Xe mixture. It should be noted here that the values of the masses of these gas species were taken from [23] and those of the diameter ratios from [19].

We report in Tables 1–12 the velocity, heat-flow, and shear-stress profiles computed for the three considered problems of pressure-driven, temperature-driven, and concentration-driven flow, using as additional input data the accommodation coefficients $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, and $\beta_2 = 0.8$ and two different values of the channel width ($2a = 0.1$ and 1.0). The numerical results reported in Tables 1–12 are thought to be correct to within ± 1 in the last reported digit and were obtained by increasing the values of the approximation parameters $\{L, M, K, N, K_s\}$ of our method in steps, until numerical convergence was observed. Here L is the kernel truncation parameter (the maximum value of n considered in the summation of (2.50)), M is the order of the Gaussian quadrature used to evaluate numerically integrals over the

TABLE 2

Pressure-driven flow: species-specific velocity, heat-flow, and shear-stress profiles for the He-Xe mixture with $2a = 0.1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, and $n_1/n_2 = 2/3$.

η	$-U_1(-a+2\eta a)$	$-U_2(-a+2\eta a)$	$Q_1(-a+2\eta a)$	$Q_2(-a+2\eta a)$	$P_1(-a+2\eta a)$	$P_2(-a+2\eta a)$
0.0	1.7251(−1)	1.7483(−1)	7.3532(−2)	4.9529(−2)	1.4254(−2)	3.5944(−2)
0.1	1.7469(−1)	1.8461(−1)	7.4540(−2)	5.4100(−2)	7.6815(−3)	2.3659(−2)
0.2	1.7559(−1)	1.8881(−1)	7.4952(−2)	5.6008(−2)	1.1142(−3)	1.1370(−2)
0.3	1.7577(−1)	1.9069(−1)	7.5037(−2)	5.6886(−2)	−5.4543(−3)	−9.1737(−4)
0.4	1.7532(−1)	1.9079(−1)	7.4839(−2)	5.6994(−2)	−1.2031(−2)	−1.3200(−2)
0.5	1.7426(−1)	1.8925(−1)	7.4367(−2)	5.6408(−2)	−1.8622(−2)	−2.5472(−2)
0.6	1.7257(−1)	1.8606(−1)	7.3610(−2)	5.5124(−2)	−2.5236(−2)	−3.7729(−2)
0.7	1.7017(−1)	1.8106(−1)	7.2531(−2)	5.3067(−2)	−3.1881(−2)	−4.9966(−2)
0.8	1.6692(−1)	1.7388(−1)	7.1059(−2)	5.0049(−2)	−3.8566(−2)	−6.2176(−2)
0.9	1.6246(−1)	1.6353(−1)	6.9031(−2)	4.5589(−2)	−4.5303(−2)	−7.4352(−2)
1.0	1.5509(−1)	1.4439(−1)	6.5634(−2)	3.6907(−2)	−5.2111(−2)	−8.6479(−2)

TABLE 3

Pressure-driven flow: species-specific velocity, heat-flow, and shear-stress profiles for the Ne-Ar mixture with $2a = 1.0$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, and $n_1/n_2 = 2/3$.

η	$-U_1(-a+2\eta a)$	$-U_2(-a+2\eta a)$	$Q_1(-a+2\eta a)$	$Q_2(-a+2\eta a)$	$P_1(-a+2\eta a)$	$P_2(-a+2\eta a)$
0.0	1.3905	1.5596	1.3575(−1)	9.4647(−2)	1.6828(−1)	4.3756(−1)
0.1	1.4709	1.8014	1.5711(−1)	1.5084(−1)	1.1953(−1)	3.0339(−1)
0.2	1.5097	1.9042	1.6796(−1)	1.7100(−1)	5.7794(−2)	1.7788(−1)
0.3	1.5257	1.9529	1.7378(−1)	1.8082(−1)	−1.0287(−2)	5.6604(−2)
0.4	1.5209	1.9600	1.7546(−1)	1.8426(−1)	−8.1812(−2)	−6.2380(−2)
0.5	1.4958	1.9292	1.7326(−1)	1.8256(−1)	−1.5489(−1)	−1.8033(−1)
0.6	1.4498	1.8612	1.6702(−1)	1.7579(−1)	−2.2800(−1)	−2.9825(−1)
0.7	1.3818	1.7538	1.5622(−1)	1.6306(−1)	−2.9965(−1)	−4.1716(−1)
0.8	1.2893	1.6006	1.3972(−1)	1.4198(−1)	−3.6806(−1)	−5.3821(−1)
0.9	1.1651	1.3840	1.1491(−1)	1.0648(−1)	−4.3065(−1)	−6.6315(−1)
1.0	9.6070(−1)	9.8737(−1)	6.7860(−2)	2.0640(−2)	−4.8170(−1)	−7.9579(−1)

TABLE 4

Pressure-driven flow: species-specific velocity, heat-flow, and shear-stress profiles for the He-Xe mixture with $2a = 1.0$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, and $n_1/n_2 = 2/3$.

