Journal of Quantitative Spectroscopy & Radiative Transfer 109 (2008) 2151–2170

Journal of Quantitative Spectroscopy & Radiative Transfer

www.elsevier.com/locate/jqsrt

Radiative transfer in a multi-layer medium subject to Fresnel boundary and interface conditions and uniform illumination by obliquely incident parallel rays

R.D.M. Garcia[a,*], C.E. Siewert[b], A.M. Yacout[c]

[a] *Instituto de Estudos Avançados, Rodovia dos Tamoios km 5.5, São José dos Campos, SP 12228-001, Brazil*
[b] *Mathematics Department, North Carolina State University, Raleigh, NC 27695-8205, USA*
[c] *Applied Research Associates, Inc., PO Box 40128, Tyndall AFB, FL 32403, USA*

Received 10 February 2008; received in revised form 26 March 2008; accepted 27 March 2008

Abstract

The ADO (analytical discrete ordinates) method, a pre-processing procedure, and the break-point analysis developed for azimuthally symmetric problems in a previous work are generalized and used to solve a radiative-transfer problem defined by a finite, plane–parallel, multi-layer medium subject to Fresnel boundary and interface conditions and uniform illumination in the form of obliquely incident parallel rays. Illumination is modeled by Dirac distributions in each of the two angles (polar and azimuthal) that define the direction of propagation of the incident rays. Accurate numerical results are tabulated for two sets of test problems.

1. Introduction

Radiative transfer in a scattering and absorbing plane–parallel medium subject to uniform illumination in the form of parallel rays impinging on one of its surfaces is basic to several areas of study [1–5]. The simplest case where it can be assumed that there is no variation in the index of refraction has been extensively studied by many authors, using various methods [6]. A few authors have been able to include the effects of changes in the index of refraction in their analyses of radiative-transfer problems involving a single Fresnel interface [7–9]. Other authors have done the same for problems defined in terms of two or more Fresnel boundaries/interfaces [10–18]. However, among the works that take into account Fresnel conditions, only Refs. [8,9,13,14] are concerned with the type of boundary condition that is of interest to us in this work, i.e. uniform illumination by obliquely incident parallel rays.

In a recent work [18], we reported a general solution for the case of a medium composed of an arbitrary number of dissimilar layers illuminated by azimuthally symmetric incident radiation. Here, we make use of our basic analysis and extend it to the case of uniform illumination by obliquely incident parallel rays.

*Corresponding author.
E-mail address: rdmgarcia@uol.com.br (R.D.M. Garcia).

0022-4073/$ - see front matter
doi:10.1016/j.jqsrt.2008.03.012

Our solution can be used for non-conservative (absorbing) layers as well as for conservative (non-absorbing) layers and is based on the analytical discrete ordinates (ADO) method [19], a pre-processing procedure [18] that allows us to write the Fresnel interface conditions in a way that avoids shifts in the polar angle as the radiation crosses the interface between two layers with different indices of refraction, and a break-point analysis [18] that results in an efficient interval partition scheme for overcoming a difficulty caused by the fact that the derivative of the radiation intensity (with respect to the polar angle) can be discontinuous at the boundaries and interfaces of a medium subject to Fresnel conditions.

In addition to our general development, we should make it clear that our goal in this work was similar to that of our previous work [18], namely to generate results of the highest possible accuracy (but obtained in a reasonable computer time) with two purposes: (i) to test the implementation of our analysis as thoroughly as possible and (ii) to provide reference results that could be used for assessing the accuracy of other implementations and methods.

2. Mathematical formulation

To define our notation, we consider that the medium is composed of K distinct layers and that in each layer the radiation field is described by the monochromatic or gray equation of transfer, which we write, for layer k, as [1]

$$\mu\frac{\partial}{\partial\tau}I_k(\tau,\mu,\phi)+I_k(\tau,\mu,\phi)=\frac{\varpi_k}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}p_k(\mathbf{\Omega}'\cdot\mathbf{\Omega})I_k(\tau,\mu',\phi')\,\mathrm{d}\phi'\,\mathrm{d}\mu', \tag{2.1}$$

for $\tau\in(a_{k-1},a_k)$, $\mu\in[-1,1]$, and $\phi\in[0,2\pi]$. Note that Eq. (2.1) applies to $k=1,2,\ldots,K$, a_0 defines the location of the surface of the first layer, $a_1,a_2,\ldots,a_{K-1}$ define the locations of the interfaces between layers, and a_K is the location of the surface of the last layer. In addition, τ is the optical variable defined for each layer as in Ref. [18], $\varpi_k\in[0,1]$ is the single-scattering albedo (for layer k) and $p_k(\mathbf{\Omega}'\cdot\mathbf{\Omega})$ is the phase function (for layer k) that we assume depends only on the scattering angle Θ defined by

$$\cos\Theta=\mathbf{\Omega}'\cdot\mathbf{\Omega}, \tag{2.2}$$

where $\mathbf{\Omega}'$ and $\mathbf{\Omega}$ are unit vectors that define the directions of propagation (of the radiation) before and after a scattering event. Note that we use the polar and azimuthal angles, $\theta=\arccos\mu$ and ϕ, to define the direction $\mathbf{\Omega}$, whereas $\theta'=\arccos\mu'$ and ϕ' define the direction $\mathbf{\Omega}'$.

We consider phase functions that can be represented by truncated Legendre-polynomial expansions of the form

$$p_k(\mathbf{\Omega}'\cdot\mathbf{\Omega})=\sum_{l=0}^{L_k}\beta_{k,l}P_l(\cos\Theta), \tag{2.3}$$

where $\{\beta_{k,l}\}$ are the expansion coefficients for layer k, so that we can use the addition theorem for the Legendre polynomials to write

$$p_k(\mathbf{\Omega}'\cdot\mathbf{\Omega})=\sum_{m=0}^{L_k}(2-\delta_{0,m})\sum_{l=m}^{L_k}\beta_{k,l}P_l^m(\mu')P_l^m(\mu)\cos[m(\phi'-\phi)], \tag{2.4}$$

where $\delta_{i,j}$ denotes the Kronecker delta and

$$P_l^m(\mu)=\left[\frac{(l-m)!}{(l+m)!}\right]^{1/2}(1-\mu^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}\mu^m}P_l(\mu) \tag{2.5}$$

is used to denote the *normalized* Legendre function.

In regard to the boundary (surface) conditions, we assume there is incident on the surface located at $\tau=a_0$, from an external medium characterized by an index of refraction n_0, a known distribution of radiation described by

$$\psi_0(\mu,\phi)=\delta(\mu-\mu_0)\delta(\phi-\phi_0), \tag{2.6}$$

for $\mu, \mu_0 \in (0, 1]$ and $\phi, \phi_0 \in [0, 2\pi]$. Here, θ_0, with $\mu_0 = \cos\theta_0$, and ϕ_0 are the polar and azimuthal angles that define the direction of propagation of the incoming parallel rays. In addition, we consider that there is no radiation incident on the surface located at $\tau = a_K$ from an external medium characterized by an index of refraction n_{K+1}. And so, using the Fresnel and Snell laws and the definitions [18]

$$X(n,\mu) = \begin{cases} G(n,\mu), & n \leqslant 1, \\ R(n,\mu), & n \geqslant 1, \end{cases} \tag{2.7}$$

and

$$Y(n,\mu) = n^2[1 - X(n,\mu)], \tag{2.8}$$

where

$$G(n,\mu) = \frac{1}{2}\left\{\left[\frac{\mu - nf(n,\mu)}{\mu + nf(n,\mu)}\right]^2 + \left[\frac{n\mu - f(n,\mu)}{n\mu + f(n,\mu)}\right]^2\right\} \tag{2.9}$$

and

$$R(n,\mu) = 1 + [G(n,\mu) - 1]H[\mu - \mu_c(n)], \tag{2.10}$$

with $H(x)$ denoting the Heaviside step function,

$$\mu_c(n) = (1 - 1/n^2)^{1/2}, \tag{2.11}$$

and

$$f(n,\mu) = [1 - n^2(1 - \mu^2)]^{1/2}, \tag{2.12}$$

we can express these conditions as

$$I_1(a_0,\mu,\phi) = X(n_{1,0},\mu)I_1(a_0,-\mu,\phi) + Y(n_{1,0},\mu)\delta[f(n_{1,0},\mu) - \mu_0]\delta(\phi - \phi_0) \tag{2.13a}$$

and

$$I_K(a_K,-\mu,\phi) = X(n_{K,K+1},\mu)I_K(a_K,\mu,\phi), \tag{2.13b}$$

for $\mu \in (0, 1]$ and $\phi \in [0, 2\pi]$. In our notation, the X and Y functions represent the Fresnel reflection and transmission coefficients and

$$n_{\alpha,\beta} = n_\alpha / n_\beta, \tag{2.14}$$

where n_α and n_β denote the indices of refraction for layers α and β, respectively. We note that by taking $n_{K,K+1} = 1$ we can simulate a black surface at $\tau = a_K$, a case of possible interest for ocean studies. We also note that the driving term in Eq. (2.13a) is zero if

$$\mu_0 \leqslant (1 - n_{1,0}^2)^{1/2} \quad \text{and} \quad n_{1,0} < 1. \tag{2.15}$$

And so, if $n_{1,0} < 1$, there is no problem to solve unless

$$\mu_0 > (1 - n_{1,0}^2)^{1/2}. \tag{2.16}$$

At each of the interfaces, the conditions are similar to Eqs. (2.13), except that intensities with shifts in the polar angle are required, and the known driving term is not present. We have [18]

$$I_k(a_k,-\mu,\phi) = X(n_{k,k+1},\mu)I_k(a_k,\mu,\phi) + Y(n_{k,k+1},\mu)I_{k+1}[a_k,-f(n_{k,k+1},\mu),\phi] \tag{2.17a}$$

and

$$I_{k+1}(a_k,\mu,\phi) = X(n_{k+1,k},\mu)I_{k+1}(a_k,-\mu,\phi) + Y(n_{k+1,k},\mu)I_k[a_k,f(n_{k+1,k},\mu),\phi], \tag{2.17b}$$

for $\mu \in (0, 1]$, $\phi \in [0, 2\pi]$, and $k = 1, 2, \ldots, K - 1$.

