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Two-Group Neutron Transport Theory in Spherical Geometry

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A set of normal modes for the two-group steady-state neutron transport equation in spherical geometry is constructed. The singular eigenfunction-expansion technique is then used to develop a rigorous solution to the isotropically emitting spherical shell-source problem in an infinite medium.

I. INTRODUCTION

The singular eigenfunction-expansion technique introduced by Case has been used extensively in the areas of neutron transport theory and radiative transfer to construct rigorous solutions to a certain class of model problems.[1–5] This method has enjoyed particular success for energy-dependent problems,[5] for time-dependent theory,[6] for anisotropic scattering models,[3] for reactor-cell calculations,[7] and for several astrophysical applications.[4,8] Although Case's normal-mode expansion technique has been found suitable for a large number of applications, one of the major restrictions of the method is the difficulty with which the extension to nonplanar geometries is made.

Mitsis, by introducing a transform technique, solved the critical-sphere problem and he made an exhaustive study of the normal modes of the one-speed equation with spherical symmetry.[9] Leonard and Mullikin[10] and Erdmann and Siewert[11] also solved several problems in spherical one-speed theory. In the latter paper, two distinct approaches to spherical problems were employed: The first relied upon the spherical-to-plane geometry transformation for the density, and the second utilized more directly the normal modes of the equation for the angular density.

The N-group formulation discussed by Davison has been employed for investigating energy-dependent problems in neutron-transport theory.[12] This model also has been examined in light of the Case technique[13]; Leonard and Ferziger,[14] Siewert and Shieh,[15] and Yoshimura and Katsuragi[16] have made contributions to the theory of multigroup neutron transport, and Metcalf and Zweifel[17] have used this work to make numerical calculations for the Milne problem in two-group theory.

The purpose of the present paper is to blend the methods of Erdmann and Siewert[11] for spherical problems with the two-group analysis of Siewert and Shieh[15] in order to solve the isotropically emitting spherical-shell source problem for the two-group model in an infinite medium. In Sec. II the basic equations for this problem are given, and the normal modes of the two-group equation in spherical geometry are constructed, while Sec. III is devoted to the solution of the considered problem.

II. GENERAL ANALYSIS

We consider the Green's function associated with an isotropically emitting spherical-shell source in an infinite medium. Thus we seek a solution to the time-independent transport equation

$$\mu \frac{\partial}{\partial r} \mathbf{\Psi}(r_0: r, \mu) + \frac{(1-\mu^2)}{r}\frac{\partial}{\partial \mu} \mathbf{\Psi}(r_0: r, \mu) + \mathbf{\Sigma}\mathbf{\Psi}(r_0: r, \mu) = \mathbf{C}\int_{-1}^{1} \mathbf{\Psi}(r_0: r, \mu')\, d\mu' + \frac{\delta(r-r_0)}{8\pi r^2}\mathbf{Q}, \quad (1)$$

subject to the constraint that $\mathbf{\Psi}(r_0;\ r, \mu)$ must be bounded for all r, since the considered medium must be nonmultiplying. Here $\mathbf{\Psi}(r_0: r, \mu)$ is a vector whose two components represent the angular neutron fluxes in each of the energy groups, μ is the direction cosine of the propagating radiation, and r is the optical variable defined in terms of the smaller of the two total cross sections. Thus, the $\mathbf{\Sigma}$ matrix takes the form

$$\mathbf{\Sigma} = \begin{vmatrix} \sigma & 0 \\ 0 & 1 \end{vmatrix}, \quad \sigma > 1, \quad (2)$$

where σ is the ratio of the total cross section in the first group to the total cross section in the second group, and $\mathbf{C}$ is the transfer matrix with elements c_{ij}. In addition the components of $\mathbf{Q}$, q_1 and q_2, are used to indicate the intensities of the two group sources.

