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Couette flow of a binary mixture of rigid-sphere gases described by the linearized Boltzmann equation

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Abstract

A concise and accurate solution to the problem of plane Couette flow for a binary mixture of rigid-sphere gases described by the linearized Boltzmann equation and general (specular-diffuse) Maxwell boundary conditions for each of the two species of gas particles is developed. An analytical version of the discrete-ordinates method is used to establish the velocity, heat-flow, and shear-stress profiles for both types of particles, as well as the particle-flow and heat-flow rates associated with each of the two species. Accurate numerical results are given for the case of a mixture of helium and argon confined between molybdenum and tantalum plates.

Keywords: Rarefied gas dynamics; Couette flow; Linearized Boltzmann equation; Binary mixtures; Rigid spheres

1. Introduction

The study of the internal gas flow induced by two infinite plates that are moving with respect to each other in parallel directions constitutes a classical problem in rarefied gas dynamics known as plane Couette flow. The problem can be adequately modeled by the linearized Boltzmann equation (LBE) if the plate velocities $v_{w,1}$ and $v_{w,2}$ can be considered small when compared to the reference Maxwellian speed $(2kT_0/m_\alpha)^{1/2}$ for all species present in the gas. Here, k is the Boltzmann constant, T_0 is the (unperturbed) gas temperature, and m_α is the mass of a gas particle of species α.

A list of all the works dedicated to the study of linearized plane Couette flow of a single gas would be too lengthy to be included here, and so, for reference, we mention the books by Cercignani [1–3], Williams [4] and Ferziger and Kaper [5], as well as the review papers by Sharipov and Seleznev [6] and Williams [7], where discussions of many important papers on the single-gas case and useful background material can be found. On the other hand, the literature on solutions of this problem for gas mixtures is scarce. We are aware of only five works [8–12] on linearized plane Couette flow of gas mixtures; however, all of these works are based on model equations. In this paper, we extend

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our recent work with the linearized Boltzmann equation (for rigid-sphere interactions) to the important problem of Couette flow for gas mixtures.

2. Basic formulation

Before starting our work that is specific to Couette flow between parallel plates, we review briefly our analytical formulation of the linearized Boltzmann equation for a binary mixture of rigid spheres. This formulation was started in Ref. [13], was further developed in Refs. [14] and [15], and was applied in Refs. [16–19] to various basic problems in rarefied gas dynamics. Considering what has gone before this work, we write the coupled linearized Boltzmann equations for variations only in the z direction (perpendicular to the confining plates) for the considered binary mixture of rigid spheres as

$$c\mu\frac{\partial}{\partial z}\boldsymbol{H}(z,\boldsymbol{c})+\varepsilon_0\boldsymbol{V}(c)\boldsymbol{H}(z,\boldsymbol{c})=\varepsilon_0\int \mathrm{e}^{-c'^2}\boldsymbol{\mathcal{K}}(\boldsymbol{c}':\boldsymbol{c})\boldsymbol{H}(z,\boldsymbol{c}')\,\mathrm{d}^3c', \tag{2.1}$$

where

$$\boldsymbol{H}(z,\boldsymbol{c})=\begin{bmatrix} h_1(z,\boldsymbol{c})\\ h_2(z,\boldsymbol{c})\end{bmatrix}, \tag{2.2}$$

$$\boldsymbol{V}(c)=(1/\varepsilon_0)\boldsymbol{\Sigma}(c), \tag{2.3}$$

and

$$\boldsymbol{\mathcal{K}}(\boldsymbol{c}':\boldsymbol{c})=(1/\varepsilon_0)\boldsymbol{K}(\boldsymbol{c}':\boldsymbol{c}). \tag{2.4}$$

Here, ε_0 is an arbitrary parameter that we choose as

$$\varepsilon_0=(n_1+n_2)\pi^{1/2}\left(\frac{n_1d_1+n_2d_2}{n_1+n_2}\right)^2, \tag{2.5}$$

and, as defined in Appendix A, $\boldsymbol{\Sigma}(c)$ is a 2×2 diagonal matrix with elements expressed in terms of the collision frequencies and $\boldsymbol{K}(\boldsymbol{c}':\boldsymbol{c})$ is a 2×2 matrix with the scattering kernels as elements. In addition, since Eq. (2.1) is written in terms of a dimensionless velocity variable $\boldsymbol{c}$, we note that the basic velocity distribution functions are given by

$$f_\alpha(z,\boldsymbol{v})=f_{\alpha,0}(v)\left[1+h_\alpha(z,\lambda_\alpha^{1/2}\boldsymbol{v})\right],\quad \alpha=1,2, \tag{2.6}$$

where $\lambda_\alpha=m_\alpha/(2kT_0)$, and where

$$f_{\alpha,0}(v)=n_\alpha(\lambda_\alpha/\pi)^{3/2}\mathrm{e}^{-\lambda_\alpha v^2} \tag{2.7}$$

is the Maxwellian distribution for n_α particles (of mass m_α and diameter d_α) in equilibrium at temperature T_0. It can be noted from Eq. (2.6) that, at this point, the particle distribution functions $f_\alpha(z,\boldsymbol{v})$ have been linearized about the absolute Maxwellian distributions $f_{\alpha,0}(v)$. Continuing, we note that we use spherical coordinates $\{c,\theta,\phi\}$, with $\mu=\cos\theta$, to describe the dimensionless velocity vector, so that

$$\boldsymbol{H}(z,\boldsymbol{c})\Leftrightarrow \boldsymbol{H}(z,c,\mu,\phi).$$

In our notation, $c\mu$ is the component of the (dimensionless) velocity vector in the positive z direction, and so if we let

$$c_x=c(1-\mu^2)^{1/2}\cos\phi \tag{2.8}$$

denote the component of the velocity in the x direction (parallel to the confining plates) of the flow, then we can express the velocity, the shear-stress, and the heat-flow profiles for the considered flow problem as

$$\boldsymbol{U}(z)=\frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi}\mathrm{e}^{-c^2}\boldsymbol{H}(z,\boldsymbol{c})c^3(1-\mu^2)^{1/2}\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.9}$$

$$\boldsymbol{P}(z)=\frac{2}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi}\mathrm{e}^{-c^2}\boldsymbol{H}(z,\boldsymbol{c})c^4\mu(1-\mu^2)^{1/2}\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.10}$$

and

$$\boldsymbol{Q}(z)=\frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi}\mathrm{e}^{-c^2}\boldsymbol{H}(z,\boldsymbol{c})(c^2-5/2)c^3(1-\mu^2)^{1/2}\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.11}$$

where the components of $\boldsymbol{U}(z)$, $\boldsymbol{P}(z)$, and $\boldsymbol{Q}(z)$ are, respectively, the functions $U_\alpha(z)$, $P_\alpha(z)$, and $Q_\alpha(z)$, $\alpha=1,2$, that can be used as in Appendix A of Ref. [17] to define macroscopic quantities for a binary mixture.