We can easily generalize the pre-processing procedure reported in Ref. [18] and express all of the boundary/interface conditions as

$$I_k(a_{k-1},\mu,\phi) - Z_k^-(\mu)I_k(a_{k-1},-\mu,\phi) = V_k^-(\mu,\phi) + W_k^-(\mu,\phi) \tag{2.18a}$$

and

$$I_k(a_k, -\mu, \phi) - Z_k^+(\mu) I_k(a_k, \mu, \phi) = W_k^+(\mu, \phi), \tag{2.18b}$$

for $\mu \in (0, 1]$, $\phi \in [0, 2\pi]$, and $k = 1, 2, \ldots, K$. Here the Z functions are defined by the following recursive schemes:

$$Z_1^-(\mu) = X(n_{1,0}, \mu), \tag{2.19a}$$

$$Z_K^+(\mu) = X(n_{K,K+1}, \mu), \tag{2.19b}$$

$$Z_k^-(\mu) = X(n_{k,k-1}, \mu) + Y(n_{k,k-1}, \mu) \Lambda_k^-[f(n_{k,k-1}, \mu)], \quad k = 2, 3, \ldots, K, \tag{2.20a}$$

and

$$Z_k^+(\mu) = X(n_{k,k+1}, \mu) + Y(n_{k,k+1}, \mu) \Lambda_k^+[f(n_{k,k+1}, \mu)], \quad k = K-1,\ \ K-2, \ldots, 1, \tag{2.20b}$$

where

$$\Lambda_k^-(\xi) = \Xi_k^-(\xi) Z_{k-1}^-(\xi) Y(n_{k-1,k}, \xi) \mathrm{e}^{-2\Delta_{k-1}/\xi} \tag{2.21a}$$

and

$$\Lambda_k^+(\xi) = \Xi_k^+(\xi) Z_{k+1}^+(\xi) Y(n_{k+1,k}, \xi) \mathrm{e}^{-2\Delta_{k+1}/\xi}, \tag{2.21b}$$

with $\Delta_k = a_k - a_{k-1}$,

$$\Xi_k^-(\xi) = [1 - Z_{k-1}^-(\xi) X(n_{k-1,k}, \xi) \mathrm{e}^{-2\Delta_{k-1}/\xi}]^{-1}, \tag{2.22a}$$

and

$$\Xi_k^+(\xi) = [1 - Z_{k+1}^+(\xi) X(n_{k+1,k}, \xi) \mathrm{e}^{-2\Delta_{k+1}/\xi}]^{-1}. \tag{2.22b}$$

For the considered case where the radiation incident on the surface is given by Eq. (2.6), we find that the V functions are given by

$$V_k^-(\mu, \phi) = F_k \delta(\mu - \mu_k) \delta(\phi - \phi_0), \tag{2.23}$$

where

$$\mu_k = f(n_{0,k}, \mu_0) \tag{2.24}$$

and

$$F_k = \left[\left(\frac{\mu_{k-1}}{n_{k,k-1}^2 \mu_k} \right) Y(n_{k,k-1}, \mu_k) \Xi_k^-(\mu_{k-1}) \mathrm{e}^{-\Delta_{k-1}/\mu_{k-1}} \right] F_{k-1}, \tag{2.25}$$

for $k = 2, 3, \ldots, K$, with

$$F_1 = \frac{\mu_0}{n_{1,0}^2 \mu_1} Y(n_{1,0}, \mu_1). \tag{2.26}$$

Note that there are two ways that $V_k^-(\mu, \phi)$ can be zero: the term $\delta(\mu - \mu_k)$ in Eq. (2.23) can be zero or F_k can be zero. We find, first of all, that

$$\delta(\mu - \mu_k) = 0, \quad \mu \in (0, 1], \tag{2.27}$$

if

$$\mu_0 \leqslant (1 - n_{k,0}^2)^{1/2}, \quad n_{k,0} < 1. \tag{2.28}$$

Now, in regard to F_k, we see from Eq. (2.25) that we can write

$$F_k = C_k Y(n_{1,0}, \mu_1) Y(n_{2,1}, \mu_2) \cdots Y(n_{k,k-1}, \mu_k), \tag{2.29}$$

where $C_k \neq 0$. It follows that F_k will be zero if

$$Y(n_{\alpha,\alpha-1}, \mu_\alpha) = 0, \tag{2.30}$$

for any $\alpha \in \{1, 2, \ldots, k\}$. For $Y(n_{\alpha,\alpha-1}, \mu_\alpha)$ to be zero, we must have $n_{\alpha-1,\alpha} < 1$. Then we must also have

$$\mu_\alpha \leqslant (1 - n^2_{\alpha-1,\alpha})^{1/2} \tag{2.31}$$

or

$$[1 - n^2_{0,\alpha}(1 - \mu_0^2)]^{1/2} \leqslant (1 - n^2_{\alpha-1,\alpha})^{1/2}. \tag{2.32}$$

And, so we must have

$$\mu_0^2 \geqslant (1 - n^2_{\alpha,0}). \tag{2.33}$$

If $n_{\alpha,0} < 1$, then

$$F_k = 0 \tag{2.34}$$

only if, for $k > 1$,

$$(1 - n^2_{\alpha,0})^{1/2} \leqslant \mu_0 \leqslant (1 - n^2_{\alpha-1,0})^{1/2}, \quad n_{\alpha,0} < 1, \quad n_{\alpha-1,0} < 1, \quad \text{and} \quad n_{\alpha-1,\alpha} < 1, \tag{2.35}$$

for any $\alpha \in \{2, 3, \ldots, k\}$. If $n_{\alpha,0} \geqslant 1$, then

$$F_k = 0 \tag{2.36}$$

only if, for $k > 1$,

$$0 < \mu_0 \leqslant (1 - n^2_{\alpha-1,0})^{1/2}, \quad n_{\alpha,0} \geqslant 1, \quad n_{\alpha-1,0} < 1, \quad \text{and} \quad n_{\alpha-1,\alpha} < 1, \tag{2.37}$$

for any $\alpha \in \{2, 3, \ldots, k\}$. In regard to Eq. (2.35), we see that the condition $n_{\alpha-1,0} < 1$ can be omitted. Note that we have used the facts that Eqs. (2.35) and (2.37) cannot be satisfied for $\alpha = 1$. To summarize: we find that

$$V_k^-(\mu, \phi) = 0 \tag{2.38}$$

if any of the conditions defined by Eqs. (2.28), (2.35), and (2.37) is satisfied.

Finally, the W functions that appear on the right-hand sides of Eqs. (2.18) are defined by

$$W_1^-(\mu, \phi) = 0, \tag{2.39a}$$

$$W_K^+(\mu, \phi) = 0, \tag{2.39b}$$

$$W_k^-(\mu, \phi) = Y(n_{k,k-1}, \mu)\Omega_k^-[f(n_{k,k-1}, \mu), \phi], \quad k = 2, 3, \ldots, K, \tag{2.40a}$$

and

$$W_k^+(\mu, \phi) = Y(n_{k,k+1}, \mu)\Omega_k^+[f(n_{k,k+1}, \mu), \phi], \quad k = K-1, K-2, \ldots, 1, \tag{2.40b}$$

where

$$\Omega_k^-(\xi, \phi) = \Xi_k^-(\xi)\{Q_{k-1}^+(\xi, \phi) + [Z_{k-1}^-(\xi)Q_{k-1}^-(\xi, \phi) + W_{k-1}^-(\xi, \phi)]\mathrm{e}^{-\Delta_{k-1}/\xi}\} \tag{2.41a}$$

and

$$\Omega_k^+(\xi, \phi) = \Xi_k^+(\xi)\{Q_{k+1}^-(\xi, \phi) + [Z_{k+1}^+(\xi)Q_{k+1}^+(\xi, \phi) + W_{k+1}^+(\xi, \phi)]\mathrm{e}^{-\Delta_{k+1}/\xi}\}. \tag{2.41b}$$

Here

$$Q_k^+(\mu, \phi) = \frac{1}{\mu}\int_{a_{k-1}}^{a_k} \mathscr{F}_k(\tau, \mu, \phi)\mathrm{e}^{-(a_k-\tau)/\mu}\,\mathrm{d}\tau \tag{2.42a}$$

and

$$Q_k^-(\mu, \phi) = \frac{1}{\mu}\int_{a_{k-1}}^{a_k} \mathscr{F}_k(\tau, -\mu, \phi)\mathrm{e}^{-(\tau-a_{k-1})/\mu}\,\mathrm{d}\tau, \tag{2.42b}$$

where

$$\mathscr{F}_k(\tau,\mu,\phi)=\frac{\varpi_k}{4\pi}\sum_{m=0}^{L_k}(2-\delta_{0,m})\sum_{l=m}^{L_k}\beta_{k,l}P_l^m(\mu)\int_{-1}^{1}\int_0^{2\pi}P_l^m(\mu')\cos[m(\phi'-\phi)]I_k(\tau,\mu',\phi')\,\mathrm{d}\phi'\,\mathrm{d}\mu'. \tag{2.43}$$

3. Problem decomposition

Following and generalizing (for problems subject to Fresnel boundary and interface conditions) the classical approach of Chandrasekhar [1], we begin by decomposing the radiation intensity in each layer into unscattered and scattered components, viz.

$$I_k(\tau,\mu,\phi)=I_k^{(0)}(\tau,\mu,\phi)+I_k^{(*)}(\tau,\mu,\phi). \tag{3.1}$$

In this expression, $I_k^{(0)}(\tau,\mu,\phi)$ denotes the unscattered component of the intensity in layer k, i.e. the solution of Eq. (2.1) with $\varpi_k=0$ subject to Eqs. (2.18) with $W_k^{\pm}(\mu,\phi)=0$. We can readily solve these equations to find

$$I_k^{(0)}(\tau,\mu,\phi)=S_k^{+}\mathrm{e}^{-(\tau-a_{k-1})/\mu_k}\delta(\mu-\mu_k)\delta(\phi-\phi_0) \tag{3.2a}$$

and

$$I_k^{(0)}(\tau,-\mu,\phi)=S_k^{-}\mathrm{e}^{-(a_k-\tau)/\mu_k}\delta(\mu-\mu_k)\delta(\phi-\phi_0), \tag{3.2b}$$

for $\tau\in[a_{k-1},a_k]$, $\mu\in(0,1]$, and $\phi\in[0,2\pi]$. Here

$$S_k^{+}=F_k\Upsilon_k(\mu_k) \tag{3.3a}$$

and

$$S_k^{-}=F_k\Upsilon_k(\mu_k)Z_k^{+}(\mu_k)\mathrm{e}^{-\Delta_k/\mu_k}, \tag{3.3b}$$

where

$$\Upsilon_k(\mu)=[1-Z_k^{-}(\mu)Z_k^{+}(\mu)\mathrm{e}^{-2\Delta_k/\mu}]^{-1}. \tag{3.4}$$

In addition, the term $I_k^{(*)}(\tau,\mu,\phi)$ in Eq. (3.1) denotes the scattered component of the intensity in layer k and can be determined by solving

$$\mu\frac{\partial}{\partial\tau}I_k^{(*)}(\tau,\mu,\phi)+I_k^{(*)}(\tau,\mu,\phi)=\frac{\varpi_k}{4\pi}\sum_{m=0}^{L_k}(2-\delta_{0,m})\sum_{l=m}^{L_k}\beta_{k,l}P_l^m(\mu)\times\int_{-1}^{1}\int_0^{2\pi}P_l^m(\mu')\cos[m(\phi'-\phi)]I_k^{(*)}(\tau,\mu',\phi')\,\mathrm{d}\phi'\,\mathrm{d}\mu'+\Psi_k(\tau,\mu,\phi), \tag{3.5}$$

for $\tau\in(a_{k-1},a_k)$, $\mu\in[-1,1]$, $\phi\in[0,2\pi]$, and

$$\Psi_k(\tau,\mu,\phi)=\frac{\varpi_k}{4\pi}\sum_{m=0}^{L_k}(2-\delta_{0,m})\sum_{l=m}^{L_k}\beta_{k,l}[S_k^{+}\mathrm{e}^{-(\tau-a_{k-1})/\mu_k}+(-1)^{l-m}S_k^{-}\mathrm{e}^{-(a_k-\tau)/\mu_k}]P_l^m(\mu)P_l^m(\mu_k)\cos[m(\phi-\phi_0)], \tag{3.6}$$

subject to

$$I_k^{(*)}(a_{k-1},\mu,\phi)-Z_k^{-}(\mu)I_k^{(*)}(a_{k-1},-\mu,\phi)=\widehat{W}_k^{-}(\mu,\phi) \tag{3.7a}$$

and

$$I_k^{(*)}(a_k,-\mu,\phi)-Z_k^{+}(\mu)I_k^{(*)}(a_k,\mu,\phi)=\widehat{W}_k^{+}(\mu,\phi), \tag{3.7b}$$

for $\mu\in(0,1]$ and $\phi\in[0,2\pi]$. Here, the $\widehat{W}$ functions are (as are the W functions) defined by Eqs. (2.39)–(2.43), except that the source term $\Psi_k(\tau,\mu,\phi)$ should be added to the right-hand side of Eq. (2.43) and $I_k^{(*)}(\tau,\mu,\phi)$ should replace $I_k(\tau,\mu,\phi)$ in the integrand of Eq. (2.43).

