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On the use of Fresnel boundary and interface conditions in radiative-transfer calculations for multilayered media

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Abstract

The ADO (analytical discrete ordinates) method is used to establish a concise and accurate solution for a multi-layer radiative-transfer problem with Fresnel boundary and interface conditions. A finite plane-parallel medium composed of a number (K) of sub-strata with different material properties is considered to be illuminated by isotropically incident radiation. While a general result is obtained, emphasis in the numerical work is given to computing accurately the currents and the intensities that exit each of the two exterior surfaces. Monochromatic forms (with anisotropic scattering) of the radiative-transfer equation are used, and numerical results are given for several specific cases. The complications introduced by the Fresnel boundary and interface conditions are well resolved, so that the numerical results obtained are thought to define a very high standard.

Keywords: Radiative transfer; Discrete ordinates; Fresnel conditions; Multilayers

1. Introduction

In two recent papers, Cassell and Williams [1,2] have pointed out the importance of using the Fresnel boundary and interface conditions for radiative-transfer applications related to optical tomography [3] and fiber optics [4]. And most interestingly, Elias and co-workers [5–7] have included the Fresnel boundary and interface conditions in their studies of the surfaces (and sub-strata) of old masterpiece paintings. While the vast majority of radiative-transfer problems studied and solved in the past has been based on the (easy to apply) classical specular/diffuse boundary conditions, the use of the Fresnel conditions introduces new mathematical and computational challenges. In addition to Refs. [1–7] already mentioned, the Fresnel formulas have been used by Aronson [8,9] in work associated with applications in the medical field and by Tanaka and Nakajima [10] in studies of the atmosphere–ocean system. A look at two useful texts, Born and Wolf [11] and Modest [12], can give a good idea about the complexity of the Fresnel/Snell laws in general.

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Our work here makes use of the contributions of Cassell and Williams [1,2] who, as others [10,13–15] have done, expressed the Fresnel boundary and interface conditions in forms appropriate for use with the monochromatic version of the equation of transfer [16]. As a first part of our work, we have rewritten the Fresnel boundary/interface conditions of Refs. [1] and [2] in what we believe to be a more "user friendly" way.

In this work, we investigate the case of a multi-layer medium with any number (K) of layers with different materials properties. And although our numerical work is based on the ADO (analytical discrete ordinates) method [17], a significant part of our analysis is carried out before any approximations are introduced. We have succeeded in formulating the multi-layer problem in such a way that the interface conditions can be expressed without any shift in the direction cosines as radiation either passes through or is reflected by an interface. We consider that this "preliminary analysis" is especially important since we are able to implement the interface conditions without interpolations (between direction cosines) as has been done, for example, by Liou and Wu [18]. Our preliminary analysis allows us to express the interface conditions in terms of simple independent variables (the direction cosines $\pm\mu \in (0,1]$ of the radiation intensity). And so, we are also able to identify the points (we call them break points) where discontinuities in the derivatives (with respect to the angular variables $\pm\mu$) are introduced into the intensities by the various Fresnel/Snell functions that are used to define all of the boundary and interface conditions. As a consequence of this preliminary analysis, we are able to define the minimum number (and location) of break points that we use to build our quadrature scheme (for a given layer). And so we do not resort to postulated quadrature schemes that either miss break points [18] or include more break points [19] than those absolutely necessary, when defining the composite quadrature scheme to be used for each layer.

While there are other works on this subject, we consider the two important papers by Liou and Wu [14,20] that report analyses of (and numerical results for) the case of isotropic scattering in each layer of a two-layer system to be the ones most closely related to our work.

2. Mathematical formulation

To define our notation, we consider that there are K distinct layers and that in each layer the radiation field is described by the monochromatic or grey equation of transfer, which we write as [16]

$$\mu\frac{\partial}{\partial\tau}I_k(\tau,\mu) + I_k(\tau,\mu) = \frac{\varpi_k}{2}\sum_{l=0}^{L_k}\beta_{k,l}P_l(\mu)\int_{-1}^{1}P_l(\mu')I_k(\tau,\mu')\,\mathrm{d}\mu', \tag{2.1}$$

for $k = 1,2,\ldots,K$, $\mu \in [-1,1]$, and $\tau \in (a_{k-1},a_k)$. Here we use a_0 to define the location of the first surface and a_K for the last surface. In addition, τ is the optical variable (defined differently for each of the K layers, as discussed in Section 9), $\varpi_k \in [0,1]$ and $\{\beta_{k,l}\}$ are, respectively, the single-scattering albedo and the coefficients in a Legendre expansion of the scattering law appropriate to each of the K layers. As we are considering, in this work, problems without azimuthal dependence, the intensities $I_k(\tau,\mu)$ depend only on τ and the polar angle $\theta = \arccos\mu$. The interior sub-strata meet at $\tau = a_k$, for $k = 1,2,\ldots,K-1$, and so it is at these values of τ that we must introduce the appropriate interface conditions.

In regard to the boundary (surface) conditions, we assume there is incident on the surface located at $\tau = a_0$, from a medium characterized by an index of refraction n_0, a known distribution of radiation described by $\psi_0(\mu)$, and similarly there is incident on the surface located at $\tau = a_K$, from a medium characterized by index of refraction n_{K+1}, a known distribution $\psi_K(\mu)$. And so, using the Fresnel and Snell laws, we express these conditions as [2,21,22]

$$I_1(a_0,\mu) = X(n_{1,0},\mu)I_1(a_0,-\mu) + Y(n_{1,0},\mu)\psi_0[f(n_{1,0},\mu)] \tag{2.2a}$$

and

$$I_K(a_K,-\mu) = X(n_{K,K+1},\mu)I_K(a_K,\mu) + Y(n_{K,K+1},\mu)\psi_K[f(n_{K,K+1},\mu)], \tag{2.2b}$$

for $\mu \in (0,1]$. Here,

$$f(n,\mu) = [1 - n^2(1-\mu^2)]^{1/2}, \tag{2.3}$$

the reflection coefficient $X(n,\mu)$ is defined as

$$X(n,\mu)=\begin{cases} G(n,\mu), & n\leqslant 1,\\ R(n,\mu), & n\geqslant 1,\end{cases} \tag{2.4}$$

and the transmission coefficient $Y(n,\mu)$ is defined as

$$Y(n,\mu)=n^2[1-X(n,\mu)]. \tag{2.5}$$

In addition, we use the Heaviside step function $H(x)$ to express the basic elements of the Fresnel and Snell laws as

$$R(n,\mu)=1+[G(n,\mu)-1]H[\mu-\mu_c(n)], \tag{2.6}$$

$$G(n,\mu)=\frac{1}{2}\left\{\left[\frac{\mu-nf(n,\mu)}{\mu+nf(n,\mu)}\right]^2+\left[\frac{n\mu-f(n,\mu)}{n\mu+f(n,\mu)}\right]^2\right\} \tag{2.7}$$

and

$$\mu_c(n)=(1-1/n^2)^{1/2}. \tag{2.8}$$

In the process of writing Eqs. (2.2), we have also used

$$n_{\alpha,\beta}=n_\alpha/n_\beta, \tag{2.9}$$

where n_α and n_β denote the indices of refraction for layers α and β, respectively. At each of the interfaces we have conditions similar to Eqs. (2.2) except that now there are no known driving terms. We have worked with the expressions reported by Cassel and Williams [2] so that we can write these conditions (in an improved notation) for the multi-layer case as

$$I_k(a_k,-\mu)=X(n_{k,k+1},\mu)I_k(a_k,\mu)+Y(n_{k,k+1},\mu)I_{k+1}[a_k,-f(n_{k,k+1},\mu)] \tag{2.10a}$$

and

$$I_{k+1}(a_k,\mu)=X(n_{k+1,k},\mu)I_{k+1}(a_k,-\mu)+Y(n_{k+1,k},\mu)I_k[a_k,f(n_{k+1,k},\mu)], \tag{2.10b}$$

for $\mu\in(0,1]$ and $k=1,2,\ldots,K-1$.

