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The linearized Boltzmann equation with Cercignani–Lampis boundary conditions: Basic flow problems in a plane channel

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ABSTRACT

A polynomial expansion procedure and the ADO (analytical discrete-ordinates) method are used to solve a collection of basic flow problems based on the linearized Boltzmann equation for rigid-sphere interactions and the Cercignani–Lampis boundary conditions with a free choice of the accommodation coefficients at each boundary. In particular, three classical problems defined by flow in a plane-parallel channel (Poiseuille, thermal-creep, and Couette flow) are solved (essentially) analytically and evaluated to a very high numerical standard. Some comparisons with known kinetic models are also reported.

1. Introduction

In a series of (somewhat) recent works [1–10], a newly introduced polynomial expansion technique (relevant to the speed variable) and an analytical discrete-ordinates (ADO) method [11] that has evolved from Chandrasekhar's work [12] in radiative transfer were used to solve most of the classical flow and heat-transfer problems for single-species gases [1–5] and for binary gas mixtures [6–10]. These works were all based on the linearized Boltzmann equation (LBE) for rigid-sphere interactions and the Maxwell (specular/diffuse) boundary conditions. While the three basic half-space problems (Kramers, temperature jump, and thermal creep) have also been well solved [3] in terms of the LBE for a single-species gas of rigid spheres and Cercignani–Lampis (C-L) boundary conditions [13], that work [3] is only now being extended to the standard problems of flow in a finite, plane-parallel channel. To the best of our knowledge, this is the first time these problems have been formulated (and solved) in terms of the LBE and C-L boundary conditions. However, as discussed next, some solutions for flow problems in plane channels formulated in terms of kinetic models and C-L boundary conditions have been reported in the literature.

In two concurrent papers published in 2002, Siewert [14] and Sharipov [15] solved the plane Poiseuille and thermal-creep flow problems formulated in terms of the S model [16] and C-L boundary conditions. The solution given in Ref. [14] was based on the ADO method, which provides a result that is continuous in both the space and speed variables, while the solution in Ref. [15] was based on an optimized discrete-velocities method with an error estimation (reported to be) of less than 0.1%. A few months later, Cercignani, Lampis, and Lorenzani [17] reported a variational solution for plane Poiseuille flow formulated in terms of the BGK model [18] and C-L boundary conditions. Shortly afterwards, these authors developed [19] a solution that relies on a finite-difference technique applied to an integral formulation of the same problem. More recently, Knackfuss and Barichello [20] have used the ADO method to solve the plane Poiseuille, thermal-creep, and Couette flow problems formulated in terms of the BGK and S models and C-L boundary conditions.

To be complete, we note that there are numerous works that base the analysis of the considered problems on the LBE with Maxwell boundary conditions. Since many of these works have been referenced in most of Refs. [1–10], they are not reviewed here other than to say that Refs. [21–32] should be consulted for alternative (to our own) treatments of the LBE (for single-species gases and for binary mixtures) with general or special forms of the Maxwell boundary conditions.

2. Mathematical formulation

The flow problems considered in this work are driven by wall movements or temperature and/or pressure gradients, and so we base our linearization of the particle distribution function about absolute conditions (for the Couette problem) and about local con-

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ditions (for the Poiseuille and thermal-creep problems). With regard to the latter, we follow Williams [33], use x to measure distance in the direction (parallel to the confining walls of the plane-parallel channel) of the mentioned gradients, and write the local Maxwellian as

$$f_0(x,v)=n(x)\left[\frac{m}{2\pi kT(x)}\right]^{3/2}\exp\left\{-\frac{mv^2}{2kT(x)}\right\}, \tag{2.1}$$

where v is the magnitude of the velocity $\boldsymbol{v}$, m is the mass of a particle, and k is the Boltzmann constant. If we now express the considered linear variations in the number density and the temperature as

$$n(x)=n(1+Rx) \tag{2.2}$$

and

$$T(x)=T_0(1+K_Tx), \tag{2.3}$$

where R and K_T are considered to be given (small) constants, we can linearize Eq. (2.1) to obtain the approximation