η	$-U_1(-a+2\eta a)$	$-U_2(-a+2\eta a)$	$Q_1(-a+2\eta a)$	$Q_2(-a+2\eta a)$	$P_1(-a+2\eta a)$	$P_2(-a+2\eta a)$
0.0	6.4867(−1)	1.8406	1.4476(−1)	9.1086(−2)	7.4981(−2)	5.2277(−1)
0.1	6.8104(−1)	2.1498	1.5355(−1)	1.7507(−1)	5.1223(−2)	3.7194(−1)
0.2	6.9653(−1)	2.2785	1.5721(−1)	2.0267(−1)	2.4189(−2)	2.2330(−1)
0.3	7.0306(−1)	2.3412	1.5884(−1)	2.1651(−1)	−4.4918(−3)	7.5753(−2)
0.4	7.0167(−1)	2.3519	1.5896(−1)	2.2175(−1)	−3.4250(−2)	−7.1074(−2)
0.5	6.9255(−1)	2.3149	1.5770(−1)	2.1998(−1)	−6.4771(−2)	−2.1739(−1)
0.6	6.7546(−1)	2.2308	1.5497(−1)	2.1132(−1)	−9.5859(−2)	−3.6334(−1)
0.7	6.4971(−1)	2.0973	1.5044(−1)	1.9468(−1)	−1.2739(−1)	−5.0898(−1)
0.8	6.1378(−1)	1.9075	1.4339(−1)	1.6716(−1)	−1.5929(−1)	−6.5438(−1)
0.9	5.6382(−1)	1.6418	1.3211(−1)	1.2119(−1)	−1.9153(−1)	−7.9956(−1)
1.0	4.7729(−1)	1.1502	1.0784(−1)	1.7932(−3)	−2.2385(−1)	−9.4468(−1)

speed variable, K is the order of the basis-function approximation introduced in (3.4) and (3.5) to take care of the speed dependence of the solution, N is the number of discrete ordinates used to represent the μ variable in $(0, 1)$, and K_s is the number

Table 5

Temperature-driven flow: species-specific velocity, heat-flow, and shear-stress profiles for the Ne-Ar mixture with $2a = 0.1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, *and* $n_1/n_2 = 2/3$.

η	$U_1(-a+2\eta a)$	$U_2(-a+2\eta a)$	$-Q_1(-a+2\eta a)$	$-Q_2(-a+2\eta a)$	$-P_1(-a+2\eta a)$	$-P_2(-a+2\eta a)$
0.0	6.9591(−2)	4.6009(−2)	3.3654(−1)	2.2923(−1)	6.9271(−5)	3.4849(−4)
0.1	7.0821(−2)	4.8693(−2)	3.4162(−1)	2.3995(−1)	9.1318(−5)	3.3379(−4)
0.2	7.1328(−2)	4.9914(−2)	3.4369(−1)	2.4483(−1)	9.9311(−5)	3.2847(−4)
0.3	7.1444(−2)	5.0468(−2)	3.4414(−1)	2.4704(−1)	9.9592(−5)	3.2828(−4)
0.4	7.1223(−2)	5.0495(−2)	3.4320(−1)	2.4716(−1)	9.5821(−5)	3.3079(−4)
0.5	7.0675(−2)	5.0036(−2)	3.4092(−1)	2.4536(−1)	9.0902(−5)	3.3407(−4)
0.6	6.9786(−2)	4.9086(−2)	3.3723(−1)	2.4160(−1)	8.7513(−5)	3.3633(−4)
0.7	6.8517(−2)	4.7594(−2)	3.3197(−1)	2.3566(−1)	8.8411(−5)	3.3573(−4)
0.8	6.6783(−2)	4.5443(−2)	3.2477(−1)	2.2707(−1)	9.6781(−5)	3.3015(−4)
0.9	6.4392(−2)	4.2344(−2)	3.1481(−1)	2.1462(−1)	1.1689(−4)	3.1675(−4)
1.0	6.0363(−2)	3.6811(−2)	2.9793(−1)	1.9222(−1)	1.5656(−4)	2.9030(−4)

Table 6

Temperature-driven flow: species-specific velocity, heat-flow, and shear-stress profiles for the He-Xe mixture with $2a = 0.1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, *and* $n_1/n_2 = 2/3$.

η	$U_1(-a+2\eta a)$	$U_2(-a+2\eta a)$	$-Q_1(-a+2\eta a)$	$-Q_2(-a+2\eta a)$	$P_1(-a+2\eta a)$	$-P_2(-a+2\eta a)$
0.0	7.6247(−2)	4.0307(−2)	3.4185(−1)	2.1217(−1)	−2.3469(−4)	1.8717(−4)
0.1	7.7279(−2)	4.3532(−2)	3.4621(−1)	2.2424(−1)	−1.4265(−4)	2.4853(−4)
0.2	7.7703(−2)	4.4892(−2)	3.4798(−1)	2.2939(−1)	−4.3015(−5)	3.1495(−4)
0.3	7.7789(−2)	4.5519(−2)	3.4834(−1)	2.3174(−1)	6.0403(−5)	3.8389(−4)
0.4	7.7585(−2)	4.5592(−2)	3.4747(−1)	2.3196(−1)	1.6554(−4)	4.5398(−4)
0.5	7.7098(−2)	4.5163(−2)	3.4541(−1)	2.3024(−1)	2.7077(−4)	5.2414(−4)
0.6	7.6316(−2)	4.4230(−2)	3.4212(−1)	2.2658(−1)	3.7457(−4)	5.9334(−4)
0.7	7.5204(−2)	4.2737(−2)	3.3743(−1)	2.2073(−1)	4.7534(−4)	6.6052(−4)
0.8	7.3690(−2)	4.0546(−2)	3.3104(−1)	2.1217(−1)	5.7114(−4)	7.2438(−4)
0.9	7.1607(−2)	3.7317(−2)	3.2222(−1)	1.9955(−1)	6.5924(−4)	7.8312(−4)
1.0	6.8134(−2)	3.1091(−2)	3.0745(−1)	1.7544(−1)	7.3409(−4)	8.3302(−4)

Table 7

Temperature-driven flow: species-specific velocity, heat-flow, and shear-stress profiles for the Ne-Ar mixture with $2a = 1.0$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, *and* $n_1/n_2 = 2/3$.