In the usual manner,[2] we need consider only the homogeneous version of Eq. (1); we thus replace the source term by the equivalent boundary condition

$$\mu[\mathbf{\Psi}(r_0: r_0^+, \mu) - \mathbf{\Psi}(r_0: r_0^-, \mu)] = (1/8\pi r_0^2)\mathbf{Q}. \quad (3)$$

We should like to construct the solution to this problem in a manner analogous to that used so successfully in plane geometry, i.e., Case's method of singular eigenfunction expansions.[1] First, a general set of solutions, denoted as normal modes, to the homogeneous transport equation is determined. The desired solution is then written as a linear sum of these normal modes, and the arbitrary expansion coefficients in this sum are selected such that the boundary conditions of the problem are satisfied. This procedure is, of course, a classical technique; however, in contrast to problems in plane geometry where the necessary completeness theorems are usually available,[2] the solution here cannot be effected quite so readily.

We begin by constructing a set of normal modes for the equation

$$\left[\mu \frac{\partial}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial}{\partial \mu} + \mathbf{\Sigma}\right]\mathbf{\Psi}(r,\mu) = \mathbf{C}\int_{-1}^{1} \mathbf{\Psi}(r,\mu')\, d\mu'. \quad (4)$$

Careful inspection of the similarities between the normal modes in one-speed transport theory for plane[2] and spherical[11] geometries suggests a form for the solutions here. In addition, the relationships between the normal modes for one-speed[2] and two-group[15] theory in plane geometry may also be used to advantage. It is therefore proposed that Eq. (4) has solutions of the form

$$\mathbf{\Psi}_\eta(r,\mu) = \sum_{m=0}^{\infty} [\tfrac{1}{2}(2m+1)]P_m(\mu)U_m(r,\eta)\mathbf{G}_m(\eta). \quad (5)$$

Here the Legendre polynomials are represented by $P_m(\mu)$, and

$$U_m(r,\eta) = A(\eta)k_m(r/\eta) + B(\eta)(-1)^m i_m(r/\eta), \quad (6)$$

with $A(\eta)$, $B(\eta)$, and, at this point, η being arbitrary. In addition,

$$i_m(x) = (\pi/2x)^{\frac{1}{2}} I_{m+\frac{1}{2}}(x) \quad (7a)$$

and

$$k_m(x) = (\pi/2x)^{\frac{1}{2}} K_{m+\frac{1}{2}}(x). \quad (7b)$$

Following Watson's notation, we have used $I_{m+\frac{1}{2}}(x)$ and $K_{m+\frac{1}{2}}(x)$ to denote the modified Bessel functions.[18]

If the proposed solution is substituted into Eq. (4), we observe that the G-vectors must be solutions of the recursion relation

$$(2m+1)\eta\mathbf{\Sigma}\mathbf{G}_m(\eta) = 2\eta\mathbf{C}\mathbf{G}_0(\eta)\delta_{0,m} + (m+1)\mathbf{G}_{m+1}(\eta) + m\mathbf{G}_{m-1}(\eta), \quad m = 0, 1, 2, \cdots. \quad (8)$$

In order to establish Eq. (8) the following expressions have been utilized[19]:

$$(2m+1)\mu P_m(\mu) = (m+1)P_{m+1}(\mu) + mP_{m-1}(\mu), \quad (9a)$$

$$(1-\mu^2)\frac{d}{d\mu}P_m(\mu) = (m+1)[\mu P_m(\mu) - P_{m+1}(\mu)], \quad (9b)$$

$$\frac{d}{dr}U_m(r,\eta) = -\frac{1}{\eta}U_{m-1}(r,\eta) - \frac{1}{r}(m+1)U_m(r,\eta), \quad (9c)$$

and

$$\eta(2m+1)U_m(r,\eta) = r[U_{m+1}(r,\eta) - U_{m-1}(r,\eta)]. \quad (9d)$$

Previous work by Siewert and Shieh[15] can now be used to find a set of G-vectors and thus to complete the justification of the solutions given by Eq. (5). In Ref. 15 an eigenvalue equation

$$(\eta\mathbf{\Sigma} - \mu\mathbf{I})\mathbf{F}(\eta,\mu) = \eta\mathbf{C}\int_{-1}^{1}\mathbf{F}(\eta,\mu')\, d\mu' \quad (10)$$