In this work, we compute the radiation intensities exiting the multi-layer system and the currents exiting each of the two external surfaces (normalized by the sum of the incoming currents at the two surfaces). To be clear, we note that the two incoming currents have not (yet) entered the layers and so are given simply by

$$J_0^+=\int_0^1\psi_0(\mu)\mu\,\mathrm{d}\mu \quad\text{and}\quad J_K^-=\int_0^1\psi_K(\mu)\mu\,\mathrm{d}\mu. \tag{2.11a,b}$$

On the other hand, it turns out that we can express the currents exiting the two surfaces as

$$J_0^-=\int_0^1[1-X(n_{1,0},\mu)]I_1(a_0,-\mu)\mu\,\mathrm{d}\mu+\int_0^1X(n_{0,1},\mu)\psi_0(\mu)\mu\,\mathrm{d}\mu \tag{2.12a}$$

and

$$J_K^+=\int_0^1[1-X(n_{K,K+1},\mu)]I_K(a_K,\mu)\mu\,\mathrm{d}\mu+\int_0^1X(n_{K+1,K},\mu)\psi_K(\mu)\mu\,\mathrm{d}\mu. \tag{2.12b}$$

Note that the first terms in Eqs. (2.12) represent radiation that has passed through the surfaces, while the second terms in those equations describe radiation that is reflected at the bounding surfaces. And so, in addition to the intensities exiting the multi-layer system, we also report

$$A=\frac{1}{N_0}J_0^- \quad\text{and}\quad B=\frac{1}{N_0}J_K^+, \tag{2.13a,b}$$

where

$$N_0 = J_0^+ + J_K^-. \tag{2.14}$$

And so, summarizing, we seek for each of the K regions solutions of Eq. (2.1), subject to the boundary and interface conditions

$$I_1(a_0,\mu) - X(n_{1,0},\mu)I_1(a_0,-\mu) = Y(n_{1,0},\mu)\psi_0[f(n_{1,0},\mu)] \tag{2.15a}$$

and

$$I_1(a_1,-\mu) - X(n_{1,2},\mu)I_1(a_1,\mu) = Y(n_{1,2},\mu)I_2[a_1,-f(n_{1,2},\mu)], \tag{2.15b}$$

for $\mu \in (0,1]$,

$$I_k(a_{k-1},\mu) - X(n_{k,k-1},\mu)I_k(a_{k-1},-\mu) = Y(n_{k,k-1},\mu)I_{k-1}[a_{k-1},f(n_{k,k-1},\mu)] \tag{2.15c}$$

and

$$I_k(a_k,-\mu) - X(n_{k,k+1},\mu)I_k(a_k,\mu) = Y(n_{k,k+1},\mu)I_{k+1}[a_k,-f(n_{k,k+1},\mu)], \tag{2.15d}$$

for $\mu \in (0,1]$ and $k = 2,3,\ldots,K-1$, and

$$I_K(a_{K-1},\mu) - X(n_{K,K-1},\mu)I_K(a_{K-1},-\mu) = Y(n_{K,K-1},\mu)I_{K-1}[a_{K-1},f(n_{K,K-1},\mu)] \tag{2.15e}$$

and

$$I_K(a_K,-\mu) - X(n_{K,K+1},\mu)I_K(a_K,\mu) = Y(n_{K,K+1},\mu)\psi_K[f(n_{K,K+1},\mu)], \tag{2.15f}$$

for $\mu \in (0,1]$.

It is clear from Eqs. (2.15) that we have a complication here that we would not encounter with the more often used specular/diffuse boundary and interface conditions: in addition to requiring the intensities evaluated at $\pm\mu \in (0,1]$, we also require

$$I_{k+1}[a_k,-f(n_{k,k+1},\mu)], \quad k = 1,2,\ldots,K-1,$$

and

$$I_{k-1}[a_{k-1},f(n_{k,k-1},\mu)], \quad k = 2,3,\ldots,K.$$

However, as developed in the following section of this work, we can carry out some preliminary analysis that will allow us to express all of the boundary and interface conditions in terms of known functions and radiation intensities evaluated at simple (angular) arguments, i.e., $I_k(a_{k-1},\pm\mu)$ and $I_k(a_k,\pm\mu)$, for $\mu \in (0,1]$ and $k = 1,2,\ldots,K$.

3. Preliminary analysis

As we consider this part of our analysis (what we call "pre-processing of the interface conditions") to be especially important, we first give an extended presentation for the case of only two layers ($K = 2$). In this way, the ideas behind this pre-processing can be seen clearly before the significant extensions required for the general case of a multi-layer ($K > 2$) medium are formulated.

3.1. The two-layer case

For $K = 2$, we have

$$\mu\frac{\partial}{\partial\tau}I_1(\tau,\mu) + I_1(\tau,\mu) = \mathscr{F}_1(\tau,\mu), \tag{3.1a}$$

for $\mu \in [-1,1]$ and $\tau \in (a_0,a_1)$, and

$$\mu\frac{\partial}{\partial\tau}I_2(\tau,\mu) + I_2(\tau,\mu) = \mathscr{F}_2(\tau,\mu), \tag{3.1b}$$

for $\mu \in [-1, 1]$ and $\tau \in (a_1, a_2)$, where,

$$\mathscr{F}_1(\tau,\mu) = \frac{\varpi_1}{2}\sum_{l=0}^{L_1} \beta_{1,l} P_l(\mu)\int_{-1}^{1} P_l(\mu') I_1(\tau,\mu')\,\mathrm{d}\mu' \tag{3.2a}$$

and

$$\mathscr{F}_2(\tau,\mu) = \frac{\varpi_2}{2}\sum_{l=0}^{L_2} \beta_{2,l} P_l(\mu)\int_{-1}^{1} P_l(\mu') I_2(\tau,\mu')\,\mathrm{d}\mu', \tag{3.2b}$$

subject to

$$I_1(a_0,\mu) - X(n_{1,0},\mu) I_1(a_0,-\mu) = Y(n_{1,0},\mu)\psi_0(\xi_{1,0}), \tag{3.3a}$$

$$I_1(a_1,-\mu) - X(n_{1,2},\mu) I_1(a_1,\mu) = Y(n_{1,2},\mu) I_2(a_1,-\xi_{1,2}), \tag{3.3b}$$

$$I_2(a_1,\mu) - X(n_{2,1},\mu) I_2(a_1,-\mu) = Y(n_{2,1},\mu) I_1(a_1,\xi_{2,1}) \tag{3.3c}$$

and

$$I_2(a_2,-\mu) - X(n_{2,3},\mu) I_2(a_2,\mu) = Y(n_{2,3},\mu)\psi_2(\xi_{2,3}), \tag{3.3d}$$

for $\mu \in (0, 1]$. Here, in writing Eqs. (3.3), we have compacted our notation by using

$$\xi_{\alpha,\beta} \Rightarrow \xi_{\alpha,\beta}(\mu) = f(n_{\alpha,\beta},\mu). \tag{3.4}$$

Looking at the interface conditions, Eqs. (3.3b) and (3.3c), we see that we require $I_2(a_1, -\xi_{1,2})$ and $I_1(a_1, \xi_{2,1})$, in addition to the usual intensities evaluated at simple angular arguments, $\pm\mu \in (0, 1]$. Our way of dealing with this complication is based on combining Eqs. (3.3) with integrated forms of Eqs. (3.1). So, for the moment, we assume that the right-hand sides of Eqs. (3.1) are known (for example, from a previous iterate when using an iterative procedure), and we integrate those equations to obtain

$$I_1(\tau,\mu) = I_1(a_0,\mu)\,\mathrm{e}^{-(\tau-a_0)/\mu} + \frac{1}{\mu}\int_{a_0}^{\tau} \mathscr{F}_1(x,\mu)\,\mathrm{e}^{-(\tau-x)/\mu}\,\mathrm{d}x \tag{3.5a}$$

and

$$I_1(\tau,-\mu) = I_1(a_1,-\mu)\,\mathrm{e}^{-(a_1-\tau)/\mu} + \frac{1}{\mu}\int_{\tau}^{a_1} \mathscr{F}_1(x,-\mu)\,\mathrm{e}^{-(x-\tau)/\mu}\,\mathrm{d}x, \tag{3.5b}$$

for $\mu \in (0, 1]$ and $\tau \in (a_0, a_1)$, and

$$I_2(\tau,\mu) = I_2(a_1,\mu)\,\mathrm{e}^{-(\tau-a_1)/\mu} + \frac{1}{\mu}\int_{a_1}^{\tau} \mathscr{F}_2(x,\mu)\,\mathrm{e}^{-(\tau-x)/\mu}\,\mathrm{d}x \tag{3.6a}$$

and

$$I_2(\tau,-\mu) = I_2(a_2,-\mu)\,\mathrm{e}^{-(a_2-\tau)/\mu} + \frac{1}{\mu}\int_{\tau}^{a_2} \mathscr{F}_2(x,-\mu)\,\mathrm{e}^{-(x-\tau)/\mu}\,\mathrm{d}x, \tag{3.6b}$$

for $\mu \in (0, 1]$ and $\tau \in (a_1, a_2)$.