η	$U_1(-a+2\eta a)$	$U_2(-a+2\eta a)$	$-Q_1(-a+2\eta a)$	$-Q_2(-a+2\eta a)$	$P_1(-a+2\eta a)$	$-P_2(-a+2\eta a)$
0.0	1.6352(−1)	1.1439(−1)	7.3626(−1)	5.3931(−1)	−1.5921(−3)	1.8955(−3)
0.1	1.7493(−1)	1.3808(−1)	7.7567(−1)	6.1916(−1)	−2.3711(−3)	1.3762(−3)
0.2	1.8001(−1)	1.4709(−1)	7.9100(−1)	6.4734(−1)	−2.2690(−3)	1.4442(−3)
0.3	1.8245(−1)	1.5156(−1)	7.9742(−1)	6.6053(−1)	−1.8145(−3)	1.7473(−3)
0.4	1.8282(−1)	1.5304(−1)	7.9733(−1)	6.6445(−1)	−1.1850(−3)	2.1669(−3)
0.5	1.8121(−1)	1.5202(−1)	7.9125(−1)	6.6080(−1)	−4.6766(−4)	2.6451(−3)
0.6	1.7752(−1)	1.4845(−1)	7.7869(−1)	6.4946(−1)	2.7989(−4)	3.1435(−3)
0.7	1.7133(−1)	1.4188(−1)	7.5812(−1)	6.2872(−1)	9.9769(−4)	3.6220(−3)
0.8	1.6181(−1)	1.3114(−1)	7.2619(−1)	5.9426(−1)	1.5901(−3)	4.0170(−3)
0.9	1.4703(−1)	1.1335(−1)	6.7487(−1)	5.3501(−1)	1.8583(−3)	4.1958(−3)
1.0	1.1654(−1)	7.1981(−2)	5.6056(−1)	3.8620(−1)	1.2125(−3)	3.7653(−3)

of spline functions used to compute (without postprocessing) the Chapman–Enskog (vector-valued) functions $\boldsymbol{A}^{(1)}(c)$, $\boldsymbol{A}^{(2)}(c)$, $\boldsymbol{B}(c)$, $\boldsymbol{D}(c)$, and $\boldsymbol{E}(c)$, as explained in de-

Table 8

Temperature-driven flow: species-specific velocity, heat-flow, and shear-stress profiles for the He-Xe mixture with $2a = 1.0$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, *and* $n_1/n_2 = 2/3$.

η	$U_1(-a+2\eta a)$	$U_2(-a+2\eta a)$	$-Q_1(-a+2\eta a)$	$-Q_2(-a+2\eta a)$	$P_1(-a+2\eta a)$	$-P_2(-a+2\eta a)$
0.0	1.6158(−1)	9.7015(−2)	6.8378(−1)	4.7673(−1)	−1.6983(−3)	1.0746(−3)
0.1	1.6960(−1)	1.2111(−1)	7.1404(−1)	5.5454(−1)	−1.5536(−3)	1.1711(−3)
0.2	1.7288(−1)	1.2977(−1)	7.2547(−1)	5.8063(−1)	−1.0929(−3)	1.4782(−3)
0.3	1.7427(−1)	1.3418(−1)	7.2994(−1)	5.9301(−1)	−5.1014(−4)	1.8667(−3)
0.4	1.7425(−1)	1.3582(−1)	7.2943(−1)	5.9702(−1)	1.4055(−4)	2.3005(−3)
0.5	1.7293(−1)	1.3512(−1)	7.2436(−1)	5.9424(−1)	8.3761(−4)	2.7652(−3)
0.6	1.7019(−1)	1.3205(−1)	7.1433(−1)	5.8455(−1)	1.5696(−3)	3.2532(−3)
0.7	1.6572(−1)	1.2617(−1)	6.9803(−1)	5.6636(−1)	2.3234(−3)	3.7557(−3)
0.8	1.5882(−1)	1.1639(−1)	6.7267(−1)	5.3564(−1)	3.0706(−3)	4.2538(−3)
0.9	1.4786(−1)	9.9969(−2)	6.3148(−1)	4.8193(−1)	3.7379(−3)	4.6988(−3)
1.0	1.2437(−1)	5.9234(−2)	5.3915(−1)	3.3921(−1)	4.0610(−3)	4.9141(−3)

Table 9

Concentration-driven flow: species-specific velocity, heat-flow, and shear-stress profiles for the Ne-Ar mixture with $2a = 0.1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, *and* $n_1/n_2 = 2/3$.

η	$-U_1(-a+2\eta a)$	$U_2(-a+2\eta a)$	$Q_1(-a+2\eta a)$	$-Q_2(-a+2\eta a)$	$P_1(-a+2\eta a)$	$P_2(-a+2\eta a)$
0.0	8.5004(−2)	3.3436(−2)	3.2208(−2)	9.7700(−3)	7.5261(−3)	−6.8152(−3)
0.1	8.6279(−2)	3.4967(−2)	3.2792(−2)	1.0403(−2)	4.0901(−3)	−4.5245(−3)
0.2	8.6795(−2)	3.5673(−2)	3.3027(−2)	1.0691(−2)	6.8986(−4)	−2.2577(−3)
0.3	8.6903(−2)	3.5997(−2)	3.3081(−2)	1.0828(−2)	−2.6950(−3)	−1.1075(−6)
0.4	8.6666(−2)	3.6015(−2)	3.2985(−2)	1.0847(−2)	−6.0786(−3)	2.2546(−3)
0.5	8.6094(−2)	3.5752(−2)	3.2742(−2)	1.0758(−2)	−9.4735(−3)	4.5179(−3)
0.6	8.5173(−2)	3.5201(−2)	3.2347(−2)	1.0559(−2)	−1.2892(−2)	6.7972(−3)
0.7	8.3857(−2)	3.4331(−2)	3.1777(−2)	1.0236(−2)	−1.6349(−2)	9.1014(−3)
0.8	8.2054(−2)	3.3072(−2)	3.0988(−2)	9.7578(−3)	−1.9858(−2)	1.1441(−2)
0.9	7.9550(−2)	3.1254(−2)	2.9878(−2)	9.0476(−3)	−2.3442(−2)	1.3830(−2)
1.0	7.5288(−2)	2.8012(−2)	2.7939(−2)	7.7216(−3)	−2.7137(−2)	1.6294(−2)

Table 10

Concentration-driven flow: species-specific velocity, heat-flow, and shear-stress profiles for the He-Xe mixture with $2a = 0.1$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, *and* $n_1/n_2 = 2/3$.