$$f_0^*(x,v)=f_0(v)\big[1+f(v)x\big], \tag{2.4}$$

where

$$f_0(v)=n(\lambda/\pi)^{3/2}\mathrm{e}^{-\lambda v^2},\quad \lambda=m/(2kT_0), \tag{2.5}$$

is the absolute Maxwellian distribution for n particles in equilibrium at temperature T_0. The function $f(v)$ in Eq. (2.4) is to be determined. If we express the pressure distribution as

$$p(x)=p_0(1+K_Px), \tag{2.6}$$

where $p_0=nkT_0$ and K_P is a given (small) constant, then using the perfect gas law

$$p(x)=n(x)kT(x), \tag{2.7}$$

we find, after neglecting 2nd-order effects,

$$R=K_P-K_T. \tag{2.8}$$

And so, we find we can use

$$f(v)=\big[mv^2/(2kT_0)-5/2\big]K_T+K_P \tag{2.9}$$

to complete Eq. (2.4). Using the variable $z\in[-z_0,z_0]$ to measure the transverse or cross-channel direction, we now write the true velocity distribution as

$$f(x,z,\boldsymbol{v})=f_0(v)\big\{1+f(v)x+h(z,\lambda^{1/2}\boldsymbol{v})\big\}, \tag{2.10}$$

where the perturbation $h(z,\lambda^{1/2}\boldsymbol{v})$ is to be determined from a form of the linearized Boltzmann equation that has an inhomogeneous driving term due to the x variation in Eq. (2.10).

And so, we proceed with an inhomogeneous form of the linearized Boltzmann equation for a single-species of rigid spheres, written as

$$S(\boldsymbol{c})+c\mu\frac{\partial}{\partial z}h(z,\boldsymbol{c})+\varepsilon_0\nu(c)h(z,\boldsymbol{c}) \\ =\varepsilon_0\int \mathrm{e}^{-c'^2}\mathcal{P}(\boldsymbol{c}':\boldsymbol{c})h(z,\boldsymbol{c}')\,\mathrm{d}^3c', \tag{2.11}$$

where

$$\varepsilon_0=n\pi^{1/2}d^2. \tag{2.12}$$

Here d is used to denote the diameter of the particles, and we express the inhomogeneous term in Eq. (2.11) as

$$S(\boldsymbol{c})=c\big(1-\mu^2\big)^{1/2}\cos\phi\big[\big(c^2-5/2\big)K_T+K_P\big]. \tag{2.13}$$

In writing Eq. (2.11), we have introduced the variable change

$$\boldsymbol{c}=\lambda^{1/2}\boldsymbol{v} \tag{2.14}$$

in order to work with the dimensionless velocity variable $\boldsymbol{c}$. Continuing, we note that we use spherical coordinates $\{c,\theta,\phi\}$, with $\mu=\cos\theta$, to describe the dimensionless velocity vector $\boldsymbol{c}$, so that

$$h(z,\boldsymbol{c})\Leftrightarrow h(z,c,\mu,\phi).$$

In our notation, $c\mu$ is the component of the (dimensionless) velocity vector in the positive z direction and

$$c_x=c\big(1-\mu^2\big)^{1/2}\cos\phi \tag{2.15}$$

is the component of velocity in the direction x (parallel to the confining surfaces) of the flow. Continuing to define Eq. (2.11), we have the collision frequency

$$\nu(c)=\frac{2c^2+1}{c}\int_0^c \mathrm{e}^{-x^2}\,\mathrm{d}x+\mathrm{e}^{-c^2} \tag{2.16}$$

and the scattering kernel

$$\mathcal{P}(\boldsymbol{c}':\boldsymbol{c})=\frac{1}{\pi}\left(\frac{2}{|\boldsymbol{c}'-\boldsymbol{c}|}\exp\left\{\frac{|\boldsymbol{c}'\times\boldsymbol{c}|^2}{|\boldsymbol{c}'-\boldsymbol{c}|^2}\right\}-|\boldsymbol{c}'-\boldsymbol{c}|\right) \tag{2.17}$$

given, for example, by Pekeris [34].