We now discuss our way of using Eqs. (3.3), (3.5), and (3.6) to express the intensities $I_2(a_1, -\xi_{1,2})$ and $I_1(a_1, \xi_{2,1})$ in terms of intensities evaluated at the simple arguments $\pm\mu \in (0, 1]$. We begin by using Eq. (3.5a) for $\tau = a_1$ in Eq. (3.3a) and Eq. (3.5b) for $\tau = a_0$ in Eq. (3.3b) to find

$$I_1(a_1,\mu) = X(n_{1,0},\mu) I_1(a_0,-\mu)\,\mathrm{e}^{-\Delta_1/\mu} + K_1^{+}(\mu) \tag{3.7a}$$

and

$$I_1(a_0,-\mu) = X(n_{1,2},\mu) I_1(a_1,\mu)\,\mathrm{e}^{-\Delta_1/\mu} + K_1^{-}(\mu), \tag{3.7b}$$

where $\Delta_1 = a_1 - a_0$ is the optical thickness of layer one,

$$K_1^{+}(\mu) = Q_1^{+}(\mu) + Y(n_{1,0},\mu)\psi_0(\xi_{1,0})\,\mathrm{e}^{-\Delta_1/\mu} \tag{3.8a}$$

and

$$K_1^-(\mu) = Q_1^-(\mu) + Y(n_{1,2},\mu)I_2(a_1,-\xi_{1,2})\,\mathrm{e}^{-\Delta_1/\mu}. \tag{3.8b}$$

Here

$$Q_1^+(\mu) = \frac{1}{\mu}\int_{a_0}^{a_1} \mathscr{F}_1(x,\mu)\,\mathrm{e}^{-(a_1-x)/\mu}\,\mathrm{d}x \tag{3.9a}$$

and

$$Q_1^-(\mu) = \frac{1}{\mu}\int_{a_0}^{a_1} \mathscr{F}_1(x,-\mu)\,\mathrm{e}^{-(x-a_0)/\mu}\,\mathrm{d}x. \tag{3.9b}$$

We can now substitute Eq. (3.7b) into the right-hand side of Eq. (3.7a) to find

$$I_1(a_1,\mu) = \Xi_2^-(\mu)[K_1^+(\mu) + X(n_{1,0},\mu)K_1^-(\mu)\,\mathrm{e}^{-\Delta_1/\mu}], \tag{3.10}$$

where

$$\Xi_2^-(\mu) = [1 - X(n_{1,0},\mu)X(n_{1,2},\mu)\,\mathrm{e}^{-2\Delta_1/\mu}]^{-1}. \tag{3.11}$$

In a similar way, using Eq. (3.6a) for $\tau = a_2$ in Eq. (3.3c) and Eq. (3.6b) for $\tau = a_1$ in Eq. (3.3d), we find, for region two,

$$I_2(a_2,\mu) = X(n_{2,1},\mu)I_2(a_1,-\mu)\,\mathrm{e}^{-\Delta_2/\mu} + K_2^+(\mu) \tag{3.12a}$$

and

$$I_2(a_1,-\mu) = X(n_{2,3},\mu)I_2(a_2,\mu)\,\mathrm{e}^{-\Delta_2/\mu} + K_2^-(\mu), \tag{3.12b}$$

where $\Delta_2 = a_2 - a_1$ is the optical thickness of layer two,

$$K_2^+(\mu) = Q_2^+(\mu) + Y(n_{2,1},\mu)I_1(a_1,\xi_{2,1})\,\mathrm{e}^{-\Delta_2/\mu} \tag{3.13a}$$

and

$$K_2^-(\mu) = Q_2^-(\mu) + Y(n_{2,3},\mu)\psi_2(\xi_{2,3})\,\mathrm{e}^{-\Delta_2/\mu}. \tag{3.13b}$$

Here

$$Q_2^+(\mu) = \frac{1}{\mu}\int_{a_1}^{a_2} \mathscr{F}_2(x,\mu)\,\mathrm{e}^{-(a_2-x)/\mu}\,\mathrm{d}x \tag{3.14a}$$

and

$$Q_2^-(\mu) = \frac{1}{\mu}\int_{a_1}^{a_2} \mathscr{F}_2(x,-\mu)\,\mathrm{e}^{-(x-a_1)/\mu}\,\mathrm{d}x. \tag{3.14b}$$

And so, substituting Eq. (3.12a) into the right-hand side of Eq. (3.12b), we find

$$I_2(a_1,-\mu) = \Xi_1^+(\mu)[K_2^-(\mu) + X(n_{2,3},\mu)K_2^+(\mu)\,\mathrm{e}^{-\Delta_2/\mu}], \tag{3.15}$$

where

$$\Xi_1^+(\mu) = [1 - X(n_{2,3},\mu)X(n_{2,1},\mu)\,\mathrm{e}^{-2\Delta_2/\mu}]^{-1}. \tag{3.16}$$

Finally, changing the argument μ to, respectively, $\xi_{2,1}$ and $\xi_{1,2}$ in Eqs. (3.10) and (3.15), we are able to establish our desired expressions for $I_1(a_1,\xi_{2,1})$ and $I_2(a_1,-\xi_{1,2})$, viz.

$$\begin{aligned} I_1(a_1,\xi_{2,1}) = \Xi_2^-(\xi_{2,1})\{&Q_1^+(\xi_{2,1}) + Y(n_{1,0},\xi_{2,1})\psi_0(\xi_{2,0})\,\mathrm{e}^{-\Delta_1/\xi_{2,1}} \\ &+ X(n_{1,0},\xi_{2,1})\,\mathrm{e}^{-\Delta_1/\xi_{2,1}}[Q_1^-(\xi_{2,1}) + Y(n_{1,2},\xi_{2,1})I_2(a_1,-\mu)\,\mathrm{e}^{-\Delta_1/\xi_{2,1}}]\} \end{aligned} \tag{3.17a}$$

and

$$I_2(a_1, -\xi_{1,2}) = \Xi_1^+(\xi_{1,2})\{Q_2^-(\xi_{1,2}) + Y(n_{2,3}, \xi_{1,2})\psi_2(\xi_{1,3})\,\mathrm{e}^{-\Delta_2/\xi_{1,2}} + X(n_{2,3}, \xi_{1,2})\,\mathrm{e}^{-\Delta_2/\xi_{1,2}}[Q_2^+(\xi_{1,2}) + Y(n_{2,1}, \xi_{1,2})I_1(a_1, \mu)\,\mathrm{e}^{-\Delta_2/\xi_{1,2}}]\}. \tag{3.17b}$$

And so, using Eqs. (3.17) on the right-hand sides of Eqs. (3.3b) and (3.3c), we can rewrite the boundary and interface conditions (for $K = 2$) as

$$I_1(a_0, \mu) - X(n_{1,0}, \mu)I_1(a_0, -\mu) = Y(n_{1,0}, \mu)\psi_0(\xi_{1,0}), \tag{3.18a}$$

$$I_1(a_1, -\mu) - Z_1^+(\mu)I_1(a_1, \mu) = V_1^+(\mu) + W_1^+(\mu), \tag{3.18b}$$

$$I_2(a_1, \mu) - Z_2^-(\mu)I_2(a_1, -\mu) = V_2^-(\mu) + W_2^-(\mu) \tag{3.18c}$$

and

$$I_2(a_2, -\mu) - X(n_{2,3}, \mu)I_2(a_2, \mu) = Y(n_{2,3}, \mu)\psi_2(\xi_{2,3}), \tag{3.18d}$$

for $\mu \in (0, 1]$. In writing Eqs. (3.18), we have introduced some additional notation:

$$Z_1^+(\mu) = X(n_{1,2}, \mu) + Y(n_{1,2}, \mu)\Xi_1^+(\xi_{1,2})X(n_{2,3}, \xi_{1,2})Y(n_{2,1}, \xi_{1,2})\,\mathrm{e}^{-2\Delta_2/\xi_{1,2}}, \tag{3.19a}$$

$$Z_2^-(\mu) = X(n_{2,1}, \mu) + Y(n_{2,1}, \mu)\Xi_2^-(\xi_{2,1})X(n_{1,0}, \xi_{2,1})Y(n_{1,2}, \xi_{2,1})\,\mathrm{e}^{-2\Delta_1/\xi_{2,1}}, \tag{3.19b}$$

$$V_1^+(\mu) = Y(n_{1,2}, \mu)\Xi_1^+(\xi_{1,2})Y(n_{2,3}, \xi_{1,2})\psi_2(\xi_{1,3})\,\mathrm{e}^{-\Delta_2/\xi_{1,2}}, \tag{3.20a}$$

$$V_2^-(\mu) = Y(n_{2,1}, \mu)\Xi_2^-(\xi_{2,1})Y(n_{1,0}, \xi_{2,1})\psi_0(\xi_{2,0})\,\mathrm{e}^{-\Delta_1/\xi_{2,1}}, \tag{3.20b}$$

$$W_1^+(\mu) = Y(n_{1,2}, \mu)\Xi_1^+(\xi_{1,2})[Q_2^-(\xi_{1,2}) + X(n_{2,3}, \xi_{1,2})Q_2^+(\xi_{1,2})\,\mathrm{e}^{-\Delta_2/\xi_{1,2}}] \tag{3.21a}$$

and

$$W_2^-(\mu) = Y(n_{2,1}, \mu)\Xi_2^-(\xi_{2,1})[Q_1^+(\xi_{2,1}) + X(n_{1,0}, \xi_{2,1})Q_1^-(\xi_{2,1})\,\mathrm{e}^{-\Delta_1/\xi_{2,1}}]. \tag{3.21b}$$