η	$-U_1(-a+2\eta a)$	$U_2(-a+2\eta a)$	$Q_1(-a+2\eta a)$	$-Q_2(-a+2\eta a)$	$P_1(-a+2\eta a)$	$P_2(-a+2\eta a)$
0.0	9.3618(−2)	3.3805(−2)	4.3904(−2)	9.3958(−3)	7.6577(−3)	−6.9794(−3)
0.1	9.4798(−2)	3.5686(−2)	4.4496(−2)	1.0264(−2)	4.1248(−3)	−4.6241(−3)
0.2	9.5276(−2)	3.6496(−2)	4.4737(−2)	1.0627(−2)	6.1580(−4)	−2.2848(−3)
0.3	9.5370(−2)	3.6870(−2)	4.4785(−2)	1.0800(−2)	−2.8838(−3)	4.8252(−5)
0.4	9.5131(−2)	3.6911(−2)	4.4669(−2)	1.0833(−2)	−6.3842(−3)	2.3819(−3)
0.5	9.4572(−2)	3.6648(−2)	4.4392(−2)	1.0741(−2)	−9.8947(−3)	4.7222(−3)
0.6	9.3677(−2)	3.6077(−2)	4.3949(−2)	1.0523(−2)	−1.3425(−2)	7.0757(−3)
0.7	9.2405(−2)	3.5164(−2)	4.3317(−2)	1.0161(−2)	−1.6985(−2)	9.4488(−3)
0.8	9.0671(−2)	3.3826(−2)	4.2455(−2)	9.6159(−3)	−2.0586(−2)	1.1850(−2)
0.9	8.8280(−2)	3.1859(−2)	4.1263(−2)	8.7885(−3)	−2.4244(−2)	1.4288(−2)
1.0	8.4274(−2)	2.8114(−2)	3.9260(−2)	7.1141(−3)	−2.7987(−2)	1.6783(−2)

tail in [8] and [11]. To be more specific, we note that we have used $20 \le L \le 95$, $100 \le M \le 400$, $20 \le K \le 35$, $20 \le N \le 50$, and $80 \le K_s - 2 \le 1280$. In addition to the profiles reported in Tables 1–12, we report in Tables 13–15 mass- and heat-flow

Table 11

Concentration-driven flow: species-specific velocity, heat-flow, and shear-stress profiles for the Ne-Ar mixture with $2a = 1.0$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, *and* $n_1/n_2 = 2/3$.

η	$-U_1(-a+2\eta a)$	$U_2(-a+2\eta a)$	$Q_1(-a+2\eta a)$	$-Q_2(-a+2\eta a)$	$P_1(-a+2\eta a)$	$P_2(-a+2\eta a)$
0.0	1.6697(−1)	6.1119(−2)	5.0453(−2)	5.4942(−3)	1.9191(−2)	−1.7106(−2)
0.1	1.7575(−1)	7.0455(−2)	5.3750(−2)	7.7513(−3)	1.1026(−2)	−1.1663(−2)
0.2	1.7865(−1)	7.4230(−2)	5.4732(−2)	8.3674(−3)	4.9631(−3)	−7.6208(−3)
0.3	1.7964(−1)	7.6312(−2)	5.5100(−2)	8.6958(−3)	−1.8189(−4)	−4.1908(−3)
0.4	1.7935(−1)	7.7304(−2)	5.5117(−2)	8.9489(−3)	−4.9955(−3)	−9.8168(−4)
0.5	1.7796(−1)	7.7372(−2)	5.4846(−2)	9.1930(−3)	−9.9215(−3)	2.3023(−3)
0.6	1.7537(−1)	7.6472(−2)	5.4254(−2)	9.4308(−3)	−1.5388(−2)	5.9467(−3)
0.7	1.7122(−1)	7.4366(−2)	5.3204(−2)	9.6061(−3)	−2.1898(−2)	1.0287(−2)
0.8	1.6470(−1)	7.0492(−2)	5.1378(−2)	9.5542(−3)	−3.0148(−2)	1.5786(−2)
0.9	1.5379(−1)	6.3444(−2)	4.7937(−2)	8.7972(−3)	−4.1284(−2)	2.3211(−2)
1.0	1.2756(−1)	4.5357(−2)	3.8020(−2)	4.3594(−3)	−5.7978(−2)	3.4340(−2)

Table 12

Concentration-driven flow: species-specific velocity, heat-flow, and shear-stress profiles for the He-Xe mixture with $2a = 1.0$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, *and* $n_1/n_2 = 2/3$.