was encountered, and the eigenvalue spectrum and corresponding eigenvectors were established. If we multiply Eq. (10) by $P_m(\mu)$, integrate over μ from -1 to 1, and make the identification

$$\mathbf{G}_m(\eta) \triangleq \int_{-1}^{1} P_m(\mu)\mathbf{F}(\eta,\mu)\, d\mu, \quad (11)$$

we note that $\mathbf{G}_m(\eta)$ will be a solution to Eq. (8). Since $\mathbf{F}(\eta,\mu)$ is known for all acceptable values of η in the complex plane,[15] the G-vectors as given by Eq. (11) are determined; more explicitly, the discrete spectrum yields

$$\mathbf{G}_m(\eta_i) = \begin{vmatrix} 2c_{12}\eta_i Q_m(\sigma\eta_i) \\ 2\eta_i[c_{22} - 2C\eta_i T(1/\sigma\eta_i)]Q_m(\eta_i) \end{vmatrix}, \quad (12)$$

where the η_i are the "positive" zeros of the dispersion function

$$\Omega(z) = 1 - 2c_{11}zT(1/\sigma z) - 2c_{22}zT(1/z) + 4Cz^2T(1/z)T(1/\sigma z), \quad (13)$$

the degenerate spectrum $\eta \in (0, 1/\sigma)$ yields

$$\mathbf{G}_{1,m}(\eta) = \begin{vmatrix} -c_{12}P_m(\sigma\eta) \\ 2CZ_m(\eta) + c_{11}P_m(\eta) \end{vmatrix}, \quad \eta \in (0, 1/\sigma), \quad (14a)$$

$$\mathbf{G}_{2,m}(\eta) = \begin{vmatrix} (2C/\sigma)Z_m(\sigma\eta) + c_{22}P_m(\sigma\eta) \\ -c_{21}P_m(\eta) \end{vmatrix}, \quad \eta \in (0, 1/\sigma), \quad (14b)$$

and the spectrum $\eta \in (1/\sigma, 1)$ leads to

$$\mathbf{G}_{3,m}(\eta) = \begin{vmatrix} 2c_{12}\eta Q_m(\sigma\eta) \\ 2[c_{22} - 2\eta CT(1/\sigma\eta)]Z_m(\eta) + P_m(\eta)[1 - c_{11}\eta T(1/\sigma\eta)] \end{vmatrix}, \quad \eta \in (1/\sigma, 1). \quad (15)$$

Here we have used the notation $T(x) = \tanh^{-1} x$ and $C = \det \mathbf{C}$; the Legendre functions of the second kind are denoted by $Q_m(x)$, and the mth-order polynomials $Z_m(x)$,

$$Z_m(x) \triangleq \tfrac{1}{2} x P \int_{-1}^{1} \frac{P_m(\mu)}{x-\mu}\, d\mu - xT(x)P_m(x), \quad (16)$$

satisfy the same recursion formula, viz., Eq. (9a), as the Legendre polynomials; they begin differently, however,

$$Z_0(x) = 0, \quad Z_1(x) = -x, \quad \text{and} \quad Z_2(x) = -\tfrac{3}{2}x^2.$$

Now that a set of normal modes for Eq. (4) has been established, and before continuing to the final section where the considered problem is solved, several additional comments can be made: (a) Although the construction here of the normal modes has not been a formal derivation, these results should follow from an analysis similar to that used by Mitsis[9] for one-speed theory. (b) By no means have we proved that all solutions to Eq. (4) are given by these results; on the other hand, we do have solutions sufficiently general for the construction of the solution to the shell-source problem. (c) We have used the results for $\mathbf{F}(\eta, \mu)$ given by Siewert and Shieh[15] for the two-group model; however, the method employed here may also be used to extend the N-group theory of Yoshimura and Katsuragi[16] to spherical geometry.