3.2. The multi-layer case

A generalization of the procedure that we have called "pre-processing of the interface conditions" (and explained in detail in the first part of this section for the case $K = 2$) was carried out for the multi-layer case. To this end, we have explicitly derived the improved interface conditions for the cases of three, four, and five layers, and then showed (rigorous proof by mathematical induction) that the same formulas hold for any number of layers. The resulting boundary/interface conditions can be written in the compact form

$$I_k(a_{k-1}, \mu) - Z_k^-(\mu)I_k(a_{k-1}, -\mu) = V_k^-(\mu) + W_k^-(\mu) \tag{3.22a}$$

and

$$I_k(a_k, -\mu) - Z_k^+(\mu)I_k(a_k, \mu) = V_k^+(\mu) + W_k^+(\mu), \tag{3.22b}$$

for $k = 1, 2, \ldots, K$. Here the $Z_k^-(\mu)$ and $Z_k^+(\mu)$ functions are defined by the following recursive schemes:

$$Z_1^-(\mu) = X(n_{1,0}, \mu), \tag{3.23a}$$

$$Z_K^+(\mu) = X(n_{K,K+1}, \mu), \tag{3.23b}$$

$$Z_k^-(\mu) = X(n_{k,k-1}, \mu) + Y(n_{k,k-1}, \mu)\Lambda_k^-[f(n_{k,k-1}, \mu)], \quad k = 2, 3, \ldots, K, \tag{3.24a}$$

and

$$Z_k^+(\mu) = X(n_{k,k+1}, \mu) + Y(n_{k,k+1}, \mu)\Lambda_k^+[f(n_{k,k+1}, \mu)], \quad k = K-1, K-2, \ldots, 1, \tag{3.24b}$$

where

$$\Lambda_k^-(\xi) = \Xi_k^-(\xi) Z_{k-1}^-(\xi) Y(n_{k-1,k}, \xi)\, \mathrm{e}^{-2\Delta_{k-1}/\xi} \tag{3.25a}$$

and

$$\Lambda_k^+(\xi) = \Xi_k^+(\xi) Z_{k+1}^+(\xi) Y(n_{k+1,k}, \xi)\, \mathrm{e}^{-2\Delta_{k+1}/\xi}, \tag{3.25b}$$

with

$$\Xi_k^-(\xi) = [1 - Z_{k-1}^-(\xi) X(n_{k-1,k}, \xi)\, \mathrm{e}^{-2\Delta_{k-1}/\xi}]^{-1} \tag{3.26a}$$

and

$$\Xi_k^+(\xi) = [1 - Z_{k+1}^+(\xi) X(n_{k+1,k}, \xi)\, \mathrm{e}^{-2\Delta_{k+1}/\xi}]^{-1}. \tag{3.26b}$$

We note that in Eqs. (3.25) and (3.26) and hereafter we use

$$\Delta_k = a_k - a_{k-1} \tag{3.27}$$

to denote the optical thickness of layer k. Similarly, the $V_k^-(\mu)$ and $V_k^+(\mu)$ functions are defined by the recursive schemes

$$V_1^-(\mu) = Y(n_{1,0}, \mu) \psi_0[f(n_{1,0}, \mu)], \tag{3.28a}$$

$$V_K^+(\mu) = Y(n_{K,K+1}, \mu) \psi_K[f(n_{K,K+1}, \mu)], \tag{3.28b}$$

$$V_k^-(\mu) = Y(n_{k,k-1}, \mu) \Pi_k^-[f(n_{k,k-1}, \mu)], \quad k = 2, 3, \ldots, K, \tag{3.29a}$$

and

$$V_k^+(\mu) = Y(n_{k,k+1}, \mu) \Pi_k^+[f(n_{k,k+1}, \mu)], \; k = K-1, K-2, \ldots, 1, \tag{3.29b}$$

where

$$\Pi_k^-(\xi) = \Xi_k^-(\xi) V_{k-1}^-(\xi)\, \mathrm{e}^{-\Delta_{k-1}/\xi} \tag{3.30a}$$

and

$$\Pi_k^+(\xi) = \Xi_k^+(\xi) V_{k+1}^+(\xi)\, \mathrm{e}^{-\Delta_{k+1}/\xi}. \tag{3.30b}$$

Finally, the $W_k^-(\mu)$ and $W_k^+(\mu)$ functions are defined by the recursive schemes

$$W_1^-(\mu) = 0, \tag{3.31a}$$

$$W_K^+(\mu) = 0, \tag{3.31b}$$

$$W_k^-(\mu) = Y(n_{k,k-1}, \mu) \Omega_k^-[f(n_{k,k-1}, \mu)], \quad k = 2, 3, \ldots, K, \tag{3.32a}$$

and

$$W_k^+(\mu) = Y(n_{k,k+1}, \mu) \Omega_k^+[f(n_{k,k+1}, \mu)], \; k = K-1, K-2, \ldots, 1, \tag{3.32b}$$

where

$$\Omega_k^-(\xi) = \Xi_k^-(\xi) \{Q_{k-1}^+(\xi) + [Z_{k-1}^-(\xi) Q_{k-1}^-(\xi) + W_{k-1}^-(\xi)]\, \mathrm{e}^{-\Delta_{k-1}/\xi}\} \tag{3.33a}$$

and

$$\Omega_k^+(\xi) = \Xi_k^+(\xi) \{Q_{k+1}^-(\xi) + [Z_{k+1}^+(\xi) Q_{k+1}^+(\xi) + W_{k+1}^+(\xi)]\, \mathrm{e}^{-\Delta_{k+1}/\xi}\}. \tag{3.33b}$$

Here

$$Q_k^+(\mu) = \frac{1}{\mu} \int_{a_{k-1}}^{a_k} \mathscr{F}_k(\tau, \mu)\, \mathrm{e}^{-(a_k - \tau)/\mu}\, \mathrm{d}\tau \tag{3.34a}$$

and

$$Q_k^-(\mu) = \frac{1}{\mu}\int_{a_{k-1}}^{a_k} \mathscr{F}_k(\tau, -\mu)\, \mathrm{e}^{-(\tau-a_{k-1})/\mu}\, \mathrm{d}\tau, \tag{3.34b}$$

where

$$\mathscr{F}_k(\tau, \mu) = \frac{\varpi_k}{2}\sum_{l=0}^{L_k} \beta_{k,l} P_l(\mu) \int_{-1}^{1} P_l(\mu') I_k(\tau, \mu')\, \mathrm{d}\mu'. \tag{3.35}$$

4. Break points in the quadrature scheme

Since our approach in this work is essentially a direct use of the ADO method [17] and the radiation intensities display, as discussed in the Introduction, discontinuities in their derivatives with respect to the angular variables $\pm\mu$, it is clear that we can compute the quantities of interest with good accuracy only if we introduce partitions in the angular interval, when defining the quadrature scheme to be used by the method. To this end, we introduce break points in [0, 1], in such a way that the angular variation of the intensity within each of the subintervals of the composite quadrature so defined be smooth. It turns out that the required break points are layer-dependent, and consequently a different quadrature set will be applied to each layer.

To establish the required set of break points, we must find all points where the Heaviside function in Eq. (2.6) can have an effect (which could be large or small depending on the data that defines a particular problem) on our solution. And so, since we have expressed the boundary/interface conditions in the form of Eqs. (3.22), we conclude that the break points can be determined by finding how the Heaviside function affects the input data given by the Z, V, and W functions. As we have been able to show that the set of break points due to the W functions contains all of the break points due to the Z and V functions, it is sufficient to consider the functions $W_k^-(\mu)$ and $W_k^+(\mu)$ when looking for the desired set of break points for layer k.

The W functions are defined recursively by Eqs. (3.31)–(3.33) and depend on Z, X, and Y functions of varying arguments. The Z functions are also defined recursively according to Eqs. (3.23)–(3.26) and depend on X and Y functions of varying arguments. Since the Y function is simply defined in terms of the X function by Eq. (2.5), we conclude that the points of discontinuity in the derivatives of the W functions are introduced by the various X functions of different arguments that appear along the recursive definition of the Z and W functions. However, the Y functions also play a role in this process: since $Y(n, \mu) = 0$ for $\mu < \mu_c(n)$ when $n \geqslant 1$, we must reject, when performing the break-point analysis for a given recursive step of definition of the Z and W functions, any break point that happens to be located in the range where the corresponding Y function is zero.