Next, taking note of the form of the source expressed by Eq. (3.6), we can use the (finite) Fourier decomposition

$$I_k^{(*)}(\tau,\mu,\phi)=\frac{1}{2\pi}\sum_{m=0}^{L_{\max}}(2-\delta_{0,m})I_k^m(\tau,\mu)\cos[m(\phi-\phi_0)], \tag{3.8}$$

where $L_{\max}=\max\{L_1,L_2,\ldots,L_K\}$, to reduce the scattered problem defined by Eqs. (3.5)–(3.7) to a set of $L_{\max}+1$ azimuthally independent problems formulated, for $m=0,1,\ldots,L_{\max}$, by

$$\mu\frac{\partial}{\partial\tau}I_k^m(\tau,\mu)+I_k^m(\tau,\mu)=\frac{\varpi_k}{2}\sum_{l=m}^{L_k}\beta_{k,l}P_l^m(\mu)\int_{-1}^{1}P_l^m(\mu')I_k^m(\tau,\mu')\,\mathrm{d}\mu'+\Psi_k^m(\tau,\mu), \tag{3.9}$$

for $\tau\in(a_{k-1},a_k)$, $\mu\in[-1,1]$, and

$$\Psi_k^m(\tau,\mu)=\frac{\varpi_k}{2}\sum_{l=m}^{L_k}\beta_{k,l}[S_k^+\mathrm{e}^{-(\tau-a_{k-1})/\mu_k}+(-1)^{l-m}S_k^-\mathrm{e}^{-(a_k-\tau)/\mu_k}]P_l^m(\mu)P_l^m(\mu_k), \tag{3.10}$$

subject to

$$I_k^m(a_{k-1},\mu)-Z_k^-(\mu)I_k^m(a_{k-1},-\mu)=W_{k,m}^-(\mu) \tag{3.11a}$$

and

$$I_k^m(a_k,-\mu)-Z_k^+(\mu)I_k^m(a_k,\mu)=W_{k,m}^+(\mu), \tag{3.11b}$$

for $\mu\in(0,1]$. Here, the W functions are to be computed from

$$W_{1,m}^-(\mu)=0, \tag{3.12a}$$

$$W_{K,m}^+(\mu)=0, \tag{3.12b}$$

$$W_{k,m}^-(\mu)=Y(n_{k,k-1},\mu)\Omega_{k,m}^-[f(n_{k,k-1},\mu)],\quad k=2,3,\ldots,K, \tag{3.13a}$$

and

$$W_{k,m}^+(\mu)=Y(n_{k,k+1},\mu)\Omega_{k,m}^+[f(n_{k,k+1},\mu)],\quad k=K-1,K-2,\ldots,1, \tag{3.13b}$$

where

$$\Omega_{k,m}^-(\xi)=\Xi_k^-(\xi)\{Q_{k-1,m}^+(\xi)+[Z_{k-1}^-(\xi)Q_{k-1,m}^-(\xi)+W_{k-1,m}^-(\xi)]\mathrm{e}^{-\Delta_{k-1}/\xi}\} \tag{3.14a}$$

and

$$\Omega_{k,m}^+(\xi)=\Xi_k^+(\xi)\{Q_{k+1,m}^-(\xi)+[Z_{k+1}^+(\xi)Q_{k+1,m}^+(\xi)+W_{k+1,m}^+(\xi)]\mathrm{e}^{-\Delta_{k+1}/\xi}\}. \tag{3.14b}$$

Here

$$Q_{k,m}^+(\mu)=\frac{1}{\mu}\int_{a_{k-1}}^{a_k}\mathscr{F}_k^m(\tau,\mu)\mathrm{e}^{-(a_k-\tau)/\mu}\,\mathrm{d}\tau \tag{3.15a}$$

and

$$Q_{k,m}^-(\mu)=\frac{1}{\mu}\int_{a_{k-1}}^{a_k}\mathscr{F}_k^m(\tau,-\mu)\mathrm{e}^{-(\tau-a_{k-1})/\mu}\,\mathrm{d}\tau, \tag{3.15b}$$

where

$$\mathscr{F}_k^m(\tau,\mu)=\Psi_k^m(\tau,\mu)+\frac{\varpi_k}{2}\sum_{l=m}^{L_k}\beta_{k,l}P_l^m(\mu)\int_{-1}^{1}P_l^m(\mu')I_k^m(\tau,\mu')\,\mathrm{d}\mu'. \tag{3.16}$$

Note that, because of the coupling between layers introduced by the W functions defined by the recursive scheme of Eqs. (3.12)–(3.16), we have to solve $L_{\max}+1$ Fourier component problems for each of the K layers. While the component problems with Fourier index $L_k<m\leqslant L_{\max}$ for layer k could, in principle, be solved

analytically, we find it more convenient (from a computational point-of-view) to use the ADO version [19] of the discrete-ordinates method to solve (in the same way) all of the problems.

4. An ADO solution for the Fourier component problems

In order to solve the set of Fourier component problems defined by Eqs. (3.9)–(3.11), we use the ADO version [19] of the discrete-ordinates method that was used a few years ago [20] to solve Chandrasekhar's basic problem in radiative transfer [1]. Since the basic material on the ADO solution that we need here has been presented in detail in previous works [19,20], our presentation of this material can be brief.

We first consider the case of a non-conservative layer ($\varpi_k \neq 1$). To obtain our discrete-ordinates version of Eq. (3.9), we start by splitting the interval of integration $[-1, 1]$ in that equation into two half-range intervals and applying the transformation $\mu \to -\mu$ to the negative interval, to be able to express the integral term of Eq. (3.9) in terms of a sum of two integral terms defined over the positive interval [0, 1]. Next, we approximate the μ-integration over the interval [0, 1] by a composite quadrature of order N_k with nodes and weights η_i and w_i, $i = 1, 2, \ldots, N_k$. Such a quadrature scheme is obtained from an application of the break-point analysis of Ref. [18], which, as mentioned in the Introduction, allows us to subdivide the interval [0, 1] into a number of subintervals, say s_k, the boundaries of which are chosen so as to avoid integration across the discontinuities in the (polar) derivatives of the radiation intensities that are caused by changes in the index of refraction. The use of a standard Gauss–Legendre quadrature of order M, mapped (linearly) onto each of the s_k subintervals, results in a composite quadrature of order $N_k = s_k M$. As we end up using a different quadrature scheme for each layer (the number and location of the break points are layer-dependent), we could have affixed an index k to the quadrature nodes $\{\eta_i\}$ and the quadrature weights $\{w_i\}$. However, we prefer not to do so in order to avoid clumsy notation.

Continuing, we write our ADO solution to the discrete-ordinates version of Eq. (3.9) so obtained as

$$I_k^m(\tau, \pm\eta_i) = \sum_{j=1}^{N_k} [A_{k,j}^m \phi_k^m(\nu_{k,j}, \pm\eta_i) \mathrm{e}^{-(\tau - a_{k-1})/\nu_{k,j}} + B_{k,j}^m \phi_k^m(\nu_{k,j}, \mp\eta_i) \mathrm{e}^{-(a_k - \tau)/\nu_{k,j}}] + \sum_{j=1}^{N_k} [\mathscr{A}_{k,j}^m(\tau) \phi_k^m(\nu_{k,j}, \pm\eta_i) + \mathscr{B}_{k,j}^m(\tau) \phi_k^m(\nu_{k,j}, \mp\eta_i)], \tag{4.1}$$

for $\tau \in [a_{k-1}, a_k]$. We note that the first summation on the right-hand side of this expression is the ADO solution of the homogenous form of the discrete-ordinates version of Eq. (3.9) and the second summation is a particular ADO solution of the discrete-ordinates version of Eq. (3.9). In addition, we note that the separation constants $\{\nu_{k,j}\}$ and the elementary solutions $\{\phi_k^m(\nu_{k,j}, \pm\eta_i)\}$ can be determined by solving an eigensystem of order N_k as explained in detail in Ref. [20], the coefficients $\{\mathscr{A}_{k,j}^m(\tau)\}$ and $\{\mathscr{B}_{k,j}^m(\tau)\}$ of the particular solution can be determined by slightly modifying the infinite-medium Green's function approach developed in Ref. [21] for treating a source located in a spatial interval defined by $\tau \in (0, \tau_0)$ and used in Ref. [20] to solve a single-layer problem, and the coefficients $\{A_{k,j}^m\}$ and $\{B_{k,j}^m\}$ of the homogeneous solution are to be determined from a discrete-ordinates version of the Fresnel boundary/interface conditions expressed by Eqs. (3.11). Again, in order to avoid clumsy notation, we have suppressed the index m that should, in principle, be affixed to the separation constants $\{\nu_{k,j}\}$. Also, as we may eventually wish to use different quadrature orders for solving component problems with different Fourier indices, we should keep in mind that the number of quadrature points per subinterval (M) and the quadrature order (N_k) may depend on m.

While we believe that the procedure used to compute the separation constants and elementary solutions has been reported with sufficient detail in Ref. [20], we intend to be more explicit here with regard to the coefficients of our particular solution. We begin by rewriting, for a source $\Psi_k^m(\tau, \mu)$ in layer k, the expressions given in Ref. [20] for these coefficients as

$$\mathscr{A}_{k,j}^m(\tau) = \frac{1}{\mathscr{N}_k^m(\nu_{k,j})} \int_{a_{k-1}}^{\tau} \sum_{i=1}^{N_k} w_i [\Psi_k^m(x, \eta_i) \phi_k^m(\nu_{k,j}, \eta_i) + \Psi_k^m(x, -\eta_i) \phi_k^m(\nu_{k,j}, -\eta_i)] \mathrm{e}^{-(\tau - x)/\nu_{k,j}} \, \mathrm{d}x \tag{4.2a}$$

and

$$\mathscr{B}^m_{k,j}(\tau) = \frac{1}{\mathscr{N}^m_k(\nu_{k,j})}\int_\tau^{a_k}\sum_{i=1}^{N_k} w_i[\Psi^m_k(x,\eta_i)\phi^m_k(\nu_{k,j},-\eta_i) + \Psi^m_k(x,-\eta_i)\phi^m_k(\nu_{k,j},\eta_i)]\mathrm{e}^{-(x-\tau)/\nu_{k,j}}\,\mathrm{d}x, \tag{4.2b}$$

where $\tau \in [a_{k-1}, a_k]$ and

$$\mathscr{N}^m_k(\nu_{k,j}) = \sum_{i=1}^{N_k} w_i\eta_i\{[\phi^m_k(\nu_{k,j},\eta_i)]^2 - [\phi^m_k(\nu_{k,j},-\eta_i)]^2\}. \tag{4.3}$$