III. SPHERICAL SHELL-SOURCE PROBLEM

We seek a bounded solution to Eq. (1) or, alternatively, a bounded solution to Eq. (4) subject to the "jump" boundary condition, Eq. (3). Since the Bessel functions $i_m(x)$ diverge as x increases without bound, and since the $k_m(x)$ behave similarly in the vicinity of the origin, we separate the desired solution in the usual manner[2]:

$$\mathbf{\Psi}(r_0: r, \mu) = \sum_{m=0}^{\infty} [\tfrac{1}{2}(2m+1)] P_m(\mu) \mathbf{R}_m^+(r), \quad r > r_0, \quad (17a)$$

and

$$\mathbf{\Psi}(r_0: r, \mu) = \sum_{m=0}^{\infty} [\tfrac{1}{2}(2m+1)] P_m(\mu) \mathbf{R}_m^-(r), \quad r < r_0, \quad (17b)$$

where

$$\mathbf{R}_m^+(r) \triangleq \sum_i A(\eta_i)\mathbf{G}_m(\eta_i)k_m(r/\eta_i) + \int_0^{1/\sigma} [A_1(\eta)\mathbf{G}_{1,m}(\eta) + A_2(\eta)\mathbf{G}_{2,m}(\eta)]k_m(r/\eta)\, d\eta + \int_{1/\sigma}^{1} A_3(\eta)\mathbf{G}_{3,m}(\eta)k_m(r/\eta)\, d\eta \quad (18a)$$

and

$$\mathbf{R}_m^-(\eta_i) \triangleq \sum_i B(\eta_i)\mathbf{G}_m(\eta_i)(-1)^m i_m(r/\eta_i) + \int_0^{1/\sigma} [B_1(\eta)\mathbf{G}_{1,m}(\eta) + B_2(\eta)\mathbf{G}_{2,m}(\eta)](-1)^m i_m(r/\eta)\, d\eta + \int_{1/\sigma}^{1} B_3(\eta)\mathbf{G}_{3,m}(\eta)(-1)^m i_m(r/\eta)\, d\eta. \quad (18b)$$

The solution given by Eqs. (17) clearly satisfies the homogeneous transport equation; there remains then only the necessity to constrain this solution to meet the condition given by Eq. (3) and thus to determine all of the unknown expansion coefficients, $A(\eta_i)$, $B(\eta_i)$, $A_\alpha(\eta)$, and $B_\alpha(\eta)$, $\alpha = 1, 2, 3$, appearing in the expression for $\mathbf{\Psi}(r, \mu)$. We therefore substitute Eqs. (17) into Eq. (3) to find

$$\sum_{m=0}^{\infty} P_m(\mu)[(m+1)\mathbf{S}_{m+1}(r_0) + m\mathbf{S}_{m-1}(r_0)] = \mathbf{Q} \quad (19)$$

or, alternatively,

$$(m+1)\mathbf{S}_{m+1}(r_0) + m\mathbf{S}_{m-1}(r_0) = \mathbf{Q}\delta_{0,m}, \quad m = 0, 1, 2, \cdots, \quad (20)$$

where we have defined

$$\mathbf{S}_m(r_0) \triangleq 4\pi r_0^2[\mathbf{R}_m^+(r_0) - \mathbf{R}_m^-(r_0)]. \quad (21)$$

Noting that the neutron flux

$$\mathbf{\phi}(r_0: r) \triangleq \int_{-1}^{1} \mathbf{\Psi}(r_0: r, \mu')\, d\mu' \quad (22)$$

is to be continuous across the surface $r = r_0$, we observe that $\mathbf{S}_0(r_0) = 0$, and thus Eq. (20) yields the following sufficiency conditions on the unknown vectors $\mathbf{S}_m(r_0)$:

$$\mathbf{S}_m(r_0) = 0, \quad \text{for } m \text{ even}, \quad (23a)$$

$$\mathbf{S}_1(r_0) = \mathbf{Q}, \quad (23b)$$

and

$$\mathbf{S}_m(r_0) = -\frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots (m-1)}{3 \cdot 5 \cdot 7 \cdot 9 \cdots (m)} (-1)^{\frac{1}{2}(m+1)} \mathbf{Q}, \quad m = 3, 5, 7, \cdots. \quad (23c)$$

Equations (23) now represent the conditions from which we must extract the necessary results for all unknown expansion coefficients. Following the procedure used by Erdmann and Siewert,[11] we shall first show how to satisfy these conditions for the cases $m = 0$ and 1; further analysis will then reveal that these conditions are met for all m.