We now introduce the dimensionless spatial variable

$$\tau=z\varepsilon_0 \tag{2.12}$$

and rewrite Eq. (2.1) as

$$c\mu\frac{\partial}{\partial\tau}\boldsymbol{H}(\tau/\varepsilon_0,\boldsymbol{c})+\boldsymbol{V}(c)\boldsymbol{H}(\tau/\varepsilon_0,\boldsymbol{c})=\int\mathrm{e}^{-c'^2}\boldsymbol{\mathcal{K}}(\boldsymbol{c}':\boldsymbol{c})\boldsymbol{H}(\tau/\varepsilon_0,\boldsymbol{c}')\,\mathrm{d}^3c'. \tag{2.13}$$

At the walls located at $\tau=-a$ and $\tau=a$, we use a combination of specular and diffuse reflection, and so, in regard to Eq. (2.13), we write the boundary conditions as [7,12]

$$\boldsymbol{H}(-a/\varepsilon_0,c,\mu,\phi)-(\boldsymbol{I}-\boldsymbol{\alpha})\boldsymbol{H}(-a/\varepsilon_0,c,-\mu,\phi)-\boldsymbol{\alpha}\boldsymbol{\mathcal{I}}_-\{\boldsymbol{H}\}(-a/\varepsilon_0)=2c_xu_{w,1}\boldsymbol{\alpha}\boldsymbol{r} \tag{2.14a}$$

and

$$\boldsymbol{H}(a/\varepsilon_0,c,-\mu,\phi)-(\boldsymbol{I}-\boldsymbol{\beta})\boldsymbol{H}(a/\varepsilon_0,c,\mu,\phi)-\boldsymbol{\beta}\boldsymbol{\mathcal{I}}_+\{\boldsymbol{H}\}(a/\varepsilon_0)=2c_xu_{w,2}\boldsymbol{\beta}\boldsymbol{r}, \tag{2.14b}$$

for $c\in[0,\infty)$, $\mu\in(0,1]$, and $\phi\in[0,2\pi]$. Here

$$\boldsymbol{\mathcal{I}}_\mp\{\boldsymbol{H}\}(z)=\frac{2}{\pi}\int_0^\infty\int_0^1\int_0^{2\pi}\mathrm{e}^{-c'^2}\boldsymbol{H}(z,c',\mp\mu',\phi')\mu'c'^3\,\mathrm{d}\phi'\,\mathrm{d}\mu'\,\mathrm{d}c', \tag{2.15}$$

$$\boldsymbol{\alpha}=\mathrm{diag}\{\alpha_1,\alpha_2\},\quad\text{and}\quad\boldsymbol{\beta}=\mathrm{diag}\{\beta_1,\beta_2\}, \tag{2.16a,b}$$

where α_1, α_2, β_1, and β_2 are the accommodation coefficients. In addition, $u_{w,1}$ and $u_{w,2}$ are dimensionless plate velocities expressed in units of $v_0=(2kT_0/m)^{1/2}$, where

$$m=\frac{n_1m_1+n_2m_2}{n_1+n_2}, \tag{2.17}$$

and $\boldsymbol{r}$ is a vector with components $r_\alpha=(m_\alpha/m)^{1/2}$, $\alpha=1,2$.

As in Refs. [17] and [19], it is clear that an expansion of $\boldsymbol{H}(z,\boldsymbol{c})$ in a Fourier series (in the angle ϕ) requires only one term – viz., one proportional to $\cos\phi$. And so, we write

$$\boldsymbol{H}(\tau/\varepsilon_0,\boldsymbol{c})=\boldsymbol{\Psi}(\tau,c,\mu)(1-\mu^2)^{1/2}\cos\phi, \tag{2.18}$$

where $\boldsymbol{\Psi}(\tau,c,\mu)$ is the (vector-valued) function to be determined. We deduce from Eqs. (2.13) and (2.14) that $\boldsymbol{\Psi}(\tau,c,\mu)$ must satisfy

$$c\mu\frac{\partial}{\partial\tau}\boldsymbol{\Psi}(\tau,c,\mu)+\boldsymbol{V}(c)\boldsymbol{\Psi}(\tau,c,\mu)=\int_0^\infty\int_{-1}^1\mathrm{e}^{-c'^2}f(\mu',\mu)\boldsymbol{\mathcal{K}}(c',\mu':c,\mu)\boldsymbol{\Psi}(\tau,c',\mu')c'^2\,\mathrm{d}\mu'\,\mathrm{d}c', \tag{2.19}$$

for $\tau\in(-a,a)$, $c\in[0,\infty)$, and $\mu\in[-1,1]$, subject to

$$\boldsymbol{\Psi}(-a,c,\mu)-(\boldsymbol{I}-\boldsymbol{\alpha})\boldsymbol{\Psi}(-a,c,-\mu)=2cu_{w,1}\boldsymbol{\alpha}\boldsymbol{r} \tag{2.20a}$$

and

$$\boldsymbol{\Psi}(a,c,-\mu)-(\boldsymbol{I}-\boldsymbol{\beta})\boldsymbol{\Psi}(a,c,\mu)=2cu_{w,2}\boldsymbol{\beta}\boldsymbol{r}, \tag{2.20b}$$

for $c\in[0,\infty)$ and $\mu\in(0,1]$. In Eq. (2.19),

$$f(\mu',\mu)=\left(\frac{1-\mu'^2}{1-\mu^2}\right)^{1/2} \tag{2.21}$$

and

$$\mathcal{K}(c', \mu' : c, \mu)\cos\phi' = \int_0^{2\pi} \mathcal{K}(\boldsymbol{c}' : \boldsymbol{c})\cos\phi \,\mathrm{d}\phi, \tag{2.22}$$

which can be expressed as

$$\mathcal{K}(c', \mu' : c, \mu) = (1/2)\sum_{n=1}^{\infty}(2n+1)P_n^1(\mu')P_n^1(\mu)\mathcal{K}_n(c', c). \tag{2.23}$$

Here, $\mathcal{K}_n(c', c)$ is defined in Appendix A and $P_n^1(x)$ is used to denote one of the normalized associated Legendre functions. More explicitly,

$$P_l^m(\mu) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2}(1-\mu^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}\mu^m}P_l(\mu), \tag{2.24}$$

where $P_l(\mu)$ is the Legendre polynomial.

Once Eqs. (2.19) and (2.20) are solved, we can immediately determine the quantities of interest from

$$\boldsymbol{U}(\tau) = \frac{1}{\pi^{1/2}}\int_0^{\infty}\int_{-1}^{1} \mathrm{e}^{-c^2}\boldsymbol{\Psi}(\tau, c, \mu)c^3(1-\mu^2)\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.25}$$

$$\boldsymbol{P}(\tau) = \frac{2}{\pi^{1/2}}\int_0^{\infty}\int_{-1}^{1} \mathrm{e}^{-c^2}\boldsymbol{\Psi}(\tau, c, \mu)c^4(1-\mu^2)\mu\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.26}$$

and

$$\boldsymbol{Q}(\tau) = \frac{1}{\pi^{1/2}}\int_0^{\infty}\int_{-1}^{1} \mathrm{e}^{-c^2}\boldsymbol{\Psi}(\tau, c, \mu)(c^2-5/2)c^3(1-\mu^2)\,\mathrm{d}\mu\,\mathrm{d}c. \tag{2.27}$$

It should be noted that, to avoid excessive notation, we have, in writing Eqs. (2.25)–(2.27), followed the (often-used) procedure of not always introducing new labels for dependent quantities (in this case $\boldsymbol{U}$, $\boldsymbol{P}$, and $\boldsymbol{Q}$) when the independent variable is changed. In addition to the profiles defined by Eqs. (2.25)–(2.27), we intend to compute "normalized" particle-flow and heat-flow rates given by

$$\boldsymbol{U} = \frac{1}{2a}\int_{-a}^{a}\boldsymbol{U}(\tau)\,\mathrm{d}\tau \tag{2.28}$$

and

$$\boldsymbol{Q} = \frac{1}{2a}\int_{-a}^{a}\boldsymbol{Q}(\tau)\,\mathrm{d}\tau, \tag{2.29}$$

where the factor $1/(2a)$ has been included in order to be consistent with definitions used when the considered Couette-flow problem was solved [12] in terms of the McCormack kinetic model [20].