Substituting Eq. (3.10) into Eqs. (4.2) and performing the integrations, we find

$$\mathscr{A}^m_{k,j}(\tau) = \frac{\mu_k\nu_{k,j}}{\mathscr{N}^m_k(\nu_{k,j})}[S^+_k C(\tau - a_{k-1} : \nu_{k,j},\mu_k)\mathscr{G}^+_{k,m}(\mu_k,\nu_{k,j}) + S^-_k \mathrm{e}^{-(a_k-\tau)/\mu_k} S(\tau - a_{k-1} : \nu_{k,j},\mu_k)\mathscr{G}^-_{k,m}(\mu_k,\nu_{k,j})] \tag{4.4a}$$

and

$$\mathscr{B}^m_{k,j}(\tau) = \frac{\mu_k\nu_{k,j}}{\mathscr{N}^m_k(\nu_{k,j})}[S^+_k \mathrm{e}^{-(\tau-a_{k-1})/\mu_k} S(a_k - \tau : \nu_{k,j},\mu_k)\mathscr{G}^-_{k,m}(\mu_k,\nu_{k,j}) + S^-_k C(a_k - \tau : \nu_{k,j},\mu_k)\mathscr{G}^+_{k,m}(\mu_k,\nu_{k,j})], \tag{4.4b}$$

where

$$S(\tau : \nu,\mu) = \frac{1 - \mathrm{e}^{-\tau/\nu}\mathrm{e}^{-\tau/\mu}}{\nu + \mu}, \tag{4.5a}$$

$$C(\tau : \nu,\mu) = \frac{\mathrm{e}^{-\tau/\nu} - \mathrm{e}^{-\tau/\mu}}{\nu - \mu}, \tag{4.5b}$$

and

$$\mathscr{G}^\pm_{k,m}(\mu,\nu) = \frac{\varpi_k}{2}\sum_{l=m}^{L_k}(\pm 1)^{l-m}\beta_{k,l}P^m_l(\mu)\Gamma^m_{k,l}(\nu), \tag{4.6}$$

with

$$\Gamma^m_{k,l}(\nu) = \sum_{i=1}^{N_k} w_i P^m_l(\eta_i)[\phi^m_k(\nu,\eta_i) + (-1)^{l-m}\phi^m_k(\nu,-\eta_i)]. \tag{4.7}$$

In order to complete the solution listed as Eq. (4.1), we must determine, for each of the K regions, the coefficients $\{A^m_{k,j}\}$ and $\{B^m_{k,j}\}$ of the homogeneous solution. For this purpose, we substitute Eq. (4.1) into discrete-ordinates versions of Eqs. (3.11) evaluated at $\mu = \eta_i$, $i = 1, 2, \ldots, N_k$, to obtain a system of $2N_k$ linear equations for layer k which we can write in matrix notation as

$$\begin{pmatrix} \boldsymbol{\Phi}^+_{k,m} - \boldsymbol{Z}^-_k\boldsymbol{\Phi}^-_{k,m} & [\boldsymbol{\Phi}^-_{k,m} - \boldsymbol{Z}^-_k\boldsymbol{\Phi}^+_{k,m}]\boldsymbol{\Delta}_k \\ [\boldsymbol{\Phi}^-_{k,m} - \boldsymbol{Z}^+_k\boldsymbol{\Phi}^+_{k,m}]\boldsymbol{\Delta}_k & \boldsymbol{\Phi}^+_{k,m} - \boldsymbol{Z}^+_k\boldsymbol{\Phi}^-_{k,m} \end{pmatrix}\begin{pmatrix} \boldsymbol{A}^m_k \\ \boldsymbol{B}^m_k \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}^-_{k,m} \\ \boldsymbol{R}^+_{k,m} \end{pmatrix}, \tag{4.8}$$

where $\boldsymbol{\Phi}^\pm_{k,m}$ are $N_k \times N_k$ matrices with ij elements $\phi^m_k(\nu_{k,j}, \pm\eta_i)$, $\boldsymbol{Z}^\pm_k$ and $\boldsymbol{\Delta}_k$ are diagonal matrices of order N_k whose i-th elements on the diagonals are, respectively, $Z^\pm_k(\eta_i)$ and $\exp[-\Delta_k/\nu_{k,i}]$, and $\boldsymbol{A}^m_k$ and $\boldsymbol{B}^m_k$ are N_k-vectors whose i-th components are, respectively, $A^m_{k,i}$ and $B^m_{k,i}$. In addition, the N_k-vectors $\boldsymbol{R}^-_{k,m}$ and $\boldsymbol{R}^+_{k,m}$ can be expressed as

$$\boldsymbol{R}^-_{k,m} = \boldsymbol{W}^-_{k,m} - [\boldsymbol{\Phi}^-_{k,m} - \boldsymbol{Z}^-_k\boldsymbol{\Phi}^+_{k,m}]\mathscr{B}^m_k(a_{k-1}) \tag{4.9a}$$

and

$$\boldsymbol{R}^+_{k,m} = \boldsymbol{W}^+_{k,m} - [\boldsymbol{\Phi}^-_{k,m} - \boldsymbol{Z}^+_k\boldsymbol{\Phi}^+_{k,m}]\mathscr{A}^m_k(a_k), \tag{4.9b}$$

where the N_k-vectors $\boldsymbol{W}^\pm_{k,m}$, $\mathscr{A}^m_k(a_k)$, and $\mathscr{B}^m_k(a_{k-1})$ have, respectively, $W^\pm_{k,m}(\eta_i)$, $\mathscr{A}^m_{k,i}(a_k)$, and $\mathscr{B}^m_{k,i}(a_{k-1})$ as i-th components.

By solving the systems of linear equations defined by Eq. (4.8) for $k = 1, 2, \dots, K$, we can find the vectors $\{\boldsymbol{A}_k^m\}$ and $\{\boldsymbol{B}_k^m\}$, and therefore the coefficients of the homogeneous solution that are needed to complete the solution given by Eq. (4.1). However, there is a difficulty: only the second terms on the right-hand sides of Eqs. (4.9) are known, and so these systems cannot be solved directly. The first terms, $\boldsymbol{W}_{k,m}^{\pm}$, are vectors that have as elements the W functions defined by the recursive schemes of Eqs. (3.12)–(3.16), and so, for a given layer, they depend on the coefficients of the homogeneous solutions of all the other layers. Since these coefficients are not known at this point, our way of solving the linear systems defined by Eq. (4.8) for $k = 1, 2, \dots, K$ will be iterative.

Our iterative procedure can be explained as follows. For a given value of m, we start by taking $\boldsymbol{W}_{k,m}^{\pm} = \boldsymbol{0}$ in Eqs. (4.9), and we then solve the resulting systems of linear algebraic equations for $k = 1, 2, \dots, K$, as in Ref. [18], using a "left–right" sweep followed by a "right–left" sweep through the layers [22] to find first estimates for $\{\boldsymbol{A}_k^m\}$ and $\{\boldsymbol{B}_k^m\}$, $k = 1, 2, \dots, K$. These estimates are then used for computing improved estimates of $\boldsymbol{W}_{k,m}^{\pm}$, by means of Eqs. (3.12)–(3.15) with the Q functions expressed as

$$Q_{k,m}^{+}(\mu) = U_{k,m}^{+}(\Delta_k : \mu, \mu_k) + \sum_{j=1}^{N_k} \nu_{k,j}[A_{k,j}^m C(\Delta_k : \nu_{k,j}, \mu)\mathscr{G}_{k,m}^{+}(\mu, \nu_{k,j}) + B_{k,j}^m S(\Delta_k : \nu_{k,j}, \mu)\mathscr{G}_{k,m}^{-}(\mu, \nu_{k,j}) + T_{k,m}^{+}(\Delta_k : \mu, \nu_{k,j})] \tag{4.10a}$$

and

$$Q_{k,m}^{-}(\mu) = U_{k,m}^{-}(\Delta_k : \mu, \mu_k) + \sum_{j=1}^{N_k} \nu_{k,j}[A_{k,j}^m S(\Delta_k : \nu_{k,j}, \mu)\mathscr{G}_{k,m}^{-}(\mu, \nu_{k,j}) + B_{k,j}^m C(\Delta_k : \nu_{k,j}, \mu)\mathscr{G}_{k,m}^{+}(\mu, \nu_{k,j}) + T_{k,m}^{-}(\Delta_k : \mu, \nu_{k,j})], \tag{4.10b}$$

where

$$U_{k,m}^{+}(\tau : \mu, \xi) = \xi[S_k^{+} C(\tau : \xi, \mu)\mathscr{P}_{k,m}^{+}(\mu, \xi) + S_k^{-} \mathrm{e}^{-(\Delta_k-\tau)/\xi} S(\tau : \xi, \mu)\mathscr{P}_{k,m}^{-}(\mu, \xi)], \tag{4.11a}$$

$$U_{k,m}^{-}(\tau : \mu, \xi) = \xi[S_k^{+} \mathrm{e}^{-(\Delta_k-\tau)/\xi} S(\tau : \xi, \mu)\mathscr{P}_{k,m}^{-}(\mu, \xi) + S_k^{-} C(\tau : \xi, \mu)\mathscr{P}_{k,m}^{+}(\mu, \xi)], \tag{4.11b}$$

and

$$T_{k,m}^{\pm}(\tau : \mu, \nu) = [S_k^{\pm}\mathscr{X}_{k,j}^m(\tau, \mu) + S_k^{\mp}\mathscr{W}_{k,j}^m(\tau, \mu)]\mathscr{G}_{k,m}^{+}(\mu, \nu) + [S_k^{\pm}\mathscr{Y}_{k,j}^m(\tau, \mu) + S_k^{\mp}\mathscr{Z}_{k,j}^m(\tau, \mu)]\mathscr{G}_{k,m}^{-}(\mu, \nu). \tag{4.12}$$

Here, we define

$$\mathscr{P}_{k,m}^{\pm}(\mu, \xi) = \frac{\varpi_k}{2}\sum_{l=m}^{L_k}(\pm 1)^{l-m}\beta_{k,l}P_l^m(\mu)P_l^m(\xi), \tag{4.13}$$

$$\mathscr{X}_{k,j}^m(\tau, \mu) = \frac{\mu_k}{\mathscr{N}_k^m(\nu_{k,j})}\left[\frac{\nu_{k,j}C(\tau : \nu_{k,j}, \mu) - \mu_k C(\tau : \mu_k, \mu)}{\nu_{k,j} - \mu_k}\right]\mathscr{G}_{k,m}^{+}(\mu_k, \nu_{k,j}), \tag{4.14a}$$

$$\mathscr{Y}_{k,j}^m(\tau, \mu) = \frac{\mu_k}{\mathscr{N}_k^m(\nu_{k,j})}\left[\frac{\mu_k C(\tau : \mu_k, \mu) - \nu_{k,j}\mathrm{e}^{-(\Delta_k-\tau)/\nu_{k,j}}\mathrm{e}^{-\Delta_k/\mu_k}S(\tau : \nu_{k,j}, \mu)}{\nu_{k,j} + \mu_k}\right]\mathscr{G}_{k,m}^{-}(\mu_k, \nu_{k,j}), \tag{4.14b}$$

$$\mathscr{Z}_{k,j}^m(\tau, \mu) = \frac{\mu_k}{\mathscr{N}_k^m(\nu_{k,j})}\left[\frac{\nu_{k,j}\mathrm{e}^{-(\Delta_k-\tau)/\nu_{k,j}}S(\tau : \nu_{k,j}, \mu) - \mu_k\mathrm{e}^{-(\Delta_k-\tau)/\mu_k}S(\tau : \mu_k, \mu)}{\nu_{k,j} - \mu_k}\right]\mathscr{G}_{k,m}^{+}(\mu_k, \nu_{k,j}), \tag{4.14c}$$

and

$$\mathscr{W}_{k,j}^m(\tau, \mu) = \frac{\mu_k}{\mathscr{N}_k^m(\nu_{k,j})}\left[\frac{\mu_k\mathrm{e}^{-(\Delta_k-\tau)/\mu_k}S(\tau : \mu_k, \mu) - \nu_{k,j}\mathrm{e}^{-\Delta_k/\mu_k}C(\tau : \nu_{k,j}, \mu)}{\nu_{k,j} + \mu_k}\right]\mathscr{G}_{k,m}^{-}(\mu_k, \nu_{k,j}). \tag{4.14d}$$

The foregoing procedure is repeated until some convergence criteria for the vectors $\{\boldsymbol{A}_k^m\}$ and $\{\boldsymbol{B}_k^m\}$ is met for all k. Doing the same for all required values of m, we obtain all of the coefficients $\{A_{k,j}^m\}$ and $\{B_{k,j}^m\}$ that are needed in our solution of the scattered problem. Finally, to complete the description of our iterative scheme of solving Eq. (4.8) for $k = 1, 2, \ldots, K$, we should make it clear that only one LU factorization [23] per layer is required, for a given value of m. The "left–right" and "right–left" sweeps through the layers involve only matrix-vector multiplications.