Referring to Eqs. (18), we note that $\mathbf{S}_0(r_0)$ can be made identically zero simply by insisting that

$$\frac{A(\eta_i)}{i_0(r_0/\eta_i)} = \frac{B(\eta_i)}{k_0(r_0/\eta_i)} \triangleq D(\eta_i) \qquad (24a)$$

and

$$\frac{A_\alpha(\eta)}{i_0(r_0/\eta)} = \frac{B_\alpha(\eta)}{k_0(r_0/\eta)} \triangleq D_\alpha(\eta), \quad \eta \in (0, 1), \quad \alpha = 1, 2, 3. \qquad (24b)$$

We now utilize Eqs. (18), (21), and (23) to write the condition on $\mathbf{S}_1(r_0)$ in the form

$$(2\pi^2)^{-1}\mathbf{Q} = \sum_i D(\eta_i)\mathbf{G}_1(\eta_i)\eta_i^2 + \int_0^{1/\sigma} [D_1(\eta)\mathbf{G}_{1,1}(\eta) + D_2(\eta)\mathbf{G}_{2,1}(\eta)]\eta^2\, d\eta + \int_{1/\sigma}^1 D_3(\eta)\mathbf{G}_{3,1}(\eta)\eta^2\, d\eta, \qquad (25)$$

where we have made use of the property

$$i_0(x)k_m(x) - (-1)^m i_m(x)k_0(x) = \frac{\pi}{4x^2}\sum_{\alpha=0}^{m}\frac{1}{(2x)^\alpha} W_\alpha^m[1 - (-1)^{m+\alpha}], \qquad (26a)$$

with[19]

$$W_\alpha^m = (m+\alpha)!/\alpha!\,(m-\alpha)!. \qquad (26b)$$

The form of Eq. (25) is suggestive of a full-range expansion in terms of the eigenvectors $\mathbf{F}(\eta, \mu)$ of Eq. (10). We therefore use the results of Siewert and Shieh[15] and write

$$(2\pi^2)^{-1}\mathbf{Q} = \mu\Big\{\sum_i D(\eta_i)[\mathbf{F}_{i+}(\mu) - \mathbf{F}_{i-}(\mu)]\eta_i^2 + \int_0^{1/\sigma} D_1(\eta)[\mathbf{F}_1^{(1)}(\eta, \mu) - \mathbf{F}_1^{(1)}(-\eta, \mu)]\eta^2\, d\eta + \int_0^{1/\sigma} D_2(\eta)[\mathbf{F}_2^{(1)}(\eta, \mu) - \mathbf{F}_2^{(1)}(-\eta, \mu)]\eta^2\, d\eta + \int_{1/\sigma}^1 D_3(\eta)[\mathbf{F}^{(2)}(\eta, \mu) - \mathbf{F}^{(2)}(-\eta, \mu)]\eta^2\, d\eta\Big\}. \qquad (27)$$

The various $\mathbf{F}$-vectors appearing in Eq. (27) are given explicitly in Ref. 15 [see Eqs. (6), (7), and (10)] and, for the sake of brevity, will not be repeated here. We note, however, that $\mathbf{F}(\eta, -\mu) = \mathbf{F}(-\eta, \mu)$, and thus the full-range completeness theorem[15] ensures that Eq. (27) has a solution. In addition the full-range orthogonality theorem[15] may be employed to obtain explicit results for the unknown coefficients in Eq. (27):

$$D(\eta_i) = [2\pi^2\eta_i^2 N(\eta_i)]^{-1}\tilde{\mathbf{G}}_0^\dagger(\eta_i)\mathbf{Q}, \qquad (28a)$$