η	$-U_1(-a+2\eta a)$	$U_2(-a+2\eta a)$	$Q_1(-a+2\eta a)$	$-Q_2(-a+2\eta a)$	$P_1(-a+2\eta a)$	$P_2(-a+2\eta a)$
0.0	1.9140(−1)	6.6006(−2)	8.6133(−2)	2.3421(−3)	2.0835(−2)	−1.8831(−2)
0.1	1.9968(−1)	7.6638(−2)	9.0237(−2)	4.9463(−3)	1.2501(−2)	−1.3275(−2)
0.2	2.0272(−1)	8.0883(−2)	9.1762(−2)	5.6234(−3)	5.6487(−3)	−8.7065(−3)
0.3	2.0383(−1)	8.3276(−2)	9.2365(−2)	6.0203(−3)	−5.6862(−4)	−4.5616(−3)
0.4	2.0357(−1)	8.4413(−2)	9.2316(−2)	6.3456(−3)	−6.6101(−3)	−5.3396(−4)
0.5	2.0206(−1)	8.4450(−2)	9.1676(−2)	6.6616(−3)	−1.2845(−2)	3.6227(−3)
0.6	1.9919(−1)	8.3338(−2)	9.0391(−2)	6.9674(−3)	−1.9640(−2)	8.1526(−3)
0.7	1.9461(−1)	8.0837(−2)	8.8283(−2)	7.2021(−3)	−2.7422(−2)	1.3341(−2)
0.8	1.8755(−1)	7.6392(−2)	8.4971(−2)	7.1972(−3)	−3.6768(−2)	1.9571(−2)
0.9	1.7615(−1)	6.8612(−2)	7.9529(−2)	6.4773(−3)	−4.8586(−2)	2.7450(−2)
1.0	1.5069(−1)	4.8837(−2)	6.7082(−2)	1.7050(−3)	−6.4896(−2)	3.8323(−2)

rates, as defined by (4.14)–(4.17), for several values of the channel width. We note that the composition and the wall interaction data used to generate Tables 13–15 were the same as those used for Tables 1–12, and that the numerical results for the flow rates are also thought to be correct to within ±1 in the last reported digit.

While an implementation of our solutions for the three considered problems requires, in general, some hours of computer time to establish the high-quality results we are reporting in our tables, solutions good enough for graphical presentation require very modest computational expense. To have an idea of the CPU time for what we might consider "practical results," we found, for example, that all of the He-Xe results given in Tables 1–15 could be obtained with essentially three figures of accuracy in less than one minute on an Apple MacBook running at 2 GHz.

Finally, we note that we have (for the three considered problems) compared numerical results from our approach based on the LBE for binary mixtures with those of the McCormack model, as developed and implemented in [23]. Due to different ways of the defining the dimensionless space variables in [23] and in this work, we have used the relationship

$$a = \xi_M a_M,$$

TABLE 13
Pressure-driven flow: mass- and heat-flow rates for the case $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, *and* $n_1/n_2 = 2/3$.

	Ne-Ar mixture			
$2a$	$-U_1$	$-U_2$	Q_1	Q_2
1.0(−2)	7.02875	4.57921	3.05198	1.90616
1.0(−1)	3.97126	3.34605	1.31567	1.01481
5.0(−1)	2.88363	3.26237	5.15113(−1)	4.87750(−1)
1.0	2.80472	3.50334	3.07392(−1)	3.07713(−1)
2.0	3.02335	4.02064	1.72501(−1)	1.77501(−1)
5.0	3.94713	5.45691	7.46936(−2)	7.77126(−2)
1.0(1)	5.49952	7.68822	3.82931(−2)	3.99837(−2)
1.0(2)	3.27996(1)	4.61398(1)	3.90604(−3)	4.09657(−3)
	He-Xe mixture			
$2a$	$-U_1$	$-U_2$	Q_1	Q_2
1.0(−2)	7.28191	4.43169	3.45590	1.72889
1.0(−1)	3.42560	3.62500	1.46027	1.05889
5.0(−1)	1.68434	3.80750	5.26626(−1)	5.60196(−1)
1.0	1.31140	4.18704	2.99703(−1)	3.63014(−1)
2.0	1.16837	4.90801	1.61420(−1)	2.13010(−1)
5.0	1.32920	6.86910	6.77017(−2)	9.41384(−2)
1.0(1)	1.80118	9.93900	3.43884(−2)	4.85365(−2)
1.0(2)	1.10778(1)	6.34092(1)	3.48652(−3)	4.98132(−3)

TABLE 14
Temperature-driven flow: mass- and heat-flow rates for the case $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, *and* $n_1/n_2 = 2/3$.

	Ne-Ar mixture			
$2a$	U_1	U_2	$-Q_1$	$-Q_2$
1.0(−2)	3.08637	1.87391	1.49378(1)	9.36209
1.0(−1)	1.38130	9.53251(−1)	6.68507	4.71728
5.0(−1)	5.65163(−1)	4.40808(−1)	2.59532	2.04733
1.0	3.41095(−1)	2.76103(−1)	1.51266	1.23229
2.0	1.91966(−1)	1.59245(−1)	8.25888(−1)	6.85471(−1)
5.0	8.29642(−2)	6.99555(−2)	3.48778(−1)	2.92661(−1)
1.0(1)	4.25182(−2)	3.60209(−2)	1.77504(−1)	1.49452(−1)
1.0(2)	4.34037(−3)	3.68920(−3)	1.80306(−2)	1.52261(−2)
	He-Xe mixture			
$2a$	U_1	U_2	$-Q_1$	$-Q_2$
1.0(−2)	3.47992	1.63717	1.59823(1)	8.73594
1.0(−1)	1.51411	8.53282(−1)	6.79083	4.41149
5.0(−1)	5.71005(−1)	3.90743(−1)	2.45191	1.85972
1.0	3.30863(−1)	2.44038(−1)	1.39410	1.10712
2.0	1.80382(−1)	1.40608(−1)	7.49405(−1)	6.11321(−1)
5.0	7.61935(−2)	6.17139(−2)	3.13429(−1)	2.59709(−1)
1.0(1)	3.87787(−2)	3.17730(−2)	1.59023(−1)	1.32410(−1)
1.0(2)	3.93853(−3)	3.25543(−3)	1.61100(−2)	1.34710(−2)

where ξ_M is the conversion factor defined by equation (7.19) of [9], to relate the channel half-width a used in this work with the a_M used in [23]. For the mass- and heat-flow rates, this is the only conversion that is required; for the profiles, in addition to the channel half-width conversion, the LBE results must be divided by ξ_M, in order to be properly compared to the results of [23]. Thus, concerning the mass-

Table 15

Concentration-driven flow: mass- and heat-flow rates for the case $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.6$, $\beta_2 = 0.8$, and $n_1/n_2 = 2/3$.