3. A solution

Our way of solving Eqs. (2.19) and (2.20) is based on an expansion of the form

$$\boldsymbol{\Psi}(\tau, c, \mu) = \sum_{k=0}^{K}\Pi_k(c)\boldsymbol{G}_k(\tau, \mu), \tag{3.1}$$

with the choice $\Pi_k(c) = P_k(2e^{-c} - 1)$, followed by a projection procedure, and an application of the analytic discrete-ordinates (ADO) method [21] for solving the resulting set of integro-differential equations and boundary conditions that are obtained for the coefficients, $\boldsymbol{G}_k(\tau, \mu)$, of the expansion. This procedure has been successfully used to solve other basic problems in rarefied gas dynamics described by the linearized Boltzmann equation for binary mixtures [16–19]. Since the details of the procedure are readily available, our discussion here can be brief.

We begin by expressing our solution of Eq. (2.19) at the N pairs of discrete ordinates $\pm\mu_i$, where $\{\mu_i\}$ are the nodes of our N-point half-range quadrature scheme, as

$$\boldsymbol{\Psi}(\tau, c, \pm\mu_i) = \boldsymbol{\Psi}_*(\tau, c, \pm\mu_i) + \boldsymbol{\Psi}_{\rm app}(\tau, c, \pm\mu_i) \tag{3.2}$$

for $i = 1, 2, \ldots, N$. We note that $\boldsymbol{\Psi}_*(\tau, c, \mu)$ is given in terms of two of the exact elementary solutions we reported in a previous work [15], i.e.

$$\boldsymbol{\Psi}_*(\tau, c, \mu) = A_1 c\boldsymbol{\Phi} + B_1\big[c\tau\boldsymbol{\Phi} - \mu\boldsymbol{B}(c)\big], \tag{3.3}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 \\ r_2/r_1 \end{bmatrix}, \tag{3.4}$$

and where $\boldsymbol{B}(c)$ is one of the generalized Chapman–Enskog (vector-valued) functions discussed in Ref. [15]. We also note that the approximate part of our solution in Eq. (3.2) is given by

$$\boldsymbol{\Psi}_{\rm app}(\tau, c, \pm\mu_i) = \boldsymbol{\Pi}(c)\sum_{j=2}^{J}\big[A_j\boldsymbol{\Phi}(\nu_j, \pm\mu_i)e^{-(a+\tau)/\nu_j} + B_j\boldsymbol{\Phi}(\nu_j, \mp\mu_i)e^{-(a-\tau)/\nu_j}\big], \tag{3.5}$$

where $J = 2N(K+1)$, the $2\times 2(K+1)$ matrix $\boldsymbol{\Pi}(c)$ is defined as

$$\boldsymbol{\Pi}(c) = \big[P_0(2e^{-c} - 1)\boldsymbol{I}\ P_1(2e^{-c} - 1)\boldsymbol{I}\cdots P_K(2e^{-c} - 1)\boldsymbol{I}\big], \tag{3.6}$$

the separation constants $\{\nu_j\}$ and the elementary solutions $\{\boldsymbol{\Phi}(\nu_j, \pm\mu_i)\}$ can be obtained as discussed in Ref. [17], and the arbitrary constants $\{A_j, B_j\}$ are to be determined from the boundary conditions applied at $\tau = \pm a$. To this end, we substitute Eq. (3.2) into discrete-ordinates versions of Eqs. (2.20), multiply the resulting equations by

$$c^2\exp\{-c^2\}\boldsymbol{\Pi}^T(c),$$

where the superscript T is used to denote the transpose operation, and integrate over c from 0 to ∞ to define a system of $2J$ linear algebraic equations for the $2J$ unspecified constants. Once this linear system is solved, we can compute the quantities of interest from

$$\boldsymbol{U}(\tau) = (1/2)(A_1 + B_1\tau)\boldsymbol{\Phi} + \sum_{j=2}^{J}\big[A_j e^{-(a+\tau)/\nu_j} + B_j e^{-(a-\tau)/\nu_j}\big]\boldsymbol{\mathcal{U}}_j, \tag{3.7a}$$

$$\boldsymbol{P}(\tau) = -(1/2)B_1\boldsymbol{\varepsilon}_p + \sum_{j=2}^{J}\big[A_j e^{-(a+\tau)/\nu_j} - B_j e^{-(a-\tau)/\nu_j}\big]\boldsymbol{\mathcal{P}}_j, \tag{3.7b}$$

and

$$\boldsymbol{Q}(\tau) = \sum_{j=2}^{J}\big[A_j e^{-(a+\tau)/\nu_j} + B_j e^{-(a-\tau)/\nu_j}\big]\boldsymbol{\mathcal{Q}}_j, \tag{3.7c}$$

where

$$\boldsymbol{\varepsilon}_p = \frac{16}{15\pi^{1/2}}\int_0^\infty e^{-c^2}\boldsymbol{B}(c)c^4\,dc, \tag{3.8}$$

$$\boldsymbol{\mathcal{U}}_j = \boldsymbol{\Pi}_1\boldsymbol{X}_j, \tag{3.9a}$$

$$\boldsymbol{\mathcal{P}}_j = 2\boldsymbol{\Pi}_2\boldsymbol{Y}_j, \tag{3.9b}$$

and

$$\mathcal{Q}_j = \big[\mathbf{\Pi}_3 - (5/2)\mathbf{\Pi}_1\big]\boldsymbol{X}_j. \tag{3.9c}$$

Here, to write Eqs. (3.9) in a compact form, we have used

$$\boldsymbol{X}_j = \frac{1}{\pi^{1/2}}\sum_{k=1}^{N} w_k(1-\mu_k^2)\big[\mathbf{\Phi}(\nu_j,\mu_k)+\mathbf{\Phi}(\nu_j,-\mu_k)\big], \tag{3.10a}$$

$$\boldsymbol{Y}_j = \frac{1}{\pi^{1/2}}\sum_{k=1}^{N} w_k\mu_k(1-\mu_k^2)\big[\mathbf{\Phi}(\nu_j,\mu_k)-\mathbf{\Phi}(\nu_j,-\mu_k)\big], \tag{3.10b}$$

where $\{w_k\}$ are the weights of our N-point half-range quadrature scheme, and

$$\mathbf{\Pi}_n = \int_0^\infty \mathrm{e}^{-c^2}\mathbf{\Pi}(c)c^{n+2}\,\mathrm{d}c. \tag{3.11}$$

Since we have the analytical expressions for the velocity and heat-flow profiles given by Eqs. (3.7a) and (3.7c), we can use those expressions in Eqs. (2.28) and (2.29) to find the normalized particle-flow and heat-flow rates, viz.

$$\boldsymbol{U} = \frac{1}{2a}\left[aA_1\mathbf{\Phi} + \sum_{j=2}^{J}\nu_j(A_j+B_j)\big(1-\mathrm{e}^{-2a/\nu_j}\big)\boldsymbol{\mathcal{U}}_j\right] \tag{3.12}$$

and

$$\boldsymbol{Q} = \frac{1}{2a}\sum_{j=2}^{J}\nu_j(A_j+B_j)\big(1-\mathrm{e}^{-2a/\nu_j}\big)\boldsymbol{\mathcal{Q}}_j. \tag{3.13}$$

As our solution is complete, we are now ready to consider a specific set of physical data so as to be able to report some numerical results.