$$D_1(\eta) = [2\pi^2\eta^2 N_1(\eta)]^{-1} \times [N_{22}(\eta)\tilde{\mathbf{G}}_{1,0}^\dagger(\eta) - N_{12}(\eta)\tilde{\mathbf{G}}_{2,0}^\dagger(\eta)]\mathbf{Q}, \qquad (28b)$$

$$D_2(\eta) = [2\pi^2\eta^2 N_1(\eta)]^{-1} \times [N_{11}(\eta)\tilde{\mathbf{G}}_{2,0}^\dagger(\eta) - N_{21}(\eta)\tilde{\mathbf{G}}_{1,0}^\dagger(\eta)]\mathbf{Q}, \qquad (28c)$$

and

$$D_3(\eta) = [2\pi^2\eta^2 N_2(\eta)]^{-1}\tilde{\mathbf{G}}_{3,0}^\dagger(\eta)\mathbf{Q}. \qquad (28d)$$

Here the superscript tilde denotes the transpose operation, the superscript dagger indicates an interchange of c_{ij} and c_{ji},

$$N_{11}(\eta) = \eta\{c_{12}c_{21} + [c_{11} - 2\eta CT(\eta)]^2 + \pi^2C^2\eta^2\}, \qquad (29a)$$

$$N_{22}(\eta) = \eta\{c_{12}c_{21} + [c_{22} - 2\eta CT(\sigma\eta)]^2 + \pi^2C^2\eta^2\}, \qquad (29b)$$

and

$$N_{ij}(\eta) = -c_{ji}\eta\{c_{11} + c_{22} - 2\eta C[T(\eta) + T(\sigma\eta)]\}, \quad \text{for} \quad i \neq j. \qquad (29c)$$

In addition,

$$N(\eta_i) = \eta_i^2\Big[c_{22} - 2C\eta_i T\Big(\frac{1}{\sigma\eta_i}\Big)\Big]\frac{d}{dz}\,\Omega(z)\Big|_{z=\eta_i}, \qquad (30a)$$

$$N_1(\eta) = \eta^2C^2(\{1 - 2\eta c_{11}T(\sigma\eta) - 2\eta c_{22}T(\eta) + \eta^2C[4T(\eta)T(\sigma\eta) - \pi^2]\}^2 + \pi^2\eta^2\{2C\eta[T(\eta) + T(\sigma\eta)] - c_{11} - c_{22}\}^2), \qquad (30b)$$

and

$$N_2(\eta) = \eta(\{1 - 2\eta c_{11}T(1/\sigma\eta) - 2\eta c_{22}T(\eta) + 4\eta^2CT(\eta)T(1/\sigma\eta)\}^2 + \pi^2\eta^2\{c_{22} - 2\eta CT(1/\sigma\eta)\}^2). \qquad (30c)$$

If Eq. (27) is integrated over μ from -1 to 1, the resulting equation is identical with Eq. (25), and thus the expressions given by Eqs. (28) for the expansion coefficients $D(\eta_i)$ and $D_\alpha(\eta)$, $\alpha = 1, 2, 3$, coupled with Eqs. (24), ensure that the conditions on $\mathbf{S}_m(r_0)$ are satisfied for $m = 0$ and 1. It remains to be shown that these coefficients are correct for all m.

We begin the proof by considering the explicit expression

$$\mathbf{S}_m(r_0) = \tfrac{1}{2}\sum_{k=0}^{m}[W_k^m/(2r_0)^k][1 - (-1)^{m+k}]\mathbf{J}_k^m, \quad m = 0, 1, 2, \cdots, \qquad (31a)$$

where we have used Eqs. (24) and (26) and defined

$$\mathbf{J}_k^m \triangleq 2\pi^2\Big\{\sum_i D(\eta_i)\mathbf{G}_m(\eta_i)\eta_i^{k+2} + \int_0^{1/\sigma}[D_1(\eta)\mathbf{G}_{1,m}(\eta) + D_2(\eta)\mathbf{G}_{2,m}(\eta)]\eta^{k+2}\,d\eta + \int_{1/\sigma}^1 D_3(\eta)\mathbf{G}_{3,m}(\eta)\eta^{k+2}\,d\eta\Big\}. \tag{31b}$$