We now discuss the modifications that are required in our solution to handle the case of a conservative layer, $\varpi_k = 1$. As discussed in previous works [18,20], the problem with the conservative case is that one of the separation constants becomes unbounded when $\varpi_k = 1$ and $m = 0$. To overcome this difficulty, we follow Refs. [18,20] and replace the terms associated with the unbounded separation constant, say $v_{k,1}$, in the homogenous ADO solution by the exact solutions 1 and $\tau - 3\mu/h_{k,1}$, where

$$h_{k,1} = 3 - \beta_{k,1}. \tag{4.15}$$

Of course, we must also replace the terms associated with the unbounded separation constant in our particular ADO solution. As the conservative case is not explicitly treated in Ref. [21], we have worked out the changes needed in the infinite-medium Green's function approach of Ref. [21] to find the required replacing terms. We thus rewrite Eq. (4.1) for $\varpi_k = 1$ and $m = 0$ as

$$\begin{aligned} I_k(\tau, \pm\eta_i) = {} & I_k^h(\tau, \pm\eta_i) + \sum_{j=2}^{N_k} [A_{k,j}\phi_k(v_{k,j}, \pm\eta_i)\mathrm{e}^{-(\tau - a_{k-1})/v_{k,j}} + B_{k,j}\phi_k(v_{k,j}, \mp\eta_i)\mathrm{e}^{-(a_k - \tau)/v_{k,j}}] \\ & + I_k^p(\tau, \pm\eta_i) + \sum_{j=2}^{N_k} [\mathscr{A}_{k,j}(\tau)\phi_k(v_{k,j}, \pm\eta_i) + \mathscr{B}_{k,j}(\tau)\phi_k(v_{k,j}, \mp\eta_i)], \end{aligned} \tag{4.16}$$

where

$$I_k^h(\tau, \mu) = A_{k,1} + B_{k,1}(\tau - 3\mu/h_{k,1}) \tag{4.17}$$

and

$$I_k^p(\tau, \mu) = -(3/2)\mu_k[(\mu_k + \mu)S_k^+ \mathrm{e}^{-(\tau - a_{k-1})/\mu_k} + (\mu_k - \mu)S_k^- \mathrm{e}^{-(a_k - \tau)/\mu_k}]. \tag{4.18}$$

Note that, to simplify our notation, we have omitted the index m in Eq. (4.16). Hereafter in this section, the absence of the index m in the notation of some quantities implies that these quantities are being considered for $m = 0$.

From this point on, we proceed in exactly the same way as for the non-conservative case. We use Eqs. (4.4) for $m = 0$ to express the coefficients $\mathscr{A}_{k,j}(\tau)$ and $\mathscr{B}_{k,j}(\tau)$, $j = 2, 3, \ldots, N_k$, that appear in Eq. (4.16) and substitute the resulting equation into discrete-ordinates versions of Eqs. (3.11) evaluated at $\mu = \eta_i$, $i = 1, 2, \ldots, N_k$, to obtain a linear system of $2N_k$ equations for the $2N_k$ coefficients $A_{k,j}$ and $B_{k,j}$, $j = 1, 2, \ldots, N_k$. Only the first rows of the four submatrices of order N_k that make up the $2N_k \times 2N_k$ coefficient matrix on the left-hand side of Eq. (4.8) and the first components of the right-hand side vectors $\boldsymbol{R}_{k,m}^-$ and $\boldsymbol{R}_{k,m}^+$ are different in this case. With regard to the computation of the first components of the $\boldsymbol{W}_{k,m}^{\pm}$ vectors by the recursive schemes of Eqs. (3.12)–(3.15), we can still use Eqs. (4.10) to express the required Q functions, provided the summations in these equations are started at $j = 2$ and the new terms

$$\begin{aligned} E_k^+(\mu) = {} & [A_{k,1} + B_{k,1}(a_{k-1} - 3\mu/h_{k,1})(1 - \mathrm{e}^{-\Delta_k/\mu}) + B_{k,1}\Delta_k \\ & - (1/2)\mu_k^2[(3\mu_k + \beta_{k,1}\mu)S_k^+ C(\Delta_k : \mu_k, \mu) + (3\mu_k - \beta_{k,1}\mu)S_k^- S(\Delta_k : \mu_k, \mu)] \end{aligned} \tag{4.19a}$$

and

$$\begin{aligned} E_k^-(\mu) = {} & [A_{k,1} + B_{k,1}(a_k + 3\mu/h_{k,1})(1 - \mathrm{e}^{-\Delta_k/\mu}) - B_{k,1}\Delta_k \\ & - (1/2)\mu_k^2[(3\mu_k - \beta_{k,1}\mu)S_k^+ S(\Delta_k : \mu_k, \mu) + (3\mu_k + \beta_{k,1}\mu)S_k^- C(\Delta_k : \mu_k, \mu)] \end{aligned} \tag{4.19b}$$

are added, respectively, to the right-hand side of Eq. (4.10a) and the right-hand side of Eq. (4.10b).

5. Quantities of interest

As shown in Section 3, the radiation intensity in layer k can be expressed as

$$I_k(\tau,\mu,\phi) = I_k^{(0)}(\tau,\mu,\phi) + \frac{1}{2\pi}\sum_{m=0}^{L_{\max}}(2-\delta_{0,m})I_k^m(\tau,\mu)\cos[m(\phi-\phi_0)], \tag{5.1}$$

for $\tau \in [a_{k-1}, a_k]$, $\pm\mu \in (0,1]$, and $\phi \in [0, 2\pi]$. Here, the term $I_k^{(0)}(\tau,\mu,\phi)$ is the unscattered intensity, as given explicitly by Eqs. (3.2), and the summation term is the Fourier-decomposed scattered intensity $I_k^{(*)}(\tau,\mu,\phi)$, as given by Eq. (3.8). The ADO solution that is expressed for $\varpi_k \neq 1$ and for $\varpi_k = 1$ and $m \neq 0$ by Eq. (4.1) and for $\varpi_k = 1$ and $m = 0$ by Eq. (4.16) can be used, after the unknown coefficients in those equations are determined as discussed in Section 4, to compute each of the Fourier components of the scattered intensity, $I_k^m(\tau,\mu)$, at the quadrature nodes $\mu = \pm\eta_i$. Here, we recall that the $\{\eta_i\}$ are layer-dependent and can also depend on m. In order to be able to compute the Fourier components of the scattered intensity for any value of μ, we can post-process Eq. (3.9) (see a complete discussion of our post-processing procedure in Ref. [18]) to obtain, for $\tau \in [a_{k-1}, a_k]$ and $\mu \in (0,1]$,

$$I_k^m(\tau,\mu) = I_k^m(a_{k-1},\mu)\mathrm{e}^{-(\tau-a_{k-1})/\mu} + \mathscr{Q}_k^m(\tau,\mu) \tag{5.2a}$$

and

$$I_k^m(\tau,-\mu) = I_k^m(a_k,-\mu)\mathrm{e}^{-(a_k-\tau)/\mu} + \mathscr{Q}_k^m(\tau,-\mu), \tag{5.2b}$$

where, for $\varpi_k \neq 1$ and for $\varpi_k = 1$ and $m \neq 0$,

$$\mathscr{Q}_k^m(\tau,\mu) = U_{k,m}^+(\tau-a_{k-1} : \mu,\mu_k) + \sum_{j=1}^{N_k} v_{k,j}[A_{k,j}^m C(\tau-a_{k-1} : v_{k,j},\mu)\mathscr{G}_{k,m}^+(\mu,v_{k,j})$$
$$+ B_{k,j}^m \mathrm{e}^{-(a_k-\tau)/v_{k,j}} S(\tau-a_{k-1} : v_{k,j},\mu)\mathscr{G}_{k,m}^-(\mu,v_{k,j}) + T_{k,m}^+(\tau-a_{k-1} : \mu,v_{k,j})] \tag{5.3a}$$

and

$$\mathscr{Q}_k^m(\tau,-\mu) = U_{k,m}^-(a_k-\tau : \mu,\mu_k) + \sum_{j=1}^{N_k} v_{k,j}[A_{k,j}^m \mathrm{e}^{-(\tau-a_{k-1})/v_{k,j}} S(a_k-\tau : v_{k,j},\mu)\mathscr{G}_{k,m}^-(\mu,v_{k,j})$$
$$+ B_{k,j}^m C(a_k-\tau : v_{k,j},\mu)\mathscr{G}_{k,m}^+(\mu,v_{k,j}) + T_{k,m}^-(a_k-\tau : \mu,v_{k,j})]. \tag{5.3b}$$

For $\varpi_k = 1$ and $m = 0$, we can still use Eqs. (5.3) to define $\mathscr{Q}_k^m(\tau,\pm\mu)$, provided we add the terms

$$\mathscr{E}_k(\tau,\mu) = [A_{k,1} + B_{k,1}(a_{k-1} - 3\mu/h_{k,1})][1-\mathrm{e}^{-(\tau-a_{k-1})/\mu}] + B_{k,1}(\tau-a_{k-1}) - (1/2)\mu_k^2$$
$$\times[(3\mu_k + \beta_{k,1}\mu)S_k^+ C(\tau-a_{k-1} : \mu_k,\mu) + (3\mu_k - \beta_{k,1}\mu)S_k^- \mathrm{e}^{-(a_k-\tau)/\mu_k} S(\tau-a_{k-1} : \mu_k,\mu)] \tag{5.4a}$$

and

$$\mathscr{E}_k(\tau,-\mu) = [A_{k,1} + B_{k,1}(a_k + 3\mu/h_{k,1})][1-\mathrm{e}^{-(a_k-\tau)/\mu}] + B_{k,1}(\tau-a_k) - (1/2)\mu_k^2$$
$$\times[(3\mu_k - \beta_{k,1}\mu)S_k^+ \mathrm{e}^{-(\tau-a_{k-1})/\mu_k} S(a_k-\tau : \mu_k,\mu) + (3\mu_k + \beta_{k,1}\mu)S_k^- C(a_k-\tau : \mu_k,\mu)], \tag{5.4b}$$

respectively, to the right-hand side of Eq. (5.3a) and the right-hand side of Eq. (5.3b), and we start the summations in Eqs. (5.3) at $j = 2$.