4. Numerical results

To begin this section, we note that the computational implementation of our ADO solution is similar to that of two of our recent works [17,19], and so we report only a summary of important computational aspects of our solution here. First of all, the kernel $\mathcal{K}(c',\mu' : c,\mu)$ defined by Eq. (2.23) was truncated at $n = L$, and the Legendre components $\mathcal{K}_n(c',c)$ that are required in Eq. (2.23) were computed using a 200-point Gauss–Legendre quadrature set with the integration algorithms reported in Appendix A of Ref. [15]. Along with the order M of the Gaussian quadrature used for integration over the speed variable, as for example in Eq. (3.11), the order K of the approximate representation of Eq. (3.1), the order N of the half-range Gaussian quadrature scheme used by our ADO approximation, and the number of spline functions K_s used to compute, without post-processing [15], the generalized Chapman–Enskog vector function $\boldsymbol{B}(c)$, the kernel truncation parameter L defines the set of five approximation parameters

$$\{L, M, K, N, K_s\},$$

upon which our numerical results are based. We should also note that the integral involving $\boldsymbol{B}(c)$ in Eq. (3.8) was performed in this work as in Ref. [15], by applying a Gaussian quadrature of order four to each of the subintervals of integration defined by two consecutive knots in the applied spline representation.

In order to facilitate a comparison with the numerical results (based on the McCormack model) that are reported in Ref. [12], we elected to use here the same test case used in that work. We thus consider the case of a He-Ar mixture confined between Mo and Ta plates, the basic data for which are:

$$m_1 = 4.0026,\quad m_2 = 39.948,\quad d_2/d_1 = 1.665,\quad n_2/n_1 = 7/3,$$

$$\alpha_1 = 0.20,\quad \alpha_2 = 0.67,\quad \beta_1 = 0.46,\quad \beta_2 = 0.78,$$

$$u_{w,1} = 1.0,\quad u_{w,2} = -1.0,\quad a_M = 1.5.$$

Table 1
The velocity, heat-flow, and shear-stress profiles for the He–Ar test case with $a = \xi_M a_M$, where $a_M = 1.5$ and $\xi_M = 0.403373063\ldots$

η	$U_1(-a+2\eta a)$	$U_2(-a+2\eta a)$	$Q_1(-a+2\eta a)$	$Q_2(-a+2\eta a)$	$P_1(-a+2\eta a)$	$P_2(-a+2\eta a)$
0.0	9.0558(−2)	5.8034(−1)	−1.3383(−2)	−5.6728(−2)	3.9485(−2)	3.8599(−1)
0.1	6.8612(−2)	3.7886(−1)	−9.6947(−3)	−2.6592(−2)	6.4093(−2)	3.7545(−1)
0.2	4.4780(−2)	2.4772(−1)	−6.9586(−3)	−1.5587(−2)	7.9490(−2)	3.6885(−1)
0.3	1.9170(−2)	1.3024(−1)	−4.7035(−3)	−8.7247(−3)	8.9669(−2)	3.6449(−1)
0.4	−7.5691(−3)	1.8526(−2)	−2.7345(−3)	−3.6880(−3)	9.6174(−2)	3.6170(−1)
0.5	−3.5015(−2)	−9.0706(−2)	−9.1534(−4)	5.7745(−4)	9.9785(−2)	3.6015(−1)
0.6	−6.2874(−2)	−1.9950(−1)	8.6957(−4)	4.7760(−3)	1.0086(−1)	3.5969(−1)
0.7	−9.0932(−2)	−3.0981(−1)	2.7351(−3)	9.5798(−3)	9.9457(−2)	3.6029(−1)
0.8	−1.1906(−1)	−4.2450(−1)	4.8173(−3)	1.5920(−2)	9.5286(−2)	3.6208(−1)
0.9	−1.4735(−1)	−5.5032(−1)	7.3178(−3)	2.5745(−2)	8.7554(−2)	3.6539(−1)
1.0	−1.7927(−1)	−7.3522(−1)	1.0822(−2)	5.1331(−2)	7.3898(−2)	3.7125(−1)

Table 2
The particle-flow and heat-flow rates for the He–Ar test case with various choices of the half-distance between plates $a = \xi_M a_M$, where $\xi_M = 0.403373063\ldots$

a_M	$-U_1$	$-U_2$	$-Q_1$	$-Q_2$
0.001	1.67457(−1)	1.38507(−1)	1.08598(−4)	3.01787(−5)
0.01	1.56800(−1)	1.37346(−1)	6.04352(−4)	4.76324(−5)
0.1	1.14610(−1)	1.32332(−1)	1.99862(−3)	−8.83751(−5)
0.5	6.57181(−2)	1.15727(−1)	2.07309(−3)	7.38842(−5)
1.0	4.68382(−2)	1.00035(−1)	1.47878(−3)	2.78915(−5)
2.0	3.12969(−2)	7.84716(−2)	7.72503(−4)	−4.40149(−5)
5.0	1.64375(−2)	4.70441(−2)	2.00374(−4)	−2.18928(−5)
10.0	9.31898(−3)	2.80046(−2)	5.90387(−5)	−6.55253(−6)
20.0	5.01357(−3)	1.54455(−2)	1.61741(−5)	−1.79517(−6)
50.0	2.10423(−3)	6.58089(−3)	2.74571(−6)	−3.04748(−7)

We note that the above masses were taken from Ref. [22], the diameter ratio from Ref. [23], and the accommodation coefficients from Ref. [24]. In addition, we note that a_M is used in this work to denote the dimensionless half-distance between plates of Ref. [12] (the subscript M stands for "McCormack"). The analogous quantity in this work (a) is related to a_M by

$$a = \xi_M a_M, \tag{4.1}$$

where ξ_M is a conversion factor explicitly defined in Refs. [16] and [17]. For the present case, $\xi_M = 0.403373063\ldots.$

In Table 1, we report our converged numerical results for the velocity, heat-flow, and shear-stress profiles for the selected test case. Note that the shear-stress profiles are defined in this work with a factor 2 not present in the definition of Ref. [12]. Thus, while the velocity and heat-flow profiles reported in Table 1 can immediately be compared with those reported in Table I of Ref. [12], the shear-stress profiles must be halved for a proper comparison with the shear-stress profiles reported in Table I of Ref. [12]. In Table 2, converged numerical results for the particle-flow and heat-flow rates are given for various choices of the dimensionless half-distance between plates a and, in Table 3, converged values of the ratio p/p_{fm} are reported for the same choices of a used in Table 2. We note that the quantities involved in the definition of the ratio p/p_{fm} reported in Table 3 are the total stress

$$p = c_1 P_1(\tau) + c_2 P_2(\tau) \tag{4.2}$$

and the free-molecular total stress

$$p_{fm} = \frac{1}{\pi^{1/2}}(u_{w,1} - u_{w,2})\left[c_1 r_1 \frac{\alpha_1\beta_1}{\alpha_1+\beta_1-\alpha_1\beta_1} + c_2 r_2 \frac{\alpha_2\beta_2}{\alpha_2+\beta_2-\alpha_2\beta_2}\right], \tag{4.3}$$

where $c_\alpha = n_\alpha/(n_1+n_2)$, $\alpha = 1, 2$. The results reported in Tables 2 and 3 of this work can be directly compared with those of Tables II and III of Ref. [12].