In light of this expression the constraints on $\mathbf{S}_m(r_0)$ given by Eqs. (23) can be imposed by requiring

$$\mathbf{J}_k^m = \mathbf{0}, \quad 0 < k < m, \quad m + k = 3, 5, 7, \cdots, \tag{32a}$$

$$\mathbf{J}_0^1 = \mathbf{Q}, \tag{32b}$$

and

$$\mathbf{J}_0^m = -\frac{2\cdot 4\cdot 6\cdots(m-1)(-1)^{\frac{1}{2}(m+1)}}{3\cdot 5\cdot 7\cdots(m)}\mathbf{Q}, \quad m = 3, 5, 7, \cdots. \tag{32c}$$

If we multiply Eq. (27) by $\mu^{k-1}P_m(\mu)$ and integrate over μ from -1 to 1, we find

$$(4\pi^2)^{-1}\int_{-1}^1 \mu^{k-1}P_m(\mu)\,d\mu\mathbf{Q} = \sum_i D(\eta_i)\eta_i^2\int_{-1}^1 P_m(\mu)\mu^k\mathbf{F}_{i+}(\mu)\,d\mu + \int_0^{1/\sigma}\left[D_1(\eta)\int_{-1}^1 P_m(\mu)\mu^k\mathbf{F}_1^{(1)}(\eta,\mu)\,d\mu + D_2(\eta)\int_{-1}^1 P_m(\mu)\mu^k\mathbf{F}_2^{(1)}(\eta,\mu)\right]\eta^2\,d\eta + \int_{1/\sigma}^1 D_3(\eta)\int_{-1}^1 P_m(\mu)\mu^k\mathbf{F}^{(2)}(\eta,\mu)\,d\mu\eta^2\,d\eta,$$

$$m > 0, \quad m + k = 1, 3, 5, \cdots. \tag{33}$$

For $k = 0$ we observe that the right-hand side of Eq. (33) is $\mathbf{J}_0^m/2\pi^2$ and thus for all odd m we obtain

$$\mathbf{J}_0^m = \frac{1}{2}\int_{-1}^1 P_m(\mu)\frac{d\mu}{\mu}\mathbf{Q}, \quad m = 1, 3, 5, \cdots. \tag{34}$$

The integral in the above result is nonsingular for m odd, and, in fact, Eq. (34) establishes Eqs. (32b) and (32c).

If we now consider $0 < k < m$, the left-hand side of Eq. (33) is clearly zero. Furthermore, inspection of the eigenvectors in Ref. 15 reveals that we may use the expression

$$\frac{\mu^k}{a\xi - \mu} = -\mu^{k-1} - a\xi\mu^{k-2} - (a\xi)^2\mu^{k-3} - \cdots + \frac{(a\xi)^k}{(a\xi - \mu)}, \quad \text{for} \quad a = \sigma \text{ or } 1, \tag{35}$$

to write

$$\int_{-1}^1 P_m(\mu)\mu^k\mathbf{F}(\xi,\mu)\,d\mu = \int_{-1}^1 P_m(\mu)\begin{vmatrix} -\mu^{k-1} - (\sigma\xi)\mu^{k-2} - \cdots - (\sigma\xi)^{k-1} \\ -\mu^{k-1} - \xi\mu^{k-2} - \cdots - \xi^{k-1}\end{vmatrix}d\mu + \xi^k\mathbf{\Sigma}^k\int_{-1}^1 P_m(\mu)\mathbf{F}(\xi,\mu)\,d\mu,$$

$$\text{for} \quad \xi = \eta_i \quad \text{or} \quad \eta \in (0, 1), \tag{36}$$

with $\mathbf{\Sigma}^k$ denoting the kth power of $\mathbf{\Sigma}$. For the values of k considered, Eq. (36) reduces to