The details of the analysis are too lengthy to be reported in this paper, and so here we only mention that the property

$$\xi_{\alpha,\beta}[\xi_{i,j}(\mu)] = [1 - n_{\alpha,\beta}^2 n_{i,j}^2 (1-\mu^2)]^{1/2} \tag{4.1}$$

and the fact that the function $X(n_{a,b}, \xi_{c,a})$ can introduce a derivative discontinuity at the critical point

$$\mu = \mu_c(n_{c,b}), \quad \text{if } n_{a,b} > 1 \text{ and } n_{c,b} > 1, \tag{4.2}$$

were used in our derivation. We found that the desired set of break points for layer k consists of the critical points

$$\mu_c(n_{k,\alpha}), \quad \alpha = 0, 1, \ldots, K+1, \tag{4.3}$$

that satisfy the conditions

$$n_{k,\alpha} \geqslant 1 \quad \text{and} \quad n_{\alpha+1,\alpha} \geqslant 1, \quad \alpha = 0 \text{ and } 1, \tag{4.4a}$$

$$n_{k,\alpha} \geqslant 1 \quad \text{and} \quad (n_{\alpha+1,\alpha} \geqslant 1 \text{ or } n_{\alpha-1,\alpha} \geqslant 1), \quad \alpha = 2, 3, \ldots, K-1, \; K > 2, \tag{4.4b}$$

$$n_{k,\alpha} \geqslant 1 \quad \text{and} \quad n_{\alpha-1,\alpha} \geqslant 1, \quad \alpha = K \text{ and } K+1, \tag{4.4c}$$

$$n_{\beta,\alpha} \geqslant 1 \quad \text{or} \quad n_{\beta,k} \geqslant 1 \quad \text{or} \quad n_{\beta,\beta+1} \geqslant 1, \quad \alpha = 0, 1, \ldots, k-2 \quad \text{and}$$

$$\beta = \alpha+1, \alpha+2, \ldots, k-1, \;\; k>1, \tag{4.5a}$$

and

$$n_{\beta,\alpha} \geqslant 1 \quad \text{or} \quad n_{\beta,k} \geqslant 1 \quad \text{or} \quad n_{\beta,\beta-1} \geqslant 1, \quad \alpha = k+2, k+3, \ldots, K+1 \quad \text{and}$$

$$\beta = k+1, k+2, \ldots, \alpha-1, \;\; k<K. \tag{4.5b}$$

And so, before accepting $\mu_c(n_{k,\alpha})$ as a break point of layer k for some value of α, we must be sure that all of the conditions applicable to that value of α are satisfied in Eqs. (4.4) and (4.5).

Finally, to complete this section, we specify how the break points are used to define the quadrature scheme for layer k in our ADO solution. To begin, we sort, by increasing order of magnitude, the nonzero critical points of Eq. (4.3) that satisfy the conditions imposed by Eqs. (4.4) and (4.5). This defines a sequence of break points b_j, $j = 1, 2, \ldots, m_k$, where m_k denotes the number of nonzero, nonrepeated critical points of Eq. (4.3) that satisfy Eqs. (4.4) and (4.5). Then, we map (linearly) a standard Gauss–Legendre quadrature of order M onto each of the subintervals

$$[0, b_1], \; [b_1, b_2], \; \ldots, \; [b_{m_k-1}, b_{m_k}] \quad \text{and} \quad [b_{m_k}, 1],$$

to obtain a composite quadrature of order $N_k = (m_k + 1)M$ for layer k.

5. A solution for nonconservative layers

Here we use the ADO version [17] of the discrete-ordinates method used [23] to solve Chandrasekhar's basic problem in radiative transfer [16]. Since much of what we require here is already available [17,23], our presentation is brief. To begin we write (for the case of a nonconservative layer: $\varpi_k \neq 1$)

$$I_k(\tau, \pm\mu_i) = \sum_{j=1}^{N_k} [A_{k,j}\phi_k(\nu_{k,j}, \pm\mu_i)\,\mathrm{e}^{-(\tau-a_{k-1})/\nu_{k,j}} + B_{k,j}\phi_k(\nu_{k,j}, \mp\mu_i)\,\mathrm{e}^{-(a_k-\tau)/\nu_{k,j}}], \tag{5.1}$$

for $\tau \in (a_{k-1}, a_k)$. We allow the use of a different quadrature scheme in each of the layers; however, to avoid cluttering our equations with too much notation, we suppress the k dependence that could be affixed to the weights and nodes $\{w_i, \mu_i\}$. Nevertheless, we do show explicitly the k dependence of the separation constants $\{\nu_{k,j}\}$ and the elementary solutions $\phi_k(\nu_{k,j}, \pm\mu_i)$, as well as the arbitrary coefficients $\{A_{k,j}\}$ and $\{B_{k,j}\}$. We note that the elementary solutions and the separation constants for each region are obtained by solving an eigensystem of order N_k that can be constructed as explained in detail in Refs. [17] and [23].

In order to complete the solution listed as Eq. (5.1) we must determine, for each of the K regions, the coefficients $\{A_{k,j}, B_{k,j}\}$. Noting the boundary/interface conditions listed as Eqs. (3.22), we conclude that, in addition to the discrete-ordinates solutions listed as Eqs. (5.1), we require the integrated quantities listed as Eqs. (3.34) in order to compute the quantities listed by Eqs. (3.33) and then the quantities listed by Eqs. (3.32) that are required in Eqs. (3.22). And so, we use Eq. (5.1) in a discrete-ordinates version of Eq. (3.35) to find

$$Q_k^+(\mu) = \frac{\varpi_k}{2}\sum_{l=0}^{L_k} \beta_{k,l}P_l(\mu)\sum_{j=1}^{N_k} \nu_{k,j}[A_{k,j}C(\Delta_k : \nu_{k,j}, \mu) + (-1)^l B_{k,j}S(\Delta_k : \nu_{k,j}, \mu)]\Gamma_{k,l}(\nu_{k,j}) \tag{5.2a}$$

and

$$Q_k^-(\mu) = \frac{\varpi_k}{2}\sum_{l=0}^{L_k} \beta_{k,l}P_l(\mu)\sum_{j=1}^{N_k} \nu_{k,j}[(-1)^l A_{k,j}S(\Delta_k : \nu_{k,j}, \mu) + B_{k,j}C(\Delta_k : \nu_{k,j}, \mu)]\Gamma_{k,l}(\nu_{k,j}), \tag{5.2b}$$

where

$$\Gamma_{k,l}(\nu_{k,j}) = \sum_{i=1}^{N_k} w_i P_l(\mu_i)[\phi_k(\nu_{k,j}, \mu_i) + (-1)^l \phi_k(\nu_{k,j}, -\mu_i)]. \tag{5.3}$$

In addition, the S and C functions are given by

$$S(\tau : x, y) = \frac{1 - e^{-\tau/x} e^{-\tau/y}}{x + y} \tag{5.4a}$$

and

$$C(\tau : x, y) = \frac{e^{-\tau/x} - e^{-\tau/y}}{x - y}. \tag{5.4b}$$

We note that the right-hand sides of Eqs. (3.22) depend on the W functions, which in turn depend on the Q functions, which, in fact, depend on the solution we seek. And so our approach will be iterative. To start, we take the Q functions to be zero, we use Eq. (5.1) in discrete-ordinates versions of Eqs. (3.22), and then we solve the resulting systems of linear algebraic equations as we use a "left-right" sweep and a "right-left" sweep [24] through the layers to find the first estimates of the coefficients $\{A_{k,j}, B_{k,j}\}$. These estimates are then used to compute new values for the Q functions, and this procedure is continued until some convergence criteria is met. On the other hand, an iterative approach can be avoided by considering the defining equations for all K regions at the same time. And while avoiding iteration is a reasonable choice for the case of, say, $K = 2$, such an approach quickly becomes impractical when K is increased and high-order discrete-ordinates solutions are sought.

Once the coefficients $\{A_{k,j}, B_{k,j}\}$ have been found, we can compute the desired A and B from Eqs. (2.13) and discrete-ordinates versions of Eqs. (2.12), viz.,

$$J_0^- = \sum_{i=1}^{N_1} w_i \mu_i [1 - X(n_{1,0}, \mu_i)] I_1(a_0, -\mu_i) + \int_0^1 X(n_{0,1}, \mu) \psi_0(\mu) \mu \, d\mu \tag{5.5a}$$

and

$$J_K^+ = \sum_{i=1}^{N_K} w_i \mu_i [1 - X(n_{K,K+1}, \mu_i)] I_K(a_K, \mu_i) + \int_0^1 X(n_{K+1,K}, \mu) \psi_K(\mu) \mu \, d\mu. \tag{5.5b}$$

6. A solution for conservative layers

The solutions listed in the previous section require modification for conservative layers. And so here we make use of the solutions, as reported in Ref. [23], for the case of a conservative layer ($\varpi_k = 1$).