All of the numerical results tabulated in this work are thought to be accurate to within ±1 in the last reported figure, except possibly some of the Q_2 results reported in Table 2, which may not be that accurate in the range

Table 3
The ratio p/p_{fm} for the He–Ar test case with various choices of the accommodation coefficients and the half-distance between plates $a = \xi_M a_M$, where $\xi_M = 0.403373063\ldots$

a_M	$\alpha_1 = \beta_1 = 0.4$ $\alpha_2 = \beta_2 = 0.7$	$\alpha_1 = \beta_1 = 0.6$ $\alpha_2 = \beta_2 = 0.8$	$\alpha_1 = \beta_1 = 0.8$ $\alpha_2 = \beta_2 = 0.9$	$\alpha_1 = \beta_1 = 1.0$ $\alpha_2 = \beta_2 = 1.0$
0.001	9.98968(−1)	9.98734(−1)	9.98452(−1)	9.98105(−1)
0.01	9.90220(−1)	9.88082(−1)	9.85531(−1)	9.82428(−1)
0.1	9.22896(−1)	9.08000(−1)	8.90860(−1)	8.70834(−1)
0.5	7.46495(−1)	7.07224(−1)	6.65114(−1)	6.19702(−1)
1.0	6.16913(−1)	5.67208(−1)	5.16846(−1)	4.65717(−1)
2.0	4.64123(−1)	4.10965(−1)	3.60915(−1)	3.13624(−1)
5.0	2.68717(−1)	2.26478(−1)	1.90326(−1)	1.58947(−1)
10.0	1.58065(−1)	1.29629(−1)	1.06501(−1)	8.72590(−2)
20.0	8.66805(−2)	6.98719(−2)	5.66239(−2)	4.58767(−2)
50.0	3.68094(−2)	2.93214(−2)	2.35445(−2)	1.89358(−2)

of a_M values where sign changes occur ($0.01 < a_M < 2.0$). Numerical convergence in the results of Tables 1–3 was achieved by running the developed computer code for different values of the approximation parameters in the intervals $60 \leqslant L \leqslant 115$, $100 \leqslant M \leqslant 400$, $20 \leqslant K \leqslant 35$, $60 \leqslant N \leqslant 120$, and $80 \leqslant K_s - 2 \leqslant 1280$, except in the case of the heat-flow rates for $a_M \leqslant 0.5$ reported in Table 2, as discussed next.

At this point, we should note that we have encountered a difficulty while generating some of the numerical results reported in Table 2: the convergence of the heat-flow rates as the ADO approximation is increased has been found to become progressively slower as $a \to 0$, i.e. when the free-molecular flow regime is approached. Thus, to accelerate the convergence of Q_α, $\alpha = 1, 2$, for small values of a, we have developed a post-processed version of our solution, following and generalizing, for the case of a binary mixture, the procedure reported in Ref. [25] for the single-gas case.

Our post-processed formula for $\boldsymbol{Q}$ was derived by using the solution expressed by Eqs. (3.2)–(3.5) on the right-hand side of Eq. (2.19) multiplied by the integrating factor $\exp[\boldsymbol{V}(c)\tau/(c\mu)]$, approximating the integral over μ' of the ADO part of our solution with the half-range quadrature scheme and integrating over τ the resulting equation, viz.

$$c\mu\frac{\partial}{\partial\tau}\big[\mathrm{e}^{\boldsymbol{V}(c)\tau/(c\mu)}\boldsymbol{\Psi}_{pp}(\tau,c,\mu)\big] = \mathrm{e}^{\boldsymbol{V}(c)\tau/(c\mu)}\Bigg\{\int_0^\infty\int_{-1}^1 \mathrm{e}^{-c'^2} f(\mu',\mu)\boldsymbol{\mathcal{K}}(c',\mu':c,\mu)\boldsymbol{\Psi}_*(\tau,c',\mu')c'^2\,\mathrm{d}\mu'\,\mathrm{d}c' + \int_0^\infty \mathrm{e}^{-c'^2}\sum_{k=1}^N w_k f(\mu_k,\mu)\big[\boldsymbol{\mathcal{K}}(c',\mu_k:c,\mu)\boldsymbol{\Psi}_{\mathrm{app}}(\tau,c',\mu_k) + \boldsymbol{\mathcal{K}}(c',-\mu_k:c,\mu)\boldsymbol{\Psi}_{\mathrm{app}}(\tau,c',-\mu_k)\big]c'^2\mathrm{d}c'\Bigg\}, \tag{4.4}$$

and its counterpart with μ changed to $-\mu$. We note that the subscript pp attached to $\boldsymbol{\Psi}(\tau,c,\mu)$ on the left side of Eq. (4.4) has the meaning that the solution of this equation yields a post-processed formula for $\boldsymbol{\Psi}(\tau,c,\mu)$. The resulting expressions for $\boldsymbol{\Psi}_{pp}(\tau,c,\mu)$ and $\boldsymbol{\Psi}_{pp}(\tau,c,-\mu)$, $\mu \in (0,1]$, were used in Eq. (2.27) to obtain a post-processed expression for $\boldsymbol{Q}(\tau)$. This expression was then used in Eq. (2.29) to yield our desired post-processed formula for $\boldsymbol{Q}$, viz.

$$\boldsymbol{Q}_{pp} = \frac{1}{2\pi^{1/2}a}\int_0^\infty\int_0^1 \mathrm{e}^{-c^2}\boldsymbol{\Gamma}(c,\mu)(c^2-5/2)c^3(1-\mu^2)\,\mathrm{d}\mu\,\mathrm{d}c, \tag{4.5}$$

where

$$\boldsymbol{\Gamma}(c,\mu) = \int_{-a}^a \big[\boldsymbol{\Psi}_{pp}(\tau,c,\mu) + \boldsymbol{\Psi}_{pp}(\tau,c,-\mu)\big]\,\mathrm{d}\tau \tag{4.6}$$

is given by

$$\boldsymbol{\Gamma}(c,\mu)=c\mu\boldsymbol{V}^{-1}(c)\big[\boldsymbol{I}-\mathrm{e}^{-2a\boldsymbol{V}(c)/(c\mu)}\big]\big\{\boldsymbol{E}(c,\mu)\boldsymbol{\Delta}(c,\mu)-2cA_1\boldsymbol{\Phi}\big\}$$
$$+\sum_{j=2}^{J}\nu_j(A_j+B_j)\big[\boldsymbol{F}_j(c,\mu)\boldsymbol{W}_j^{+}(c,\mu)+\boldsymbol{G}_j(c,\mu)\boldsymbol{W}_j^{-}(c,\mu)\big]. \tag{4.7}$$