$$\int_{-1}^1 P_m(\mu)\mu^k\mathbf{F}(\xi,\mu)\,d\mu = \xi^k\mathbf{\Sigma}^k\mathbf{G}_m(\xi),$$

$$\text{for} \quad \xi = \eta_i \quad \text{or} \quad \eta\in(0,1), \quad 0 < k < m, \quad m + k = 3, 5, 7, \cdots, \tag{37}$$

where $\mathbf{G}_m(\xi)$ is defined by Eq. (11). We note that Eq. (33) now may be written as

$$\mathbf{\Sigma}^k\mathbf{J}_k^m = \mathbf{0}, \quad 0 < k < m, \quad m + k = 3, 5, 7, \cdots. \tag{38}$$

Since $\mathbf{\Sigma}$ is a nonsingular matrix, Eq. (32a) follows directly, and the proof is complete.

Having successfully determined all the unknown expansion coefficients, we thus have a complete solution for the angular flux $\mathbf{\Psi}(r_0\colon r, \mu)$, and explicit expressions for the flux $\boldsymbol{\phi}(r_0\colon r)$ and the current,

$$\mathbf{j}(r_0\colon r) \triangleq \int_{-1}^1 \mathbf{\Psi}(r_0\colon r, \mu)\mu\,d\mu, \tag{39}$$

are immediately available:

$$\boldsymbol{\phi}(r_0\colon r) = (\pi/2rr_0)\Big\{\sum_i D(\eta_i)\mathbf{G}_0(\eta_i)\eta_i^2\sinh(r_0/\eta_i)e^{-r/\eta_i} + \int_0^{1/\sigma}[D_1(\eta)\mathbf{G}_{1,0}(\eta) + D_2(\eta)\mathbf{G}_{2,0}(\eta)]\sinh(r_0/\eta)e^{-r/\eta}\eta^2\,d\eta + \int_{1/\sigma}^1 D_3(\eta)\mathbf{G}_{3,0}(\eta)\sinh(r_0/\eta)e^{-r/\eta}\eta^2\,d\eta\Big\}, \quad r > r_0, \tag{40}$$

and

$$\mathbf{j}(r_0\colon r) = \frac{\pi(\mathbf{\Sigma} - 2\mathbf{C})}{2rr_0}\Big\{\sum_i D(\eta_i)\mathbf{G}_0(\eta_i)\eta_i^3 H\left(\frac{r_0}{\eta_i}:\frac{r}{\eta_i}\right) + \int_0^{1/\sigma}[D_1(\eta)\mathbf{G}_{1,0}(\eta) + D_2(\eta)\mathbf{G}_{2,0}(\eta)]H\left(\frac{r_0}{\eta}:\frac{r}{\eta}\right)\eta^3\,d\eta + \int_{1/\sigma}^1 D_3(\eta)\mathbf{G}_{3,0}(\eta)H\left(\frac{r_0}{\eta}:\frac{r}{\eta}\right)\eta^3\,d\eta\Big\}, \tag{41a}$$

where

$$H(x_0 \colon x) \triangleq \sinh x_0 e^{-x}[(1/x) + 1], \qquad x > x_0, \tag{41b}$$

$$H(x_0 \colon x) \triangleq \sinh x e^{-x_0}[(1/x) - \coth x], \quad x < x_0. \tag{41c}$$

We note that the result for $\boldsymbol{\phi}(r_0 \colon r)$ $r < r_0$ is obtained by interchanging r_0 and r in Eq. (40); in addition, we have used the relation

$$\mathbf{G}_1(\xi) = \xi(\mathbf{\Sigma} - 2\mathbf{C})\mathbf{G}_0(\xi), \quad \xi = \eta_i \quad \text{or} \quad \eta \in (0, 1), \tag{42}$$

to obtain Eq. (41a). Higher moments of $\Psi(r_0; r, \mu)$ may be obtained in a similar manner by integrating Eqs. (17), and finally, results for the two-group point-source problem are obtained by observing the limit as $r_0 \to 0$ in Eqs. (17a), (40), (41a), and (41b).

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