	Ne-Ar mixture			
$2a$	$-U_1$	U_2	Q_1	$-Q_2$
1.0(−2)	3.96894	1.59880	1.75432	6.58115(−1)
1.0(−1)	1.68853	6.87292(−1)	6.40069(−1)	2.04304(−1)
5.0(−1)	6.06366(−1)	2.50910(−1)	1.95828(−1)	4.19489(−2)
1.0	3.41892(−1)	1.43652(−1)	1.05381(−1)	1.73285(−2)
2.0	1.82236(−1)	7.77918(−2)	5.47167(−2)	6.98903(−3)
5.0	7.57830(−2)	3.25972(−2)	2.23853(−2)	2.28975(−3)
1.0(1)	3.84251(−2)	1.64751(−2)	1.12765(−2)	1.05773(−3)
1.0(2)	3.89650(−3)	1.65480(−3)	1.13525(−3)	9.79098(−5)
	He-Xe mixture			
$2a$	$-U_1$	U_2	Q_1	Q_2
1.0(−2)	4.30486	1.52721	2.07209	−5.94298(−1)
1.0(−1)	1.85963	7.02653(−1)	8.71981(−1)	−2.01990(−1)
5.0(−1)	6.81859(−1)	2.67760(−1)	3.11395(−1)	−3.71345(−2)
1.0	3.89145(−1)	1.56308(−1)	1.76153(−1)	−1.22203(−2)
2.0	2.09485(−1)	8.63759(−2)	9.43221(−2)	−3.14174(−3)
5.0	8.76311(−2)	3.70753(−2)	3.93363(−2)	−3.25368(−4)
1.0(1)	4.44688(−2)	1.89313(−2)	1.99386(−2)	6.323 (−7)
1.0(2)	4.50724(−3)	1.91922(−3)	2.01820(−3)	1.48122(−5)

flow rates reported in Tables 1–9 of [23], we have found maximum relative deviations (with respect to our LBE results) of 33%, 62%, and 33% for the problems driven by pressure, temperature, and concentration gradients, respectively. For the heat-flow rates reported in these same tables of [23], we have found maximum relative deviations of 40%, 34%, and 300%, respectively, for the pressure-, temperature-, and concentration-driven problems. In all cases but one, the maximum deviations were found to occur for the following input parameters considered in [23]: the heaviest gas particle (Xe), the widest channel ($2a_M = 100$), and the largest concentration of the lighter species ($c_1 = 0.9$). Large maximum relative deviations between the McCormack profiles reported in Tables 10–18 of [23] and those computed with our current (LBE) approach were also observed.

6. Onsager relationships. In [23], Siewert and Valougeorgis established three independent (generalized) Onsager relationships relevant to the flow of binary gas mixtures in a plane-parallel channel driven by pressure, temperature, and concentration gradients. While the derivations reported in [23] were based on the McCormack kinetic model [15], little work is required to establish those same relationships [23] for the LBE (for rigid-sphere interactions) used in this work. For that purpose, we follow here a procedure described in detail for half-space flow problems in [9]. However, before starting our derivation, we should mention that, to denote the solutions and the driving terms of two different problems (among the three that can be defined by considering separately pressure, temperature, and concentration gradients), we attach subscripts X and Y to $\boldsymbol{\Psi}(\tau, c, \mu)$ and to $\boldsymbol{\Upsilon}(c)$.

In short, using the fact that the kernel defined by (2.25) is such that

$$\boldsymbol{S}\boldsymbol{K}^T(\boldsymbol{c} : \boldsymbol{c}') = \boldsymbol{K}(\boldsymbol{c}' : \boldsymbol{c})\boldsymbol{S}, \tag{6.1}$$

where

$$\boldsymbol{S} = \begin{bmatrix} c_2 & 0 \\ 0 & c_1 a_{1,2} \end{bmatrix} \tag{6.2}$$

and $a_{1,2}$ is given by (2.23), we can multiply (2.47) with μ changed to $-\mu$ and subscript Y added to $\boldsymbol{\Psi}(\tau, c, \mu)$ and $\boldsymbol{\Upsilon}(c)$ by

$$c^2(1-\mu^2)\mathrm{e}^{-c^2}\boldsymbol{\Psi}_X^T(\tau, c, \mu)\boldsymbol{S}^{-1},$$

multiply (2.47) with subscript X added to $\boldsymbol{\Psi}(\tau, c, \mu)$ and $\boldsymbol{\Upsilon}(c)$ by

$$c^2(1-\mu^2)\mathrm{e}^{-c^2}\boldsymbol{\Psi}_Y^T(\tau, c, -\mu)\boldsymbol{S}^{-1},$$

subtract the resulting equations, one from the other, and integrate the result of this operation over all μ, over all c, and over τ from $-a$ to a to find, after using (2.62),

$$\int_{-a}^{a}\int_0^\infty\int_{-1}^{1} \mathrm{e}^{-c^2}c^2(1-\mu^2)\big[\boldsymbol{\Psi}_X^T(\tau, c, \mu)\boldsymbol{S}^{-1}\boldsymbol{\Upsilon}_Y(c) - \boldsymbol{\Psi}_Y^T(\tau, c, -\mu)\boldsymbol{S}^{-1}\boldsymbol{\Upsilon}_X(c)\big]\mathrm{d}\mu\mathrm{d}c\mathrm{d}\tau = 0. \tag{6.3}$$