Here,

$$\boldsymbol{E}(c,\mu)=\big\{\boldsymbol{I}-\big[\boldsymbol{I}-\boldsymbol{\rho}(c,\mu,\boldsymbol{\alpha})\big]\big[\boldsymbol{I}-\boldsymbol{\rho}(c,\mu,\boldsymbol{\beta})\big]\big\}^{-1}, \tag{4.8}$$

$$\boldsymbol{F}_j(c,\mu)=\boldsymbol{V}^{-1}(c)\big\{\boldsymbol{I}-\big[\nu_j\boldsymbol{V}(c)-c\mu\boldsymbol{I}\big]^{-1}\big[\nu_j\boldsymbol{V}(c)\mathrm{e}^{-2a/\nu_j}-c\mu\mathrm{e}^{-2a\boldsymbol{V}(c)/(c\mu)}\big]\big\}, \tag{4.9}$$

$$\boldsymbol{G}_j(c,\mu)=\boldsymbol{V}^{-1}(c)\big\{\big[\nu_j\boldsymbol{V}(c)+c\mu\boldsymbol{I}\big]^{-1}\big[\nu_j\boldsymbol{V}(c)+c\mu\mathrm{e}^{-2a/\nu_j}\mathrm{e}^{-2a\boldsymbol{V}(c)/(c\mu)}\big]-\boldsymbol{I}\mathrm{e}^{-2a/\nu_j}\big\}, \tag{4.10}$$

and

$$\boldsymbol{W}_j^{\pm}(c,\mu)=(1/2)\sum_{l=1}^{L}(2l+1)(\pm1)^{l-1}\boldsymbol{J}_l(c)\boldsymbol{Z}_{j,l}(\mu), \tag{4.11}$$

where the definitions

$$\boldsymbol{\rho}(c,\mu,\boldsymbol{x})=\boldsymbol{I}+(\boldsymbol{I}-\boldsymbol{x})\mathrm{e}^{-2a\boldsymbol{V}(c)/(c\mu)}, \tag{4.12}$$

$$\boldsymbol{J}_l(c)=\int_0^\infty \mathrm{e}^{-c'^2}\boldsymbol{\mathcal{K}}_l(c',c)\boldsymbol{\Pi}(c')c'^2\,\mathrm{d}c', \tag{4.13}$$

and

$$\boldsymbol{Z}_{j,l}(\mu)=(1-\mu^2)^{-1/2}P_l^1(\mu)\sum_{k=1}^{N}w_k(1-\mu_k^2)^{1/2}P_l^1(\mu_k)\big[\boldsymbol{\Phi}(\nu_j,\mu_k)+(-1)^{l-1}\boldsymbol{\Phi}(\nu_j,-\mu_k)\big] \tag{4.14}$$

have been used. Finally, the term $\boldsymbol{\Delta}(c,\mu)$ in Eq. (4.7) can be written as

$$\boldsymbol{\Delta}(c,\mu)=\boldsymbol{\Delta}_{bcs}(c,\mu)+\boldsymbol{\Delta}_{*}(c,\mu)+\boldsymbol{\Delta}_{\mathrm{app}}(c,\mu), \tag{4.15}$$

where the terms on the right side come, respectively, from the boundary conditions, the exact part of the solution, and the approximate part of the solution. These three terms are explicitly given by

$$\boldsymbol{\Delta}_{bcs}(c,\mu)=2c\big[u_{w,1}\boldsymbol{\alpha}\boldsymbol{\rho}(c,\mu,\boldsymbol{\beta})+u_{w,2}\boldsymbol{\beta}\boldsymbol{\rho}(c,\mu,\boldsymbol{\alpha})\big]\boldsymbol{r}, \tag{4.16a}$$

$$\boldsymbol{\Delta}_{*}(c,\mu)=cA_1\big[(\boldsymbol{I}-\boldsymbol{\beta})\boldsymbol{\rho}(c,\mu,\boldsymbol{\alpha})+(\boldsymbol{I}-\boldsymbol{\alpha})\boldsymbol{\rho}(c,\mu,\boldsymbol{\beta})\big]\big[\boldsymbol{I}-\mathrm{e}^{-2a\boldsymbol{V}(c)/(c\mu)}\big]\boldsymbol{\Phi}$$
$$+B_1(\boldsymbol{\alpha}-\boldsymbol{\beta})\big\{ac\big[\boldsymbol{I}+\mathrm{e}^{-2a\boldsymbol{V}(c)/(c\mu)}\big]\boldsymbol{\Phi}-\mu\big[\boldsymbol{I}-\mathrm{e}^{-2a\boldsymbol{V}(c)/(c\mu)}\big]\boldsymbol{B}(c)\big\}, \tag{4.16b}$$

and

$$\boldsymbol{\Delta}_{\mathrm{app}}(c,\mu)=\sum_{j=2}^{J}\nu_j\big\{\big[(\boldsymbol{I}-\boldsymbol{\beta})\boldsymbol{\rho}(c,\mu,\boldsymbol{\alpha})A_j+(\boldsymbol{I}-\boldsymbol{\alpha})\boldsymbol{\rho}(c,\mu,\boldsymbol{\beta})B_j\big]\boldsymbol{C}_j(c,\mu)\boldsymbol{W}_j^{+}(c,\mu)$$
$$+\big[(\boldsymbol{I}-\boldsymbol{\alpha})\boldsymbol{\rho}(c,\mu,\boldsymbol{\beta})A_j+(\boldsymbol{I}-\boldsymbol{\beta})\boldsymbol{\rho}(c,\mu,\boldsymbol{\alpha})B_j\big]\boldsymbol{S}_j(c,\mu)\boldsymbol{W}_j^{-}(c,\mu)\big\}, \tag{4.16c}$$

where

$$\boldsymbol{C}_j(c,\mu)=\big[\nu_j\boldsymbol{V}(c)-c\mu\boldsymbol{I}\big]^{-1}\big[\mathrm{e}^{-2a/\nu_j}\boldsymbol{I}-\mathrm{e}^{-2a\boldsymbol{V}(c)/(c\mu)}\big] \tag{4.17a}$$

and

$$\boldsymbol{S}_j(c,\mu)=\big[\nu_j\boldsymbol{V}(c)+c\mu\boldsymbol{I}\big]^{-1}\big[\boldsymbol{I}-\mathrm{e}^{-2a/\nu_j}\mathrm{e}^{-2a\boldsymbol{V}(c)/(c\mu)}\big]. \tag{4.17b}$$

We have used the post-processed formula expressed by Eqs. (4.5)–(4.17) to compute the heat-flow rates reported in Table 2 for $a_M \leqslant 0.5$. For this purpose, all approximation parameters were varied as mentioned before, except L, the upper limit of which was extended to 285. The μ-integration in Eq. (4.5) was performed by a shifted Gauss–Legendre quadrature of order 380.

5. Concluding remarks

We have reported in this work what we believe to be a concise and accurate solution of the plane Couette-flow problem, as described by the (vector) linearized Boltzmann equation for a binary mixture of rigid-sphere gases. In addition to reporting, for a specific set of data, the velocity, heat-flow, and shear-stress profiles for a fixed value of the channel thickness, we have listed the particle-flow and the heat-flow rates for the same data and channel widths that vary from $a_M = 0.001$ to $a_M = 50.0$.