For conservative layers, the largest separation constant becomes unbounded, and so in our ADO solution, we simply ignore the ADO solution associated with the largest separation constant, say $v_{k,1}$, and use instead two exact solutions listed in Ref. [23]. And so here we replace Eq. (5.1) with

$$I_k(\tau, \pm\mu_i) = I_k^*(\tau, \pm\mu_i) + \sum_{j=2}^{N_k} [A_{k,j} \phi_k(v_{k,j}, \pm\mu_i) e^{-(\tau - a_{k-1})/v_{k,j}} + B_{k,j} \phi_k(v_{k,j}, \mp\mu_i) e^{-(a_k - \tau)/v_{k,j}}], \tag{6.1}$$

for $\tau \in (a_{k-1}, a_k)$, where the exact elements of Eq. (6.1) are given by

$$I_k^*(\tau, \mu) = A_{k,1} + B_{k,1}(\tau - 3\mu/h_{k,1}) \tag{6.2}$$

with

$$h_{k,1} = 3 - \beta_{k,1}. \tag{6.3}$$

Consequently, for conservative layers we replace Eqs. (5.2) with

$$Q_k^+(\mu) = E_k^+(\mu) + \frac{1}{2} \sum_{l=0}^{L_k} \beta_{k,l} P_l(\mu) \sum_{j=2}^{N_k} v_{k,j} [A_{k,j} C(\Delta_k : v_{k,j}, \mu) + (-1)^l B_{k,j} S(\Delta_k : v_{k,j}, \mu)] \Gamma_{k,l}(v_{k,j}) \tag{6.4a}$$

and

$$Q_k^-(\mu) = E_k^-(\mu) + \frac{1}{2}\sum_{l=0}^{L_k} \beta_{k,l} P_l(\mu) \sum_{j=2}^{N_k} \nu_{k,j}[(-1)^l A_{k,j} S(\Delta_k : \nu_{k,j}, \mu) + B_{k,j} C(\Delta_k : \nu_{k,j}, \mu)]\Gamma_{k,l}(\nu_{k,j}), \tag{6.4b}$$

where

$$E_k^+(\mu) = [A_{k,1} + B_{k,1}(a_{k-1} - 3\mu/h_{k,1})](1 - \mathrm{e}^{-\Delta_k/\mu}) + B_{k,1}\Delta_k \tag{6.5a}$$

and

$$E_k^-(\mu) = [A_{k,1} + B_{k,1}(a_k + 3\mu/h_{k,1})](1 - \mathrm{e}^{-\Delta_k/\mu}) - B_{k,1}\Delta_k. \tag{6.5b}$$

7. Boundary and interface intensities

Once the coefficients $\{A_{k,j}, B_{k,j}\}$ are available, we can use Eq. (5.1) [or Eq. (6.1), if the layer is conservative] to compute the radiation intensities $I_k(\tau, \pm\mu_i)$ for any value of $\tau \in [a_{k-1}, a_k]$. However, if we wish to know the intensities for values of μ other than $\{\pm\mu_i\}$, we need to add a post-processing step to our approach, as described next for the boundary/interface intensities and in Section 8 for the interior intensities.

We start by integrating Eq. (2.1) and a form of Eq. (2.1) with μ changed to $-\mu$ over τ, from a_{k-1} to a_k, to obtain

$$I_k(a_{k-1}, -\mu) = I_k(a_k, -\mu)\,\mathrm{e}^{-\Delta_k/\mu} + Q_k^-(\mu) \tag{7.1a}$$

and

$$I_k(a_k, \mu) = I_k(a_{k-1}, \mu)\,\mathrm{e}^{-\Delta_k/\mu} + Q_k^+(\mu), \tag{7.1b}$$

for $\mu \in (0, 1]$. After we use Eqs. (7.1) to eliminate the $I_k(a_k, -\mu)$ and $I_k(a_{k-1}, \mu)$ terms in Eqs. (3.22), we obtain, for the radiation intensities exiting layer k,

$$I_k(a_{k-1}, -\mu) = \Upsilon_k(\mu) C_k^-(\mu) \tag{7.2a}$$

and

$$I_k(a_k, \mu) = \Upsilon_k(\mu) C_k^+(\mu), \tag{7.2b}$$

for $\mu \in (0, 1]$. Here

$$\Upsilon_k(\mu) = [1 - Z_k^-(\mu) Z_k^+(\mu)\,\mathrm{e}^{-2\Delta_k/\mu}]^{-1} \tag{7.3}$$

and

$$C_k^\pm(\mu) = Q_k^\pm(\mu) + \mathrm{e}^{-\Delta_k/\mu}[V_k^\mp(\mu) + W_k^\mp(\mu)] + Z_k^\mp(\mu)\,\mathrm{e}^{-\Delta_k/\mu}\{Q_k^\mp(\mu) + \mathrm{e}^{-\Delta_k/\mu}[V_k^\pm(\mu) + W_k^\pm(\mu)]\}. \tag{7.4}$$

Next we substitute Eqs. (7.2) into Eqs. (3.22) to find the the radiation intensities entering layer k, viz.

$$I_k(a_{k-1}, \mu) = \Upsilon_k(\mu) D_k^+(\mu) \tag{7.5a}$$

and

$$I_k(a_k, -\mu) = \Upsilon_k(\mu) D_k^-(\mu), \tag{7.5b}$$

for $\mu \in (0, 1]$. Here

$$D_k^\pm(\mu) = Z_k^\mp(\mu)\,\mathrm{e}^{-\Delta_k/\mu}[Z_k^\pm(\mu) Q_k^\pm(\mu) + V_k^\pm(\mu) + W_k^\pm(\mu)] + Z_k^\mp(\mu) Q_k^\mp(\mu) + V_k^\mp(\mu) + W_k^\mp(\mu). \tag{7.6}$$

To conclude this section, we note that the radiation intensities exiting the multi-layer system

$$\widehat{I}(a_0, -\mu) = Y(n_{0,1}, \mu) I_1[a_0, -f(n_{0,1}, \mu)] + X(n_{0,1}, \mu)\psi_0(\mu) \tag{7.7a}$$

and

$$\widehat{I}(a_K, \mu) = Y(n_{K+1,K}, \mu) I_K[a_K, f(n_{K+1,K}, \mu)] + X(n_{K+1,K}, \mu)\psi_K(\mu), \tag{7.7b}$$

for $\mu \in (0, 1]$, are usually the most important for applications. It can be mentioned that the first terms in Eqs. (7.7) describe the radiation that exits from inside the multi-layer system through the external boundaries, while the second terms in Eqs. (7.7) define those parts of the incoming radiation that are reflected externally. Simpler formulas expressed in terms of a single argument μ can be derived for these quantities by following a procedure similar to that used to derive Eqs. (3.22). We find

$$\widehat{I}(a_0, -\mu) = Z_0^+(\mu)\psi_0(\mu) + V_0^+(\mu) + W_0^+(\mu) \tag{7.8a}$$

and

$$\widehat{I}(a_K, \mu) = Z_{K+1}^-(\mu)\psi_K(\mu) + V_{K+1}^-(\mu) + W_{K+1}^-(\mu), \tag{7.8b}$$

for $\mu \in (0, 1]$. Here the functions $Z_0^+(\mu)$, $V_0^+(\mu)$, and $W_0^+(\mu)$ are defined simply by carrying the recursive schemes of Eqs. (3.24b), (3.29b), and (3.32b) one step further, i.e., by stopping the use of these equations at $k = 0$ instead of at $k = 1$. Similarly, the functions $Z_{K+1}^-(\mu)$, $V_{K+1}^-(\mu)$, and $W_{K+1}^-(\mu)$ are defined by stopping the use of Eqs. (3.24a), (3.29a), and (3.32a) at $k = K + 1$ instead of at $k = K$.

8. Interior intensities

To complete the presentation of our solution, we now wish to report our way of computing the intensities $I_k(\tau, \pm\mu)$ for any $\tau \in (a_{k-1}, a_k)$, $\mu \in (0, 1]$, and $k = 1, 2, \dots, K$. Since the boundary/interface intensities are now explicitly available from Eqs. (7.2) and (7.5), we can obtain expressions for the desired interior intensities simply by post-processing a version of Eq. (2.1) where the integral over μ' is approximated by the quadrature rule for layer k and the solution expressed by Eq. (5.1) [or Eq. (6.1) if the layer is conservative] is used to define the resulting right-hand side. Considering $\mu > 0$ and integrating the resulting equation between a_{k-1} and an arbitrary interior point $\tau \in (a_{k-1}, a_k)$, we find, for $\varpi_k \neq 1$ and $\mu \in (0, 1]$,

$$I_k(\tau, \mu) = I_k(a_{k-1}, \mu)\,\mathrm{e}^{-(\tau - a_{k-1})/\mu} + \mathcal{Q}_k(\tau, \mu), \tag{8.1}$$

where

$$\mathcal{Q}_k(\tau, \mu) = \frac{\varpi_k}{2}\sum_{l=0}^{L_k}\beta_{k,l}P_l(\mu)\sum_{j=1}^{N_k}\nu_{k,j}[A_{k,j}C(\tau - a_{k-1} : \nu_{k,j}, \mu) + (-1)^l B_{k,j}\,\mathrm{e}^{-(a_k-\tau)/\nu_{k,j}}S(\tau - a_{k-1} : \nu_{k,j}, \mu)]\Gamma_{k,l}(\nu_{k,j}). \tag{8.2}$$