Taking all possible combinations of X and Y (with the restriction that $X \neq Y$) when these subscripts are set equal to P, T, and C in (6.3) and using the forms of the driving terms appropriate to each problem, we find the relationships

$$K_T\,[c_1a_{1,2} \quad c_2]\,\boldsymbol{Q}_P = K_P\,[c_1a_{1,2} \quad c_2]\,\boldsymbol{U}_T, \tag{6.4a}$$

$$K_T\,[c_1a_{1,2} \quad c_2]\,\boldsymbol{Q}_C = c_1c_2K_C\,[a_{1,2} \quad -1]\,\boldsymbol{U}_T, \tag{6.4b}$$

and

$$c_1c_2K_C\,[a_{1,2} \quad -1]\,\boldsymbol{U}_P = K_P\,[c_1a_{1,2} \quad c_2]\,\boldsymbol{U}_C, \tag{6.4c}$$

where we have added subscripts P, T, C to the quantities defined by (4.14) and (4.15) as tags for the problems driven, respectively, by pressure, temperature, and concentration gradients. As a (minor) test of our computations, we have confirmed the three identities listed as (6.4) for the data sets used to define the numerical results reported in this work. Moreover, since Kosuge et al. [13] have tabulated numerical results related to our (6.4), we include in Table 16 our numerical results for the quantities used in [13] to express the (generalized) Onsager relationships. We note that the results listed in Table 16 are relevant to the special case [13] of strictly diffuse reflection at both walls, equal-diameter particles, mass ratio $m_2/m_1 = 2$, and density ratio $n_2/n_1 = 1$ at equilibrium. In order to compare with [13], we used the channel half-width

$$a = \left(\frac{1}{k}\right)\left[\frac{(c_1 + c_2d_2/d_1)^2}{4(2^{1/2})c_1 + c_2(1+d_1/d_2)^2(1+m_1/m_2)^{1/2}}\right], \tag{6.5}$$

where k is the Knudsen number used in [13], and the following expressions (valid for

Table 16

The quantities Λ_{XY}, $X \neq Y$, $X, Y = P, T, C$ as defined in [13] *for various values of k with $\alpha_1 = 1.0$, $\alpha_2 = 1.0$, $\beta_1 = 1.0$, $\beta_2 = 1.0$, $m_2/m_1 = 2$, $d_2/d_1 = 1$, and $n_2/n_1 = 1$.*

k	Λ_{TP}	Λ_{PT}	$-\Lambda_{CP}$	$-\Lambda_{PC}$	Λ_{CT}	Λ_{TC}
0.05	4.622713(−2)	4.622713(−2)	1.248352(−2)	1.248352(−2)	8.743064(−3)	8.743064(−3)
0.10	8.584004(−2)	8.584004(−2)	2.371949(−2)	2.371949(−2)	1.646907(−2)	1.646907(−2)
1.00	3.497536(−1)	3.497536(−1)	1.202181(−1)	1.202181(−1)	7.322853(−2)	7.322853(−2)
10.0	7.076139(−1)	7.076139(−1)	2.785816(−1)	2.785816(−1)	1.472278(−1)	1.472278(−1)
20.0	8.340453(−1)	8.340453(−1)	3.317800(−1)	3.317800(−1)	1.717606(−1)	1.717606(−1)

K_P, K_T, and K_C set equal to ε_0) for the quantities defined in [13]:

(6.6a)
$$\Lambda_{PT} = \frac{1}{2c_1} a_{2,1} \, [c_1 a_{1,2} \quad c_2] \, \boldsymbol{U}_T,$$

(6.6b)
$$\Lambda_{PC} = \frac{1}{2c_1 c_2} a_{2,1} \, [c_1 a_{1,2} \quad c_2] \, \boldsymbol{U}_C,$$

(6.6c)
$$\Lambda_{TP} = \frac{1}{2c_1} a_{2,1} \, [c_1 a_{1,2} \quad c_2] \, \boldsymbol{Q}_P,$$

(6.6d)
$$\Lambda_{TC} = \frac{1}{2c_1 c_2} a_{2,1} \, [c_1 a_{1,2} \quad c_2] \, \boldsymbol{Q}_C,$$

(6.6e)
$$\Lambda_{CP} = \frac{1}{2} a_{2,1} \, [a_{1,2} \quad -1] \, \boldsymbol{U}_P,$$

and

(6.6f)
$$\Lambda_{CT} = \frac{1}{2} a_{2,1} \, [a_{1,2} \quad -1] \, \boldsymbol{U}_T.$$

Note that subscript D is used in [13] with the same meaning as subscript C in this work (i.e., a tag for the concentration-driven problem). To be clear, we have listed identical results in various columns of Table 16 in order to emphasize that all quantities were computed as defined.

Finally, we note that we have also confirmed that

$$p_* = [c_1 \quad c_2] \, \boldsymbol{P}(\tau) + (K_P/\varepsilon_0)\tau,$$

where the second term on the right-hand side should not be taken into account for the cases of temperature and concentration gradients, is a (problem-dependent) constant.

7. Concluding remarks. We have reported in this work what we believe to be a compact, fast, and accurate method of solving channel-flow problems driven by pressure, temperature, and concentration gradients and described by the (vector) LBE for a binary mixture of rigid spheres. Accurate numerical results were given for the velocity, heat-flow, and shear-stress profiles, as well as for the mass- and heat-flow rates, for selected cases based on Ne-Ar and He-Xe mixtures.

Table 17
Refined results for Tables 10, 11, *and* 12 *of* [21] *in the notation of* [21].