In comparing the results of our previous work [12] based on the McCormack kinetic model [20] with our current results (both relevant to the case of rigid-sphere interactions and based on the same test case defined in terms of a He–Ar mixture confined between Mo and Ta plates), we have seen that the results of the McCormack model can be considered as reasonable (1–3 figures of accuracy) for the velocity profile for the heavier species in the mixture (argon), the shear-stress profiles for both types of particles, and the ratio p/p_{fm}. However, the results of the McCormack model for the velocity profile for the lighter species in the mixture (helium) and especially the heat-flow profiles for both types of particles display large errors (well above 100% for some entries). With regard to the heat-flow rates which, due to the asymmetry of the problem (plates of different materials), are small but not zero, we can see that the results from the McCormack model compare very poorly with our results. We note that the fact that heat-flow quantities are the most difficult to reproduce by approaches based on model equations for plane Couette flow has also been observed in the single-gas case (see, for example, Refs. [26] and [27]).

In addition to the comparisons with numerical results from the McCormack kinetic model, we have also performed comparisons with the single-gas LBE results of Ref. [27], using three different ways of achieving the single-gas limit in our formulation:

$$\text{(i) } c_1 = 0, \quad \text{(ii) } c_2 = 0, \quad \text{or} \quad \text{(iii) } m_1 = m_2,\ d_1 = d_2,\ \alpha_1 = \alpha_2, \quad \text{and} \quad \beta_1 = \beta_2.$$

We note that to convert our results to the same spatial units used in Ref. [27] we made use of the factor

$$\xi_{S,p} = 0.449027806\ldots,$$

which (for the single-species case) is the ratio between our dimensionless spatial variable, as defined by Eqs. (2.5) and (2.12), and that of Ref. [27]. Doing this, we found good but not perfect agreement with the five-figure results for the "half-channel" particle-flow and heat-flow rates, the five-figure results for the velocity and heat-flow profiles, and the six-figure results for a component of the reduced pressure tensor P_{xy} (see definition in Ref. [27]) that are tabulated in Ref. [27]. While we found at least five-figure agreement for P_{xy}, we did find some cases where the half-channel flow rates and the profiles of Ref. [27] are good only to three figures. We have confirmed that the loss of accuracy in Tables 6–9 of Ref. [27] was due to using $L = 8$ in those computations. To make available our current results (based on $L = 90$), we list in Tables 4–6 improved versions of Tables 6–9 of Ref. [27]. To be clear, we note that in Ref. [27] the mean-free path was defined in terms of viscosity, and so the conversion factor $\xi_{S,p}$ was used. In addition, to be consistent with Tables 7 and 8 of Ref. [27], we have made use of the slightly different normalization factor $1/(2a^2)$ to generate the results in our Table 5, instead of the normalization factor $1/(2a)$ used for the flow rates $\boldsymbol{U}$ and $\boldsymbol{Q}$ defined

Table 4
Single-species gas: P_{xy}

$2a$	$\alpha = 0.1$	$\alpha = 1.0$
1.0(−7)	2.96942(−2)	5.64190(−1)
1.0(−3)	2.96927(−2)	5.63647(−1)
1.0(−1)	2.95534(−2)	5.20872(−1)
1.0	2.85927(−2)	3.40502(−1)
1.0(1)	2.26781(−2)	8.35098(−2)
1.0(3)	9.67029(−4)	9.98029(−4)
1.0(7)	9.99997(−8)	1.00000(−7)

Table 5
Single-species gas: the half-channel particle-flow and heat-flow rates

	$\alpha = 0.1$		$\alpha = 1.0$	
$2a$	$-U_h$	Q_h	$-U_h$	Q_h
0.10	5.3171(−2)	9.3507(−3)	7.2916(−1)	1.1882(−1)
1.00	2.3114(−2)	3.2990(−3)	2.2737(−1)	2.2450(−2)
10.0	1.1584(−2)	1.7731(−4)	4.2192(−2)	3.0697(−4)

Table 6
Single-species gas: velocity and heat-flow profiles for the case $2a = 1$

	$\alpha = 0.1$		$\alpha = 1.0$	
τ/a	$-U(\tau)$	$Q(\tau)$	$-U(\tau)$	$Q(\tau)$
0.0	0.0	0.0	0.0	0.0
0.1	4.2247(−3)	5.7606(−4)	4.3188(−2)	4.0654(−3)
0.2	8.4806(−3)	1.1590(−3)	8.6559(−2)	8.1678(−3)
0.3	1.2802(−2)	1.7564(−3)	1.3031(−1)	1.2347(−2)
0.4	1.7229(−2)	2.3767(−3)	1.7469(−1)	1.6647(−2)
0.5	2.1814(−2)	3.0311(−3)	2.1998(−1)	2.1125(−2)
0.6	2.6631(−2)	3.7347(−3)	2.6662(−1)	2.5854(−2)
0.7	3.1798(−2)	4.5104(−3)	3.1525(−1)	3.0945(−2)
0.8	3.7524(−2)	5.3993(−3)	3.6704(−1)	3.6586(−2)
0.9	4.4291(−2)	6.4940(−3)	4.2461(−1)	4.3189(−2)
1.0	5.4765(−2)	8.3159(−3)	5.0205(−1)	5.2951(−2)

by Eqs. (2.28) and (2.29). In conclusion, we note that we have computed the half-channel particle-flow rate U_h and the half-channel heat-flow rate Q_h reported in Table 5 of this work from

$$U_h = \frac{\xi_{S,p}}{2a^2} \int_0^a U_\alpha(\tau)\,\mathrm{d}\tau \tag{5.1}$$

and

$$Q_h = \frac{\xi_{S,p}}{2a^2} \int_0^a Q_\alpha(\tau)\,\mathrm{d}\tau, \tag{5.2}$$

where the appropriate components ($\alpha = 1$ or 2) were used, depending on the way the single-gas case was approached in our calculation.

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Appendix A. Basic definitions

In this appendix, we follow our previous work reported in Refs. [14] and [15] and provide some basic definitions that are needed to complete our formulation of Section 2.

First of all, the 2×2 diagonal matrix $\mathbf{\Sigma}(c)$ that appears on the right-hand side of Eq. (2.3) is defined as [14,15]

$$\mathbf{\Sigma}(c) = \begin{bmatrix} \varpi_1(c) & 0 \\ 0 & \varpi_2(c) \end{bmatrix}, \tag{A.1}$$

with

$$\varpi_\alpha(c) = \varpi_\alpha^{(1)}(c) + \varpi_\alpha^{(2)}(c) \tag{A.2}$$

and

$$\varpi_\alpha^{(\beta)}(c) = 4\pi^{1/2} n_\beta \sigma_{\alpha,\beta} a_{\beta,\alpha} \nu(a_{\alpha,\beta} c). \tag{A.3}$$

Here, n_β denotes the equilibrium particle density of species β,

$$\nu(c) = \frac{2c^2+1}{c} \int_0^c \mathrm{e}^{-x^2}\,\mathrm{d}x + \mathrm{e}^{-c^2}, \tag{A.4}$$

and

$$a_{\alpha,\beta} = (m_\beta/m_\alpha)^{1/2}, \quad \alpha,\beta = 1,2, \tag{A.5}$$

where m_1 and m_2 are the masses of the two types of gas particles. In addition, we use $\sigma_{\alpha,\beta}$ to denote the differential-scattering cross section, which for the case of rigid-sphere scattering that is isotropic in the center-of-mass system, we can write as [28]

$$\sigma_{\alpha,\beta} = \frac{1}{4}\left(\frac{d_\alpha + d_\beta}{2}\right)^2, \tag{A.6}$$

where d_1 and d_2 are the atomic diameters of the two types of gas particles.