Similarly, changing μ to $-\mu$ in the approximated version of Eq. (2.1) and integrating the resulting equation between an arbitrary interior point $\tau \in (a_{k-1}, a_k)$ and a_k, we find, for $\varpi_k \neq 1$ and $\mu \in (0, 1]$,

$$I_k(\tau, -\mu) = I_k(a_k, -\mu)\,\mathrm{e}^{-(a_k - \tau)/\mu} + \mathcal{Q}_k(\tau, -\mu), \tag{8.3}$$

where

$$\mathcal{Q}_k(\tau, -\mu) = \frac{\varpi_k}{2}\sum_{l=0}^{L_k}\beta_{k,l}P_l(\mu)\sum_{j=1}^{N_k}\nu_{k,j}[(-1)^l A_{k,j}\,\mathrm{e}^{-(\tau-a_{k-1})/\nu_{k,j}}S(a_k - \tau : \nu_{k,j}, \mu) + B_{k,j}C(a_k - \tau : \nu_{k,j}, \mu)]\Gamma_{k,l}(\nu_{k,j}). \tag{8.4}$$

Repeating the above procedure with Eq. (6.1) in place of Eq. (5.1), we find that our post-processed expressions for the interior intensities for $\varpi_k = 1$ and $\mu \in (0, 1]$ can still be expressed as Eqs. (8.1) and (8.3)), but now, for a conservative case, we must use

$$\mathcal{Q}_k(\tau, \mu) = \mathcal{E}_k(\tau, \mu) + \frac{1}{2}\sum_{l=0}^{L_k}\beta_{k,l}P_l(\mu)\sum_{j=2}^{N_k}\nu_{k,j}[A_{k,j}C(\tau - a_{k-1} : \nu_{k,j}, \mu) + (-1)^l B_{k,j}\,\mathrm{e}^{-(a_k-\tau)/\nu_{k,j}}S(\tau - a_{k-1} : \nu_{k,j}, \mu)]\Gamma_{k,l}(\nu_{k,j}) \tag{8.5a}$$

and

$$\mathcal{Q}_k(\tau,-\mu) = \mathcal{E}_k(\tau,-\mu) + \frac{1}{2}\sum_{l=0}^{L_k} \beta_{k,l} P_l(\mu) \sum_{j=2}^{N_k} \nu_{k,j}[(-1)^l A_{k,j}\, \mathrm{e}^{-(\tau-a_{k-1})/\nu_{k,j}} S(a_k-\tau:\nu_{k,j},\mu) + B_{k,j} C(a_k-\tau:\nu_{k,j},\mu)]\Gamma_{k,l}(\nu_{k,j}), \tag{8.5b}$$

where

$$\mathcal{E}_k(\tau,\mu) = A_{k,1}[1-\mathrm{e}^{-(\tau-a_{k-1})/\mu}] + B_{k,1}\{\tau - a_{k-1} + (a_{k-1} - 3\mu/h_{k,1})[1-\mathrm{e}^{-(\tau-a_{k-1})/\mu}]\} \tag{8.6a}$$

and

$$\mathcal{E}_k(\tau,-\mu) = A_{k,1}[1-\mathrm{e}^{-(a_k-\tau)/\mu}] + B_{k,1}\{\tau - a_k + (a_k + 3\mu/h_{k,1})[1-\mathrm{e}^{-(a_k-\tau)/\mu}]\}, \tag{8.6b}$$

for $\mu \in (0,1]$, are used in Eqs. (8.5) to denote the parts of the solution that result from the unbounded separation constant $\nu_{k,1}$. To conclude this section, we note that Eqs. (8.1) and (8.3) require the intensities entering layer k, $I_k(a_{k-1},\mu)$ and $I_k(a_k,-\mu)$ for $\mu \in (0,1]$, which are available from Eqs. (7.5).

9. Numerical results

In this work, which we consider to be a "methods paper," we do not report extensive numerical results or present any graphical material. Instead, we report, first of all in Tables 1 and 2, some basic results (that we consider to be of a very high standard) for numerous cases defined for two-layer media. Here, we consider that the "bi-slab" is surrounded by vacuum, so that $n_0 = 1$ and $n_3 = 1$. While most of the defining data for the numerical results listed in Tables 1 and 2 are given within the titles of the tables, we still must define the scattering laws used. In order to make our results accessible to anyone seeking to make use of our solution or to compare to some independent calculation, we use scattering laws that are easy to implement: we introduce the binomial scattering law for both regions. To be clear, we use [25]

$$p(\cos\Theta) = \frac{L+1}{2^L}(1+\cos\Theta)^L \tag{9.1}$$

Table 1
The normalized exiting currents for two-layer problems: $\psi_0(\mu) = 2$ and $\psi_2(\mu) = 0$, with $\Delta_1 = 0.4$, $\Delta_2 = 0.6$, $\varpi_1 = 0.9$, $\varpi_2 = 0.99$, $L_1 = 10$, and $L_2 = 100$

	A			*B*		
n_1	$n_2 = 4/3$	$n_2 = 1.6$	$n_2 = 1.9$	$n_2 = 4/3$	$n_2 = 1.6$	$n_2 = 1.9$
1.1	1.229044(–1)	1.700122(–1)	2.207497(–1)	7.740266(–1)	7.177744(–1)	6.571845(–1)
1.4	1.557771(–1)	1.840196(–1)	2.206218(–1)	7.259716(–1)	6.889740(–1)	6.422040(–1)
1.8	2.141451(–1)	2.260156(–1)	2.475060(–1)	6.464151(–1)	6.270166(–1)	5.956496(–1)
2.0	2.428920(–1)	2.493952(–1)	2.653249(–1)	6.076314(–1)	5.943900(–1)	5.690639(–1)

Table 2
The normalized exiting currents for two-layer problems: $\psi_0(\mu) = 2$ and $\psi_2(\mu) = 0$, with $\Delta_1 = 5.0$, $\Delta_2 = 7.0$, $\varpi_1 = 1.0$, $\varpi_2 = 0.99$, $L_1 = 200$, and $L_2 = 100$

	A			*B*		
n_1	$n_2 = 4/3$	$n_2 = 1.6$	$n_2 = 1.9$	$n_2 = 4/3$	$n_2 = 1.6$	$n_2 = 1.9$
1.1	1.805408(–1)	2.274339(–1)	2.685735(–1)	6.358881(–1)	5.471086(–1)	4.610808(–1)
1.4	2.128375(–1)	2.372061(–1)	2.613893(–1)	6.015017(–1)	5.305965(–1)	4.571950(–1)
1.8	2.801588(–1)	2.843818(–1)	2.937984(–1)	5.437957(–1)	4.877804(–1)	4.276678(–1)
2.0	3.136204(–1)	3.101404(–1)	3.140405(–1)	5.157670(–1)	4.655563(–1)	4.102323(–1)

to describe the scattering process, with $L = L_1$ in region one and $L = L_2$ in region two. We note that Θ is the scattering angle and that L is a nonnegative integer. It is known that Eq. (9.1) can be rewritten in terms of Legendre polynomials as

$$p(\cos\Theta) = \sum_{l=0}^{L} \beta_l P_l(\cos\Theta), \tag{9.2}$$

where, as reported by McCormick and Sanchez [26], the β coefficients can be computed from the recursion formula

$$\beta_l = \left(\frac{2l+1}{2l-1}\right)\left(\frac{L+1-l}{L+1+l}\right)\beta_{l-1}, \tag{9.3}$$

for $l = 1, 2, \ldots, L$, with $\beta_0 = 1$.

In regard to the challenging multi-layer problems, we report in Tables 4–6 numerical results (also thought to be of a very high standard) for the five test cases defined in terms of the data given in Table 3.

For these test cases too, we consider that the multi-layer media are surrounded by vacuum so that $n_0 = 1$ and $n_{K+1} = 1$. Again, we use the binomial scattering law with $L = L_k$ in each layer and we consider the case of (isotropic) radiation incident only on the surface located at a_0, i.e.,

$$\psi_0(\mu) = 2 \quad \text{and} \quad \psi_K(\mu) = 0. \tag{9.4a,b}$$

To complete the data required for our calculation, we must note the way in which the interfaces and the boundaries of the considered multi-layer problem are defined. As Eq. (2.1) is written in terms of the same spatial variable τ, we must make clear how this variable is defined in each of the layers. To start, we consider that z is the spatial variable (in physical units), that the boundaries and interfaces are located at $z_0, z_1, \ldots, z_K$, and that the total cross-section (extinction coefficient) is σ_k for region k, $k = 1, 2, \ldots, K$. And so, in writing Eq. (2.1) in terms of the τ variable, we have used

$$\tau = a_{k-1} + \sigma_k(z - z_{k-1}), \quad z_{k-1} < z < z_k, \tag{9.5}$$

Table 3
Basic data for multi-layer problems

Layer #	Δ	ϖ	L	n
1	1.0	0.95	40	1.65
2	1.2	0.94	60	2.00
3	1.3	0.93	30	1.70
4	0.6	0.96	70	1.60
5	1.9	0.90	20	1.80
6	1.4	0.92	50	1.85
7	0.5	0.97	80	1.55
8	0.3	0.98	90	1.50
9	1.6	0.91	10	1.75
10	5.2	1.00	100	1.30

Table 4
The normalized exiting currents for multi-layer problems

Problem	Layers	A	B
I	1–3	1.950158(–1)	3.769879(–1)
II	6–10	2.157887(–1)	2.739887(–1)
III	4–10	1.498979(–1)	1.631278(–1)
IV	1–9	1.450329(–1)	9.725498(–2)
V	1–10	1.452098(–1)	9.790916(–2)