However, our job is not yet complete as the "boundary" terms in Eqs. (5.2), i.e. $I_k^m(a_{k-1},\mu)$ and $I_k^m(a_k,-\mu)$ for $\mu \in (0,1]$, are not known. To determine these unknown terms, we first take $\tau = a_k$ in Eq. (5.2a) and $\tau = a_{k-1}$ in Eq. (5.2b) to obtain

$$I_k^m(a_k,\mu) = I_k^m(a_{k-1},\mu)\mathrm{e}^{-\Delta_k/\mu} + Q_{k,m}^+(\mu) \tag{5.5a}$$

and

$$I_k^m(a_{k-1},-\mu) = I_k^m(a_k,-\mu)\mathrm{e}^{-\Delta_k/\mu} + Q_{k,m}^-(\mu), \tag{5.5b}$$

for $\mu \in (0,1]$, and we then use Eqs. (3.11) to eliminate the terms $I_k^m(a_{k-1},-\mu)$ and $I_k^m(a_k,\mu)$ from Eqs. (5.5). We thus obtain, for the unknown "boundary" terms in Eqs. (5.2):

$$I_k^m(a_{k-1},\mu) = \Upsilon_k(\mu) D_{k,m}^+(\mu) \tag{5.6a}$$

and

$$I_k^m(a_k,-\mu) = \Upsilon_k(\mu) D_{k,m}^-(\mu), \tag{5.6b}$$

where $\Upsilon_k(\mu)$ is defined by Eq. (3.4) and

$$D_{k,m}^{\pm} = Z_k^{\mp}(\mu) \mathrm{e}^{-\Delta_k/\mu} [Z_k^{\pm}(\mu) Q_{k,m}^{\pm}(\mu) + W_{k,m}^{\pm}(\mu)] + Z_k^{\mp}(\mu) Q_{k,m}^{\mp}(\mu) + W_{k,m}^{\mp}(\mu). \tag{5.7}$$

Most important for applications (especially inverse problems) are the radiation intensities exiting the multi-layer system, $\widehat{I}(a_0,-\mu,\phi)$ and $\widehat{I}(a_K,\mu,\phi)$ for $\mu \in (0,1]$ and $\phi \in [0,2\pi]$, which we can write for the problem solved in this work as

$$\widehat{I}(a_0,-\mu,\phi) = X(n_{0,1},\mu)\delta(\mu-\mu_0)\delta(\phi-\phi_0) + Y(n_{0,1},\mu) I_1[a_0,-f(n_{0,1},\mu),\phi] \tag{5.8a}$$

and

$$\widehat{I}(a_K,\mu,\phi) = Y(n_{K+1,K},\mu) I_K[a_K, f(n_{K+1,K},\mu),\phi], \tag{5.8b}$$

for $\mu \in (0,1]$ and $\phi \in [0,2\pi]$. To make clear the meaning of the right-hand sides of these expressions, we note that the terms that include intensities at shifted polar angles describe radiation that exits the multi-layer system crossing the external boundaries, while the term that includes delta functions in Eq. (5.8a) describes the part of the incoming radiation that is reflected externally at the $\tau = a_0$ boundary. As in Ref. [18], we can use a procedure similar to that used to derive Eqs. (2.18) to rewrite Eqs. (5.8) in forms that do not involve intensities with shifts in the polar angle, viz.

$$\widehat{I}(a_0,-\mu,\phi) = Z_0^+(\mu)\delta(\mu-\mu_0)\delta(\phi-\phi_0) + \frac{1}{2\pi}\sum_{m=0}^{L_{\max}} (2-\delta_{0,m}) W_{0,m}^+(\mu)\cos[m(\phi-\phi_0)] \tag{5.9a}$$

and

$$\widehat{I}(a_K,\mu,\phi) = F_{K+1}\delta(\mu-\mu_{K+1})\delta(\phi-\phi_0) + \frac{1}{2\pi}\sum_{m=0}^{L_{\max}} (2-\delta_{0,m}) W_{K+1,m}^-(\mu)\cos[m(\phi-\phi_0)], \tag{5.9b}$$

for $\mu \in (0,1]$ and $\phi \in [0,2\pi]$. Here, the functions $Z_0^+(\mu)$ and $W_{0,m}^+(\mu)$ can be computed by carrying the recursion one step further in Eqs. (2.20b) and (3.13b), i.e. by stopping the use of these equations only at $k=0$. In a similar way, the factor F_{K+1} and the function $W_{K+1,m}^-(\mu)$ can be computed by carrying the recursion one step further in Eqs. (2.25) and (3.13a), i.e. by using these equations up to $k=K+1$.

In this work, we also compute the currents (partial fluxes) exiting each of the two external surfaces, normalized by the incoming current at $\tau = a_0$. The incoming current at $\tau = a_0$ is defined as

$$J_0^+ = \int_0^1 \int_0^{2\pi} \psi_0(\mu,\phi)\,\mathrm{d}\phi\,\mu\,\mathrm{d}\mu, \tag{5.10}$$

which, in view of Eq. (2.6), reduces to

$$J_0^+ = \mu_0. \tag{5.11}$$

On the other hand, we can, in the manner of Ref. [18], write the currents exiting the two surfaces as

$$J_0^- = \int_0^1 \int_0^{2\pi} X(n_{0,1},\mu)\psi_0(\mu,\phi)\,\mathrm{d}\phi\,\mu\,\mathrm{d}\mu + \int_0^1 \int_0^{2\pi} [1 - X(n_{1,0},\mu)] I_1(a_0,-\mu,\phi)\,\mathrm{d}\phi\,\mu\,\mathrm{d}\mu \tag{5.12a}$$

and

$$J_K^+ = \int_0^1 \int_0^{2\pi} [1 - X(n_{K,K+1},\mu)] I_K(a_K,\mu,\phi)\,\mathrm{d}\phi\,\mu\,\mathrm{d}\mu. \tag{5.12b}$$

Using Eqs. (2.6), (3.1), (3.2), and (3.8) and approximating the μ-integration in the last integrals of Eqs. (5.12a) and (5.12b) with our quadrature schemes for, respectively, layers one and K, we find that we can express these currents as

$$J_0^- = \mu_0 X(n_{1,0},\mu_0) + \mu_1[1 - X(n_{1,0},\mu_1)]S_1^- \mathrm{e}^{-\Delta_1/\mu_1} + \sum_{i=1}^{N_1} w_i\eta_i[1 - X(n_{1,0},\eta_i)]I_1(a_0,-\eta_i) \tag{5.13a}$$

and

$$J_K^+ = \mu_K[1 - X(n_{K,K+1},\mu_K)]S_K^+ \mathrm{e}^{-\Delta_K/\mu_K} + \sum_{i=1}^{N_K} w_i\eta_i[1 - X(n_{K,K+1},\eta_i)]I_K(a_K,\eta_i). \tag{5.13b}$$

Here, $I_1(a_0,-\eta_i)$ is given by Eq. (4.1) for $m=0$ when $\varpi_1 \neq 1$ and by Eq. (4.16) when $\varpi_1 = 1$. Similarly, $I_K(a_K,\eta_i)$ is given by Eq. (4.1) for $m=0$ when $\varpi_K \neq 1$ and by Eq. (4.16) when $\varpi_K = 1$. And so, in addition to the intensities exiting the multi-layer system, we also report in this work our numerical results for

$$A = J_0^-/J_0^+ \quad \text{and} \quad B = J_K^+/J_0^+. \tag{5.14a, b}$$

6. Numerical results

In this section, we report numerical results for two sets of test cases that were introduced and used in Ref. [18] to illustrate the use of the ADO method for the case of isotropically incident radiation. As mentioned in the Introduction, our computational work was carried out in a way that allowed us to generate numerical results of very high accuracy, aimed at testing our implementation of the method extensively and providing high-quality results that could be used for assessing the accuracy of other implementations and methods.

Our first set of test cases is defined by a collection of two-layer problems [18], the data for which are mostly specified in the titles of Tables 1 and 2 , where our converged ADO numerical results for the normalized currents A and B are reported. It can be seen that each layer has a specific single-scattering albedo, scattering order, optical thickness, and index of refraction. In addition, we consider, as in Ref. [18], that the bi-slab is surrounded by vacuum, so that $n_0 = 1$ and $n_3 = 1$, and that scattering in the layers can be described by the

Table 1
The normalized exiting currents for two-layer problems with $\mu_0 = 1/2$, $\Delta_1 = 0.4$, $\Delta_2 = 0.6$, $\varpi_1 = 0.9$, $\varpi_2 = 0.99$, $L_1 = 10$, and $L_2 = 100$

	A			B		
n_1	$n_2 = 4/3$	$n_2 = 1.6$	$n_2 = 1.9$	$n_2 = 4/3$	$n_2 = 1.6$	$n_2 = 1.9$
1.1	1.530894(−1)	1.993133(−1)	2.425181(−1)	7.105255(−1)	6.519729(−1)	5.975008(−1)
1.4	1.795227(−1)	2.077481(−1)	2.394270(−1)	6.689800(−1)	6.285308(−1)	5.855564(−1)
1.8	2.336399(−1)	2.460546(−1)	2.643833(−1)	5.989121(−1)	5.750348(−1)	5.454289(−1)
2.0	2.594919(−1)	2.667189(−1)	2.800710(−1)	5.653177(−1)	5.472260(−1)	5.229367(−1)

Table 2
The normalized exiting currents for two-layer problems with $\mu_0 = 1/2$, $\Delta_1 = 5.0$, $\Delta_2 = 7.0$, $\varpi_1 = 1.0$, $\varpi_2 = 0.99$, $L_1 = 200$, and $L_2 = 100$

	A			B		
n_1	$n_2 = 4/3$	$n_2 = 1.6$	$n_2 = 1.9$	$n_2 = 4/3$	$n_2 = 1.6$	$n_2 = 1.9$
1.1	2.115864(−1)	2.495993(−1)	2.806585(−1)	5.686461(−1)	4.922176(−1)	4.214831(−1)
1.4	2.402545(−1)	2.539177(−1)	2.683090(−1)	5.378869(−1)	4.782080(−1)	4.189347(−1)
1.8	3.075822(−1)	3.016417(−1)	3.043548(−1)	4.920528(−1)	4.414941(−1)	3.910300(−1)
2.0	3.386581(−1)	3.253997(−1)	3.236070(−1)	4.713876(−1)	4.249295(−1)	3.768150(−1)

binomial law [24]

$$p(\cos\Theta) = \frac{L+1}{2^L}(1+\cos\Theta)^L, \tag{6.1}$$

where L is a non-negative integer that we take to be equal to L_1 in layer one and L_2 in layer two (see the titles of Tables 1 and 2 for the selected values of L_1 and L_2). As required by our approach and done by others, we rewrite Eq. (6.1) in the form of a finite (order L) Legendre-polynomial expansion, viz.

$$p(\cos\Theta) = \sum_{l=0}^{L} \beta_l P_l(\cos\Theta), \tag{6.2}$$

where, as noted by McCormick and Sanchez [25], the $\{\beta_l\}$ coefficients are defined by the recursive scheme

$$\beta_l = \left(\frac{2l+1}{2l-1}\right)\left(\frac{L+1-l}{L+1+l}\right)\beta_{l-1}, \tag{6.3}$$

for $l = 1, 2, \ldots, L$, with $\beta_0 = 1$. It should be noted that while the solutions developed in this work do depend on the form of Eq. (6.2), the particular choice of the $\{\beta_l\}$ defined in Eq. (6.3) was made here only to avoid a required listing of copious input data that would be needed in order to test our solutions for extreme cases of anisotropic scattering. By taking the cosine of the polar angle of incidence to be $\mu_0 = 1/2$ for all cases (the azimuthal angle of incidence can be chosen arbitrarily), we complete the specification of our two-layer problems.

Our second set of test cases consists of five multi-layer problems defined in terms of the data listed in Table 3, as shown in the first two columns of Table 4. With regard to these problems, we report our converged ADO results for the normalized currents A and B in Table 4 and for the scattered components of the exiting intensities $\widehat{I}(a_0, -\mu, \phi)$ and $\widehat{I}(a_K, \mu, \phi)$ at selected values of μ and ϕ in Tables 5–10. Once again, we consider that the multi-layers are surrounded by vacuum, so that $n_0 = 1$ and $n_{K+1} = 1$, and that the cosine of the polar angle of incidence is $\mu_0 = 1/2$ for all cases. For these problems, we also use the binomial scattering law in the form of Eq. (6.2) with $L = L_k$ in layer k, as specified in Table 3.