Continuing to follow Refs. [14] and [15], we write the 2×2 matrix $\boldsymbol{K}(\boldsymbol{c}' : \boldsymbol{c})$ that appears on the right-hand side of Eq. (2.4) as

$$\boldsymbol{K}(\boldsymbol{c}' : \boldsymbol{c}) = \begin{bmatrix} K_{1,1}(\boldsymbol{c}' : \boldsymbol{c}) & K_{1,2}(\boldsymbol{c}' : \boldsymbol{c}) \\ K_{2,1}(\boldsymbol{c}' : \boldsymbol{c}) & K_{2,2}(\boldsymbol{c}' : \boldsymbol{c}) \end{bmatrix}, \tag{A.7}$$

where

$$K_{1,1}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_1\sigma_{1,1}\pi^{1/2}\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) + n_2\sigma_{1,2}\pi^{1/2}\mathcal{F}_{1,2}(\boldsymbol{c}' : \boldsymbol{c}), \tag{A.8a}$$

$$K_{1,2}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_2\sigma_{1,2}\pi^{1/2}\mathcal{G}_{1,2}(\boldsymbol{c}' : \boldsymbol{c}), \tag{A.8b}$$

$$K_{2,1}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_1\sigma_{2,1}\pi^{1/2}\mathcal{G}_{2,1}(\boldsymbol{c}' : \boldsymbol{c}), \tag{A.8c}$$

and

$$K_{2,2}(\boldsymbol{c}' : \boldsymbol{c}) = 4n_2\sigma_{2,2}\pi^{1/2}\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) + n_1\sigma_{2,1}\pi^{1/2}\mathcal{F}_{2,1}(\boldsymbol{c}' : \boldsymbol{c}). \tag{A.8d}$$

Here,

$$\mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{\pi}\left(\frac{2}{|\boldsymbol{c}'-\boldsymbol{c}|}\exp\left\{\frac{|\boldsymbol{c}'\times\boldsymbol{c}|^2}{|\boldsymbol{c}'-\boldsymbol{c}|^2}\right\} - |\boldsymbol{c}'-\boldsymbol{c}|\right) \tag{A.9}$$

is the basic single-gas kernel used by Pekeris [29]. In addition,

$$\mathcal{F}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c}) = \mathcal{F}(a_{\alpha,\beta}; \boldsymbol{c}' : \boldsymbol{c}) \tag{A.10}$$

and

$$\mathcal{G}_{\alpha,\beta}(\boldsymbol{c}' : \boldsymbol{c}) = \mathcal{G}(a_{\alpha,\beta}; \boldsymbol{c}' : \boldsymbol{c}), \tag{A.11}$$

where

$$\mathcal{F}(a; \boldsymbol{c}' : \boldsymbol{c}) = \frac{(a^2+1)^2}{a^3\pi|\boldsymbol{c}'-\boldsymbol{c}|}\exp\left\{a^2\frac{|\boldsymbol{c}'\times\boldsymbol{c}|^2}{|\boldsymbol{c}'-\boldsymbol{c}|^2} - \frac{(1-a^2)^2(c'^2+c^2)}{4a^2} - \frac{(a^4-1)\boldsymbol{c}'\cdot\boldsymbol{c}}{2a^2}\right\} \tag{A.12}$$

and

$$\mathcal{G}(a; \boldsymbol{c}' : \boldsymbol{c}) = \frac{1}{a\pi}|\boldsymbol{c}' - a\boldsymbol{c}|\big[J(a; \boldsymbol{c}' : \boldsymbol{c}) - 1\big], \tag{A.13}$$

with

$$J(a; \boldsymbol{c}' : \boldsymbol{c}) = \frac{(a+1/a)^2}{2\Delta(a; \boldsymbol{c}' : \boldsymbol{c})}\exp\left\{\frac{-2C(a; \boldsymbol{c}' : \boldsymbol{c})}{(a-1/a)^2}\right\}\sinh\left\{\frac{2\Delta(a; \boldsymbol{c}' : \boldsymbol{c})}{(a-1/a)^2}\right\}, \quad a \neq 1, \tag{A.14a}$$

or

$$J(a;\boldsymbol{c}':\boldsymbol{c}) = \frac{1}{|\boldsymbol{c}'-\boldsymbol{c}|^2}\exp\left\{\frac{|\boldsymbol{c}'\times\boldsymbol{c}|^2}{|\boldsymbol{c}'-\boldsymbol{c}|^2}\right\}, \quad a=1. \tag{A.14b}$$

We note that, to write Eq. (A.14a), we have used the definitions [14,15]

$$\Delta(a;\boldsymbol{c}':\boldsymbol{c}) = \left\{C^2(a;\boldsymbol{c}':\boldsymbol{c}) + (a-1/a)^2|\boldsymbol{c}'\times\boldsymbol{c}|^2\right\}^{1/2} \tag{A.15}$$

and

$$C(a;\boldsymbol{c}':\boldsymbol{c}) = c'^2 + c^2 - (a+1/a)\boldsymbol{c}'\cdot\boldsymbol{c}. \tag{A.16}$$

Finally, the 2×2 coefficient matrices $\{\boldsymbol{\mathcal{K}}_n(c',c)\}$ in the expansion given by Eq. (2.23) are defined as [15]

$$\boldsymbol{\mathcal{K}}_n(c',c) = \begin{bmatrix} \mathcal{K}_n^{(1,1)}(c',c) & \mathcal{K}_n^{(1,2)}(c',c) \\ \mathcal{K}_n^{(2,1)}(c',c) & \mathcal{K}_n^{(2,2)}(c',c) \end{bmatrix}, \tag{A.17}$$

with

$$\mathcal{K}_n^{(1,1)}(c',c) = p_1\mathcal{P}^{(n)}(c',c) + (g_2/4)\mathcal{F}^{(n)}(a_{1,2};c',c), \tag{A.18a}$$

$$\mathcal{K}_n^{(1,2)}(c',c) = g_2\mathcal{G}^{(n)}(a_{1,2};c',c), \tag{A.18b}$$

$$\mathcal{K}_n^{(2,1)}(c',c) = g_1\mathcal{G}^{(n)}(a_{2,1};c',c), \tag{A.18c}$$

and

$$\mathcal{K}_n^{(2,2)}(c',c) = p_2\mathcal{P}^{(n)}(c',c) + (g_1/4)\mathcal{F}^{(n)}(a_{2,1};c',c). \tag{A.18d}$$

Here, the Legendre moments $\mathcal{P}^{(n)}(c',c)$, $\mathcal{F}^{(n)}(a;c',c)$, and $\mathcal{G}^{(n)}(a;c',c)$ can be computed as discussed in Appendix A of Ref. [15],

$$p_\alpha = c_\alpha\left(\frac{nd_\alpha}{n_1d_1+n_2d_2}\right)^2, \quad \alpha=1,2, \tag{A.19a}$$

and

$$g_\alpha = c_\alpha\left(\frac{nd_{\rm avg}}{n_1d_1+n_2d_2}\right)^2, \quad \alpha=1,2, \tag{A.19b}$$

where

$$c_\alpha = n_\alpha/n, \quad n = n_1+n_2, \quad \text{and} \quad d_{\rm avg} = (d_1+d_2)/2. \tag{A.20a,b,c}$$

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