	$-U_P$			Q_P		
$2a$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1.0$
0.10	2.0244(1)	4.3874	1.9504	4.1702	1.5684	7.9969(−1)
1.00	1.7564(1)	3.3264	1.5067	7.1258(−1)	5.2875(−1)	3.8908(−1)
10.0	1.8743(1)	4.5346	2.7296	7.9139(−2)	8.4299(−2)	8.9950(−2)
	U_T			$-Q_T$		
$2a$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1.0$
0.10	4.1702	1.5684	7.9969(−1)	2.0650(1)	7.7804	3.9044
1.00	7.1258(−1)	5.2875(−1)	3.8908(−1)	3.4557	2.5138	1.7830
10.0	7.9139(−2)	8.4299(−2)	8.9950(−2)	3.7488(−1)	3.6167(−1)	3.4674(−1)

In addition to the comparisons with the numerical results of the McCormack model that are discussed in section 5, we have also performed comparisons with the single-gas LBE results of [21], using three different ways of achieving the single-gas limit in our formulation:

$$\text{(i) } c_1 = 0, \quad \text{(ii) } c_2 = 0, \quad \text{or} \quad \text{(iii) } m_1 = m_2,\ d_1 = d_2,\ \alpha_1 = \alpha_2, \text{ and } \beta_1 = \beta_2.$$

We note that to convert our results to the same spatial units used in [21] we made use of the factor

$$\xi_{S,p} = 0.449027806\ldots,$$

which (for a single-species case) is the ratio between our dimensionless spatial variable, as defined by (2.41) and (2.42), and that used in [21] for channel-flow problems. Doing this, we found good but not perfect agreement with the five-figure results for the mass- and heat-flow rates and for the velocity and heat-flow profiles that are tabulated in [21]. In regard to the flow rates, while we found at most a difference of one unit in the fifth digit listed in Table 10 of [21], where the accommodation coefficients are taken to be equal to 0.1, we did find a maximum difference of 7 units in the fifth digit listed in Table 11 of [21] (case with accommodation coefficients equal to 0.5) and a maximum difference of 4 units in the fourth digit listed in Table 12 of [21] (case with accommodation coefficients equal to 1.0). The largest differences always occurred for the smallest channel width considered in Tables 10–12 of [21]. For the velocity and heat-flow profiles listed in Tables 13 and 14 of [21], we have observed, respectively, maximum differences of 5 and 3 units in the fifth digit listed in these tables. The maximum differences for the profiles were found to always occur at the channel walls. We have confirmed that the loss of accuracy in Tables 10–14 of [21] was due to using $L = 8$ in those computations, and so we list in Tables 17–19 our improved results (based on $L = 30$) for the cases studied in Tables 10–14 of [21].

Finally, we should like to mention that, considering the large deviations between the numerical results from the LBE and those from the McCormack model that were observed in this and other [9, 10] works, we are of the opinion that the McCormack model has a limited value as an economical alternative to the LBE for gas mixtures.

Table 18
Refined results for Table 13 *of* [21] *in the notation of* [21].

	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 1.0$	
τ/a	$-u_P(\tau)$	$q_P(\tau)$	$-u_P(\tau)$	$q_P(\tau)$	$-u_P(\tau)$	$q_P(\tau)$
0.0	8.8693	3.7271(−1)	1.7574	2.8921(−1)	8.5378(−1)	2.2669(−1)
0.1	8.8671	3.7230(−1)	1.7549	2.8859(−1)	8.5116(−1)	2.2589(−1)
0.2	8.8602	3.7106(−1)	1.7475	2.8672(−1)	8.4327(−1)	2.2348(−1)
0.3	8.8486	3.6895(−1)	1.7350	2.8355(−1)	8.2994(−1)	2.1938(−1)
0.4	8.8320	3.6592(−1)	1.7172	2.7898(−1)	8.1090(−1)	2.1348(−1)
0.5	8.8101	3.6187(−1)	1.6935	2.7288(−1)	7.8568(−1)	2.0559(−1)
0.6	8.7822	3.5667(−1)	1.6635	2.6501(−1)	7.5357(−1)	1.9539(−1)
0.7	8.7473	3.5006(−1)	1.6258	2.5499(−1)	7.1335(−1)	1.8239(−1)
0.8	8.7035	3.4160(−1)	1.5785	2.4212(−1)	6.6281(−1)	1.6568(−1)
0.9	8.6461	3.3023(−1)	1.5167	2.2483(−1)	5.9696(−1)	1.4323(−1)
1.0	8.5500	3.1009(−1)	1.4143	1.9464(−1)	4.8982(−1)	1.0466(−1)

Table 19
Refined results for Table 14 *of* [21] *in the notation of* [21].

	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 1.0$	
τ/a	$u_T(\tau)$	$-q_T(\tau)$	$u_T(\tau)$	$-q_T(\tau)$	$u_T(\tau)$	$-q_T(\tau)$
0.0	3.6061(−1)	1.7429	2.8168(−1)	1.3193	2.2268(−1)	9.9636(−1)
0.1	3.6050(−1)	1.7425	2.8125(−1)	1.3178	2.2198(−1)	9.9383(−1)
0.2	3.6018(−1)	1.7414	2.7995(−1)	1.3132	2.1987(−1)	9.8616(−1)
0.3	3.5963(−1)	1.7395	2.7775(−1)	1.3054	2.1629(−1)	9.7311(−1)
0.4	3.5883(−1)	1.7368	2.7457(−1)	1.2942	2.1113(−1)	9.5424(−1)
0.5	3.5777(−1)	1.7332	2.7032(−1)	1.2790	2.0422(−1)	9.2884(−1)
0.6	3.5640(−1)	1.7284	2.6484(−1)	1.2593	1.9530(−1)	8.9575(−1)
0.7	3.5466(−1)	1.7223	2.5785(−1)	1.2340	1.8392(−1)	8.5314(−1)
0.8	3.5242(−1)	1.7144	2.4886(−1)	1.2011	1.6928(−1)	7.9764(−1)
0.9	3.4941(−1)	1.7036	2.3677(−1)	1.1561	1.4960(−1)	7.2184(−1)
1.0	3.4412(−1)	1.6844	2.1575(−1)	1.0763	1.1583(−1)	5.8840(−1)

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