Table 5
The exiting intensity $\widehat{I}(a_0, -\mu)$ for multi-layer problems

μ	Problem I	Problem II	Problem III	Problem IV	Problem V
0.00	2.0	2.0	2.0	2.0	2.0
0.05	1.5685	1.5649	1.5380	1.5349	1.5350
0.10	1.2555	1.2557	1.2025	1.1982	1.1983
0.15	1.0242	1.0307	9.5492(–1)	9.4997(–1)	9.5012(–1)
0.20	8.5108(–1)	8.6391(–1)	7.6980(–1)	7.6444(–1)	7.6462(–1)
0.25	7.1996(–1)	7.3844(–1)	6.3009(–1)	6.2429(–1)	6.2450(–1)
0.30	6.1982(–1)	6.4302(–1)	5.2395(–1)	5.1762(–1)	5.1785(–1)
0.35	5.4282(–1)	5.6985(–1)	4.4295(–1)	4.3601(–1)	4.3625(–1)
0.40	4.8329(–1)	5.1341(–1)	3.8099(–1)	3.7339(–1)	3.7365(–1)
0.45	4.3707(–1)	4.6972(–1)	3.3358(–1)	3.2530(–1)	3.2558(–1)
0.50	4.0102(–1)	4.3584(–1)	2.9737(–1)	2.8843(–1)	2.8872(–1)
0.55	3.7278(–1)	4.0956(–1)	2.6982(–1)	2.6027(–1)	2.6057(–1)
0.60	3.5052(–1)	3.8920(–1)	2.4900(–1)	2.3890(–1)	2.3923(–1)
0.65	3.3284(–1)	3.7347(–1)	2.3342(–1)	2.2287(–1)	2.2321(–1)
0.70	3.1863(–1)	3.6136(–1)	2.2191(–1)	2.1102(–1)	2.1137(–1)
0.75	3.0704(–1)	3.5208(–1)	2.1358(–1)	2.0246(–1)	2.0284(–1)
0.80	2.9738(–1)	3.4499(–1)	2.0771(–1)	1.9650(–1)	1.9690(–1)
0.85	2.8912(–1)	3.3959(–1)	2.0376(–1)	1.9258(–1)	1.9300(–1)
0.90	2.8186(–1)	3.3549(–1)	2.0128(–1)	1.9026(–1)	1.9070(–1)
0.95	2.7527(–1)	3.3236(–1)	1.9992(–1)	1.8919(–1)	1.8967(–1)
1.00	2.6913(–1)	3.2994(–1)	1.9942(–1)	1.8910(–1)	1.8960(–1)

Table 6
The exiting intensity $\widehat{I}(a_K, \mu)$ for multi-layer problems

μ	Problem I	Problem II	Problem III	Problem IV	Problem V
0.00	0.0	0.0	0.0	0.0	0.0
0.05	1.4852(–1)	1.3826(–1)	8.5044(–2)	4.5866(–2)	5.1805(–2)
0.10	2.5861(–1)	2.3750(–1)	1.4604(–1)	7.8980(–2)	8.8944(–2)
0.15	3.4258(–1)	3.0972(–1)	1.9031(–1)	1.0356(–1)	1.1588(–1)
0.20	4.0859(–1)	3.6303(–1)	2.2286(–1)	1.2227(–1)	1.3565(–1)
0.25	4.6215(–1)	4.0305(–1)	2.4713(–1)	1.3688(–1)	1.5034(–1)
0.30	5.0711(–1)	4.3374(–1)	2.6553(–1)	1.4856(–1)	1.6144(–1)
0.35	5.4618(–1)	4.5791(–1)	2.7981(–1)	1.5814(–1)	1.6999(–1)
0.40	5.8136(–1)	4.7759(–1)	2.9119(–1)	1.6621(–1)	1.7674(–1)
0.45	6.1411(–1)	4.9424(–1)	3.0057(–1)	1.7321(–1)	1.8224(–1)
0.50	6.4555(–1)	5.0895(–1)	3.0862(–1)	1.7945(–1)	1.8689(–1)
0.55	6.7653(–1)	5.2253(–1)	3.1582(–1)	1.8518(–1)	1.9098(–1)
0.60	7.0772(–1)	5.3559(–1)	3.2252(–1)	1.9058(–1)	1.9472(–1)
0.65	7.3962(–1)	5.4857(–1)	3.2902(–1)	1.9580(–1)	1.9827(–1)
0.70	7.7264(–1)	5.6184(–1)	3.3551(–1)	2.0097(–1)	2.0177(–1)
0.75	8.0707(–1)	5.7567(–1)	3.4216(–1)	2.0617(–1)	2.0530(–1)
0.80	8.4310(–1)	5.9026(–1)	3.4910(–1)	2.1151(–1)	2.0894(–1)
0.85	8.8086(–1)	6.0579(–1)	3.5644(–1)	2.1703(–1)	2.1275(–1)
0.90	9.2035(–1)	6.2239(–1)	3.6427(–1)	2.2282(–1)	2.1677(–1)
0.95	9.6150(–1)	6.4014(–1)	3.7267(–1)	2.2892(–1)	2.2105(–1)
1.00	1.0042	6.5912(–1)	3.8170(–1)	2.3539(–1)	2.2563(–1)

along with the definition

$$a_k = a_{k-1} + \sigma_k(z_k - z_{k-1}), \tag{9.6}$$

where, for both of Eqs. (9.5) and (9.6), $k = 1, 2, \ldots, K$. Note that a_0 is arbitrary in these definitions.

In regard to the numerical results reported in Tables 1,2, 4–6, we have carried out several calculations using different orders of the discrete-ordinates quadrature schemes (and, for the two-layer cases, with and without iteration). In this way we have found the converged results listed in the tables. And, in order to have results based on an alternative methodology, we have used the Monte Carlo method to confirm, in most cases with three or four figures of agreement (but with a minimum of two figures), the numerical results listed in our tables.

It is important now to make a few comments about the value of M, the number of Gauss quadrature points used in each of the subintervals into which the overall half-range interval [0,1] is partitioned. For a given accuracy level, we have found that the value M to be used in our solution depends weakly on the numerous physical issues (the number of layers, the amount of absorption and the scattering law in each layer, the optical thickness of each layer, and the values of the indices of refraction used to define the problem). We have concluded that $M = 20$ was sufficient to obtain results valid to at least three significant figures for all of the solved problems. On the other hand, we have used a maximum value of $M = 720$ to obtain the results (considered to be definitive) listed in our tables.

Finally, we have confirmed all five figures of the numerical results for A and B reported by Wu and Liou [14], for the case of $K = 2$ and isotropic scattering, in their Table 7.

10. Concluding remarks

In this work we have solved a very basic problem in radiative transfer, viz., the case of a multi-layer, plane-parallel medium (illuminated by azimuthally-independent impinging radiation) for which the Fresnel/Snell laws are used to define the two boundary conditions and all of the interface conditions. The radiation field in each of the K layers is described by an equation of transfer for a grey medium, and anisotropic scattering (of order L_k) is allowed in each ($k = 1, 2, \ldots, K$) of the layers.

While the solution developed and evaluated in this work is based on the ADO method, there are two important aspects of our analysis that could be used to great effect for other numerical/analytical studies. First of all, the preliminary analysis discussed in Section 3 makes it possible to formulate the boundary and interface conditions for the unknown intensities in terms of a single simple variable (the direction cosine μ). This "pre-processing" procedure allows us to avoid interpolations, as have been used by some workers, between direction cosines in the different layers. Our pre-processing procedure has also made it possible to analyze carefully the boundary and/or interface conditions to the extent that we could establish (and have confidence in) our definitions of all the important break points (see what follows) used to construct our quadrature schemes for each individual layer in the composite medium.

A second general and very important component of our work here is the study of the break points discussed in Section 4. Since the Fresnel/Snell boundary and interface functions can have discontinuous derivatives, these discontinuities can be introduced into the boundary/interface intensities. And so any discrete-ordinates method, or in fact any way of representing the intensities, should take into account these induced discontinuities. We account for these discontinuities by finding the break points (for each layer) and then using these break points in order to partition our interval of integration. For emphasis, we note that the classical spherical-harmonics method, for example, where the intensities are expressed in terms of global polynomials, cannot be expected to recover the mentioned angular discontinuities.

It is clear to us that we could not have obtained the very high-quality results reported here without the use of our "pre-processing" procedure and our "break-point" analysis.

To conclude, we believe that we have provided a general theory for an arbitrary number of different layers, each with a given albedo for single scattering, a specified optical thickness, a general scattering law, and an index of refraction. We have also implemented the established ADO solution to find five figures of accuracy for the intensities exiting the medium and seven figures of accuracy for the exiting currents. And so we believe this work could be used to great effect for important applications like those discussed, for example, by Elias and co-workers [6,7].

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