Table 3
Basic data for multi-layer problems with $\mu_0 = 1/2$

Layer #	Δ	ϖ	L	n
1	1.0	0.95	40	1.65
2	1.2	0.94	60	2.00
3	1.3	0.93	30	1.70
4	0.6	0.96	70	1.60
5	1.9	0.90	20	1.80
6	1.4	0.92	50	1.85
7	0.5	0.97	80	1.55
8	0.3	0.98	90	1.50
9	1.6	0.91	10	1.75
10	5.2	1.00	100	1.30

Table 4
The normalized exiting currents for multi-layer problems

Problem	Layers	A	B
I	1–3	2.005097(−1)	3.234892(−1)
II	6–10	2.179189(−1)	2.447012(−1)
III	4–10	1.486857(−1)	1.480895(−1)
IV	1–9	1.442142(−1)	8.902468(−2)
V	1–10	1.443597(−1)	8.985415(−2)

Table 5
The scattered component of the exiting intensity $\widehat{I}(a_0, -\mu, \phi)$ for $\phi - \phi_0 = 0$

μ	Problem I	Problem II	Problem III	Problem IV	Problem V
0.00	0.0	0.0	0.0	0.0	0.0
0.05	9.8174(−3)	5.7816(−3)	3.0604(−3)	2.8336(−3)	2.8340(−3)
0.10	1.6739(−2)	9.8745(−3)	5.2812(−3)	4.8763(−3)	4.8770(−3)
0.15	2.1702(−2)	1.2824(−2)	6.9176(−3)	6.3661(−3)	6.3671(−3)
0.20	2.5287(−2)	1.4971(−2)	8.1357(−3)	7.4578(−3)	7.4592(−3)
0.25	2.7865(−2)	1.6537(−2)	9.0464(−3)	8.2548(−3)	8.2566(−3)
0.30	2.9685(−2)	1.7669(−2)	9.7252(−3)	8.8281(−3)	8.8304(−3)
0.35	3.0912(−2)	1.8469(−2)	1.0224(−2)	9.2270(−3)	9.2298(−3)
0.40	3.1663(−2)	1.9005(−2)	1.0577(−2)	9.4862(−3)	9.4897(−3)
0.45	3.2015(−2)	1.9328(−2)	1.0808(−2)	9.6299(−3)	9.6341(−3)
0.50	3.2021(−2)	1.9471(−2)	1.0931(−2)	9.6745(−3)	9.6796(−3)
0.55	3.1714(−2)	1.9456(−2)	1.0954(−2)	9.6307(−3)	9.6368(−3)
0.60	3.1112(−2)	1.9300(−2)	1.0880(−2)	9.5045(−3)	9.5118(−3)
0.65	3.0223(−2)	1.9011(−2)	1.0710(−2)	9.2981(−3)	9.3069(−3)
0.70	2.9044(−2)	1.8591(−2)	1.0444(−2)	9.0108(−3)	9.0212(−3)
0.75	2.7563(−2)	1.8041(−2)	1.0086(−2)	8.6393(−3)	8.6516(−3)
0.80	2.5758(−2)	1.7355(−2)	9.6426(−3)	8.1779(−3)	8.1926(−3)
0.85	2.3592(−2)	1.6520(−2)	9.1268(−3)	7.6185(−3)	7.6359(−3)
0.90	2.0990(−2)	1.5508(−2)	8.5540(−3)	6.9468(−3)	6.9675(−3)
0.95	1.7749(−2)	1.4238(−2)	7.9238(−3)	6.1246(−3)	6.1495(−3)
1.00	1.1516(−2)	1.1742(−2)	6.8689(−3)	4.6364(−3)	4.6700(−3)

Table 6
The scattered component of the exiting intensity $\widehat{I}(a_0, -\mu, \phi)$ for $\phi - \phi_0 = \pi/2$

μ	Problem I	Problem II	Problem III	Problem IV	Problem V
0.00	0.0	0.0	0.0	0.0	0.0
0.05	3.5942(−3)	3.2149(−3)	1.7437(−3)	1.1631(−3)	1.1674(−3)
0.10	6.1646(−3)	5.5022(−3)	3.0106(−3)	2.0049(−3)	2.0123(−3)
0.15	8.0382(−3)	7.1702(−3)	3.9473(−3)	2.6251(−3)	2.6349(−3)
0.20	9.4215(−3)	8.4112(−3)	4.6493(−3)	3.0880(−3)	3.0997(−3)
0.25	1.0450(−2)	9.3489(−3)	5.1812(−3)	3.4371(−3)	3.4504(−3)
0.30	1.1215(−2)	1.0066(−2)	5.5876(−3)	3.7021(−3)	3.7167(−3)
0.35	1.1780(−2)	1.0618(−2)	5.9000(−3)	3.9042(−3)	3.9199(−3)
0.40	1.2191(−2)	1.1046(−2)	6.1412(−3)	4.0587(−3)	4.0755(−3)
0.45	1.2480(−2)	1.1375(−2)	6.3280(−3)	4.1768(−3)	4.1946(−3)
0.50	1.2670(−2)	1.1626(−2)	6.4726(−3)	4.2671(−3)	4.2860(−3)
0.55	1.2778(−2)	1.1813(−2)	6.5843(−3)	4.3361(−3)	4.3560(−3)
0.60	1.2818(−2)	1.1948(−2)	6.6702(−3)	4.3889(−3)	4.4100(−3)
0.65	1.2799(−2)	1.2036(−2)	6.7355(−3)	4.4298(−3)	4.4520(−3)
0.70	1.2728(−2)	1.2084(−2)	6.7843(−3)	4.4621(−3)	4.4856(−3)
0.75	1.2612(−2)	1.2097(−2)	6.8197(−3)	4.4888(−3)	4.5137(−3)
0.80	1.2454(−2)	1.2077(−2)	6.8443(−3)	4.5128(−3)	4.5390(−3)
0.85	1.2260(−2)	1.2029(−2)	6.8599(−3)	4.5365(−3)	4.5644(−3)
0.90	1.2035(−2)	1.1954(−2)	6.8682(−3)	4.5629(−3)	4.5925(−3)
0.95	1.1784(−2)	1.1857(−2)	6.8707(−3)	4.5950(−3)	4.6265(−3)
1.00	1.1516(−2)	1.1742(−2)	6.8689(−3)	4.6364(−3)	4.6700(−3)

The results reported in Tables 1, 2, and 4–10 are thought to be accurate to within ±1 in the last figure given and were obtained by running our FORTRAN code several times with increasing orders of the discrete-ordinates quadrature schemes until convergence was achieved. To this end, M, the number of quadrature points per subinterval, was varied typically between 20 and 500 in our calculations. In addition, to

Table 7
The scattered component of the exiting intensity $\widehat{I}(a_0, -\mu, \phi)$ for $\phi - \phi_0 = \pi$

μ	Problem I	Problem II	Problem III	Problem IV	Problem V
0.00	0.0	0.0	0.0	0.0	0.0
0.05	1.2077(−3)	2.1569(−3)	1.2316(−3)	7.0007(−4)	7.0798(−4)
0.10	2.0777(−3)	3.6954(−3)	2.1293(−3)	1.2091(−3)	1.2227(−3)
0.15	2.7208(−3)	4.8246(−3)	2.7983(−3)	1.5881(−3)	1.6061(−3)
0.20	3.2072(−3)	5.6745(−3)	3.3068(−3)	1.8765(−3)	1.8978(−3)
0.25	3.5832(−3)	6.3287(−3)	3.7008(−3)	2.1007(−3)	2.1246(−3)
0.30	3.8805(−3)	6.8430(−3)	4.0120(−3)	2.2787(−3)	2.3047(−3)
0.35	4.1218(−3)	7.2558(−3)	4.2629(−3)	2.4233(−3)	2.4511(−3)
0.40	4.3235(−3)	7.5937(−3)	4.4698(−3)	2.5438(−3)	2.5731(−3)
0.45	4.4984(−3)	7.8761(−3)	4.6447(−3)	2.6472(−3)	2.6778(−3)
0.50	4.6570(−3)	8.1172(−3)	4.7968(−3)	2.7387(−3)	2.7705(−3)
0.55	4.8085(−3)	8.3279(−3)	4.9331(−3)	2.8227(−3)	2.8556(−3)
0.60	4.9615(−3)	8.5172(−3)	5.0594(−3)	2.9027(−3)	2.9366(−3)
0.65	5.1257(−3)	8.6927(−3)	5.1804(−3)	2.9821(−3)	3.0170(−3)
0.70	5.3127(−3)	8.8621(−3)	5.3007(−3)	3.0642(−3)	3.1000(−3)
0.75	5.5381(−3)	9.0339(−3)	5.4248(−3)	3.1531(−3)	3.1897(−3)
0.80	5.8257(−3)	9.2198(−3)	5.5583(−3)	3.2541(−3)	3.2915(−3)
0.85	6.2160(−3)	9.4379(−3)	5.7093(−3)	3.3759(−3)	3.4139(−3)
0.90	6.7900(−3)	9.7230(−3)	5.8923(−3)	3.5356(−3)	3.5739(−3)
0.95	7.7636(−3)	1.0165(−2)	6.1447(−3)	3.7790(−3)	3.8169(−3)
1.00	1.1516(−2)	1.1742(−2)	6.8689(−3)	4.6364(−3)	4.6700(−3)

Table 8
The scattered component of the exiting intensity $\widehat{I}(a_K, \mu, \phi)$ for $\phi - \phi_0 = 0$

μ	Problem I	Problem II	Problem III	Problem IV	Problem V
0.00	0.0	0.0	0.0	0.0	0.0
0.05	3.7751(−2)	2.0384(−2)	9.1460(−3)	4.5382(−3)	4.7188(−3)
0.10	6.5069(−2)	3.4996(−2)	1.5698(−2)	7.8109(−3)	8.0994(−3)
0.15	8.5352(−2)	4.5586(−2)	2.0440(−2)	1.0234(−2)	1.0547(−2)
0.20	1.0075(−1)	5.3340(−2)	2.3905(−2)	1.2069(−2)	1.2337(−2)
0.25	1.1264(−1)	5.9071(−2)	2.6462(−2)	1.3489(−2)	1.3659(−2)
0.30	1.2196(−1)	6.3340(−2)	2.8365(−2)	1.4610(−2)	1.4648(−2)
0.35	1.2932(−1)	6.6536(−2)	2.9795(−2)	1.5511(−2)	1.5396(−2)
0.40	1.3509(−1)	6.8923(−2)	3.0878(−2)	1.6249(−2)	1.5970(−2)
0.45	1.3951(−1)	7.0675(−2)	3.1704(−2)	1.6863(−2)	1.6419(−2)
0.50	1.4269(−1)	7.1896(−2)	3.2332(−2)	1.7380(−2)	1.6776(−2)
0.55	1.4463(−1)	7.2637(−2)	3.2801(−2)	1.7819(−2)	1.7064(−2)
0.60	1.4526(−1)	7.2909(−2)	3.3135(−2)	1.8193(−2)	1.7299(−2)
0.65	1.4440(−1)	7.2685(−2)	3.3344(−2)	1.8507(−2)	1.7490(−2)
0.70	1.4182(−1)	7.1914(−2)	3.3425(−2)	1.8763(−2)	1.7643(−2)
0.75	1.3719(−1)	7.0519(−2)	3.3367(−2)	1.8955(−2)	1.7757(−2)
0.80	1.3012(−1)	6.8397(−2)	3.3141(−2)	1.9070(−2)	1.7825(−2)
0.85	1.2011(−1)	6.5398(−2)	3.2697(−2)	1.9085(−2)	1.7834(−2)
0.90	1.0645(−1)	6.1264(−2)	3.1940(−2)	1.8952(−2)	1.7752(−2)
0.95	8.7728(−2)	5.5375(−2)	3.0637(−2)	1.8554(−2)	1.7500(−2)
1.00	4.9361(−2)	4.1513(−2)	2.6568(−2)	1.6792(−2)	1.6307(−2)

gain confidence in the correctness of our implementation, we have used the Monte Carlo method to confirm, with a minimum of three figures of agreement, the numerical results for A and B that are listed in Tables 1, 2, and 4.

Table 9
The scattered component of the exiting intensity $\widehat{I}(a_K, \mu, \phi)$ for $\phi - \phi_0 = \pi/2$

μ	Problem I	Problem II	Problem III	Problem IV	Problem V
0.00	0.0	0.0	0.0	0.0	0.0
0.05	6.9129(−3)	9.1268(−3)	6.0344(−3)	3.3037(−3)	3.7711(−3)
0.10	1.1928(−2)	1.5666(−2)	1.0359(−2)	5.6882(−3)	6.4742(−3)
0.15	1.5685(−2)	2.0402(−2)	1.3495(−2)	7.4570(−3)	8.4338(−3)
0.20	1.8589(−2)	2.3870(−2)	1.5794(−2)	8.8023(−3)	9.8708(−3)
0.25	2.0909(−2)	2.6440(−2)	1.7501(−2)	9.8503(−3)	1.0938(−2)
0.30	2.2826(−2)	2.8374(−2)	1.8788(−2)	1.0687(−2)	1.1742(−2)
0.35	2.4470(−2)	2.9858(−2)	1.9778(−2)	1.1371(−2)	1.2360(−2)
0.40	2.5936(−2)	3.1030(−2)	2.0559(−2)	1.1945(−2)	1.2847(−2)
0.45	2.7296(−2)	3.1986(−2)	2.1196(−2)	1.2440(−2)	1.3242(−2)
0.50	2.8608(−2)	3.2801(−2)	2.1734(−2)	1.2880(−2)	1.3574(−2)
0.55	2.9920(−2)	3.3532(−2)	2.2209(−2)	1.3282(−2)	1.3864(−2)
0.60	3.1275(−2)	3.4220(−2)	2.2648(−2)	1.3659(−2)	1.4129(−2)
0.65	3.2713(−2)	3.4899(−2)	2.3070(−2)	1.4023(−2)	1.4380(−2)
0.70	3.4275(−2)	3.5599(−2)	2.3490(−2)	1.4382(−2)	1.4626(−2)
0.75	3.6006(−2)	3.6343(−2)	2.3921(−2)	1.4743(−2)	1.4874(−2)
0.80	3.7956(−2)	3.7154(−2)	2.4374(−2)	1.5113(−2)	1.5130(−2)
0.85	4.0183(−2)	3.8053(−2)	2.4858(−2)	1.5498(−2)	1.5397(−2)
0.90	4.2761(−2)	3.9062(−2)	2.5379(−2)	1.5903(−2)	1.5681(−2)
0.95	4.5781(−2)	4.0206(−2)	2.5947(−2)	1.6332(−2)	1.5983(−2)
1.00	4.9361(−2)	4.1513(−2)	2.6568(−2)	1.6792(−2)	1.6307(−2)

Table 10
The scattered component of the exiting intensity $\widehat{I}(a_K, \mu, \phi)$ for $\phi - \phi_0 = \pi$

μ	Problem I	Problem II	Problem III	Problem IV	Problem V
0.00	0.0	0.0	0.0	0.0	0.0
0.05	1.8660(−3)	5.0509(−3)	4.1559(−3)	2.4548(−3)	3.0070(−3)
0.10	3.2260(−3)	8.6759(−3)	7.1379(−3)	4.2286(−3)	5.1637(−3)
0.15	4.2573(−3)	1.1313(−2)	9.3059(−3)	5.5482(−3)	6.7301(−3)
0.20	5.0725(−3)	1.3259(−2)	1.0904(−2)	6.5572(−3)	7.8826(−3)
0.25	5.7466(−3)	1.4723(−2)	1.2103(−2)	7.3500(−3)	8.7436(−3)
0.30	6.3320(−3)	1.5851(−2)	1.3022(−2)	7.9910(−3)	9.3995(−3)
0.35	6.8676(−3)	1.6749(−2)	1.3748(−2)	8.5252(−3)	9.9120(−3)
0.40	7.3843(−3)	1.7497(−2)	1.4343(−2)	8.9854(−3)	1.0326(−2)
0.45	7.9090(−3)	1.8154(−2)	1.4855(−2)	9.3961(−3)	1.0674(−2)
0.50	8.4674(−3)	1.8768(−2)	1.5320(−2)	9.7764(−3)	1.0981(−2)
0.55	9.0866(−3)	1.9379(−2)	1.5767(−2)	1.0142(−2)	1.1267(−2)
0.60	9.7988(−3)	2.0022(−2)	1.6220(−2)	1.0506(−2)	1.1547(−2)
0.65	1.0644(−2)	2.0734(−2)	1.6703(−2)	1.0882(−2)	1.1833(−2)
0.70	1.1680(−2)	2.1556(−2)	1.7237(−2)	1.1281(−2)	1.2139(−2)
0.75	1.2988(−2)	2.2539(−2)	1.7850(−2)	1.1720(−2)	1.2477(−2)
0.80	1.4700(−2)	2.3755(−2)	1.8575(−2)	1.2216(−2)	1.2862(−2)
0.85	1.7050(−2)	2.5325(−2)	1.9464(−2)	1.2798(−2)	1.3314(−2)
0.90	2.0517(−2)	2.7478(−2)	2.0610(−2)	1.3514(−2)	1.3869(−2)
0.95	2.6402(−2)	3.0808(−2)	2.2241(−2)	1.4480(−2)	1.4609(−2)
1.00	4.9361(−2)	4.1513(−2)	2.6568(−2)	1.6792(−2)	1.6307(−2)

While an implementation of our solution for a considered problem, say problem V—our most challenging case, requires, in general, some hours of computer time to establish the high-quality results we are reporting in Tables 4–10, a solution good enough for graphical presentation requires very modest computational expense. To have an idea of the CPU time for what we might consider "practical results," we found, for example, that the results given in Tables 4–10 for this case could be obtained (without making a serious effort to use the code in the most efficient way in low order and considering all 101 Fourier-component problems) with essentially three figures of accuracy in less than 2 min on an Intel Core 2 Duo machine running at 2.6 GHz.

To close this section, we note that we considered, as an additional test of our computations, a version of each of Problems I–V for which all values of ϖ_k were taken to be unity. For these conservative cases, we used $M = 200$ to find $A + B = 0.99999999$ (or better) which clearly compares well to the exact result $A + B = 1$.

7. Concluding remarks

In this work, we have used the ADO method to develop a general solution of the equation of transfer for the case of a multi-layer medium subject to Fresnel boundary and interface conditions and illuminated by obliquely incident parallel rays. We note that the unscattered solution given by Eqs. (3.2) is expressed in a simple form which is very convenient for computation. In addition, while the way in which Chandrasekhar [1] found a simple particular solution, relevant to the source term introduced when the unscattered and scattered components of the intensity are separated, could have been generalized here, as done in other works [8,14,17], we have taken an alternative route so that the particular solution developed here does not suffer from the limitations on the solution that result when the simple approach is used (see Ref. [20, Section 5] for a discussion of this point).

Finally, we believe that the highly accurate numerical results that are reported in Section 6 illustrate well the power of the method and provide a high standard against which the accuracy of other implementations/methods can be reliably assessed.

Acknowledgments

The work of one of the authors (RDMG) was supported by CNPq of Brazil. The authors are grateful to Professor C.-Y. Wu for suggesting the useful accuracy test mentioned at the end of Section 6.

References

[1] Chandrasekhar S. Radiative transfer. London: Oxford University Press; 1950.
[2] Liou KN. An introduction to atmospheric radiation. New York: Academic Press; 1980.
[3] Mobley CD. Light and water; radiative transfer in natural waters. San Diego: Academic Press; 1994.
[4] Ishimaru A. Wave propagation and scattering in random media. Piscataway, NJ: IEEE Press; 1997.
[5] Simonot L, Elias M, Charron E. Special visual effect of art glazes explained by the radiative transfer equation. Appl Opt 2004;43:2580–7.
[6] Lenoble J (ed.). Radiative transfer in scattering and absorbing atmospheres: standard computational procedures, Hampton, VA: A. Deepak; 1985.
[7] Tanaka M, Nakajima T. Effects of oceanic turbidity and index of refraction of hydrosols on the flux of solar radiation in the atmosphere-ocean system. JQSRT 1977;18:93–111.
[8] Jin Z, Stamnes K. Radiative transfer in nonuniformly refracting layered media: atmosphere-ocean system. Appl Opt 1994;33:431–42.
[9] Yan B, Stamnes K. Fast yet accurate computation of the complete radiance distribution in the coupled atmosphere-ocean system. JQSRT 2003;76:207–23.
[10] Wu C-Y, Liou B-T. Radiative transfer in a two-layer slab with Fresnel interfaces. JQSRT 1996;56:573–89.
[11] Liou B-T, Wu C-Y. Composite discrete-ordinate solutions for radiative transfer in a two-layer medium with Fresnel interfaces. Numer Heat Transfer A 1996;30:739–51.
[12] Liou B-T, Wu C-Y. Radiative transfer in a multi-layer medium with Fresnel interfaces. Heat Mass Transfer 1996;32:103–7.
[13] Elias M, Elias G. Radiative transfer in inhomogeneous stratified scattering media with use of the auxiliary function method. J Opt Soc Am A 2004;21:580–9.
[14] Caron J, Andraud C, Lafait J. Radiative transfer calculations in multilayer systems with smooth or rough interfaces. J Mod Opt 2004;51:575–95.
[15] Williams MMR. On an integral equation arising in the transport of radiation through a slab involving internal reflection II. Generalisation and numerical results for the Fresnel case. Eur Phys J B 2006;53:245–55.
[16] Siewert CE. A note on radiative transfer in a finite layer. J Phys A: Math Theor 2007;40:1785–9.
[17] Elaloufi R, Arridge S, Pierrat R, Carminati R. Light propagation in multilayered scattering media beyond the diffusive regime. Appl Opt 2007;46:2528–39.

[18] Garcia RDM, Siewert CE, Yacout AM. On the use of Fresnel boundary and interface conditions in radiative transfer calculations for multilayered media. JQSRT 2008;109:752–69.
[19] Barichello LB, Siewert CE. A discrete-ordinates solution for a non-grey model with complete frequency redistribution. JQSRT 1999; 62:665–75.
[20] Siewert CE. A concise and accurate solution to Chandrasekhar's basic problem in radiative transfer. JQSRT 2000;64:109–30.
[21] Barichello LB, Garcia RDM, Siewert CE. Particular solutions for the discrete-ordinates method. JQSRT 2000;64:219–26.
[22] Garcia RDM, Siewert CE. Multislab multigroup transport theory with *L*th order anisotropic scattering. J Comput Phys 1983;50:181–92.
[23] Dongarra JJ, Bunch JR, Moler CD, Stewart GW. LINPACK users' guide. Philadelphia: SIAM; 1979.
[24] Kaper HG, Shultis JK, Veninga JG. Numerical evaluation of the slab albedo problem solution in one-speed anisotropic transport theory. J Comput Phys 1970;6:288–313.
[25] McCormick NJ, Sanchez R. Inverse problem transport calculations for anisotropic scattering coefficients. J Math Phys 1981;22: 199–208.