As we wish to formulate the boundary conditions simultaneously for all three problems (Poiseuille, thermal creep, and Couette flow), we follow [33] and write, first for the Maxwell case, the boundary conditions at $z=\pm z_0$ as

$$h(-z_0,c,\mu,\phi)-(1-\alpha_1)h(-z_0,c,-\mu,\phi)-\alpha_1\mathcal{I}_1\{h\}(-z_0) \\ =2\alpha_1u_{w,1}c\big(1-\mu^2\big)^{1/2}\cos\phi \tag{2.18a}$$

and

$$h(z_0,c,-\mu,\phi)-(1-\alpha_2)h(z_0,c,\mu,\phi)-\alpha_2\mathcal{I}_2\{h\}(z_0) \\ =2\alpha_2u_{w,2}c\big(1-\mu^2\big)^{1/2}\cos\phi, \tag{2.18b}$$

for $c\in[0,\infty)$, $\mu\in(0,1]$, and $\phi\in[0,2\pi]$. Here the wall velocities (for Couette flow) are $u_{w,1}$ and $u_{w,2}$, α_1 and α_2 are accommodation coefficients,

$$\mathcal{I}_1\{h\}(-z_0) \\ =\frac{2}{\pi}\int_0^\infty\int_0^1\int_0^{2\pi}\mathrm{e}^{-c'^2}h(-z_0,c',-\mu',\phi')\mu'c'^3\,\mathrm{d}\phi'\,\mathrm{d}\mu'\,\mathrm{d}c', \tag{2.19a}$$

and

$$\mathcal{I}_2\{h\}(z_0) \\ =\frac{2}{\pi}\int_0^\infty\int_0^1\int_0^{2\pi}\mathrm{e}^{-c'^2}h(z_0,c',\mu',\phi')\mu'c'^3\,\mathrm{d}\phi'\,\mathrm{d}\mu'\,\mathrm{d}c'. \tag{2.19b}$$

Switching now to the case of the C-L boundary conditions, we write, following Refs. [3] and [35],

$$h(-z_0,c,\mu,\phi)-\mathcal{J}_1\{h\}(-z_0,c,\mu,\phi) \\ =2\alpha_{t,1}u_{w,1}c\big(1-\mu^2\big)^{1/2}\cos\phi \tag{2.20a}$$

and

$$h(z_0,c,-\mu,\phi)-\mathcal{J}_2\{h\}(z_0,c,-\mu,\phi) \\ =2\alpha_{t,2}u_{w,2}c\big(1-\mu^2\big)^{1/2}\cos\phi, \tag{2.20b}$$

for $c\in[0,\infty)$, $\mu\in(0,1]$, and $\phi\in[0,2\pi]$. Here

$$\mathcal{J}_1\{h\}(-z_0,c,\mu,\phi)=\int_0^\infty\int_0^1\int_0^{2\pi}h(-z_0,c',-\mu',\phi')\times R_1(c',-\mu',\phi':c,\mu,\phi)c'^2\,\mathrm{d}\phi'\,\mathrm{d}\mu'\,\mathrm{d}c' \tag{2.21a}$$

and

$$\mathcal{J}_2\{h\}(z_0,c,-\mu,\phi)=\int_0^\infty\int_0^1\int_0^{2\pi}h(z_0,c',\mu',\phi')\times R_2(c',\mu',\phi':c,-\mu,\phi)c'^2\,\mathrm{d}\phi'\,\mathrm{d}\mu'\,\mathrm{d}c'. \tag{2.21b}$$

Since we are allowing the two surfaces that confine the gas to be different, we must be careful with our definitions of the C-L functions, and so we follow Ref. [3] and write, for $\mu',\mu\in[0,1]$,

$$R(c',\mp\mu',\phi':c,\pm\mu,\phi)=\frac{2c'\mu'}{\hat{\alpha}\alpha_n\pi}S(c',\mp\mu':c,\pm\mu)T(c',\mp\mu',\phi':c,\pm\mu,\phi), \tag{2.22}$$

where $\hat{\alpha}=\alpha_t(2-\alpha_t)$, and where

$$S(c',\mp\mu':c,\pm\mu)=\exp\bigl\{-\bigl[(c'\mu')^2+(1-\alpha_n)(c\mu)^2\bigr]/\alpha_n\bigr\}\times I_0\bigl[2(1-\alpha_n)^{1/2}c'c\mu'\mu/\alpha_n\bigr], \tag{2.23a}$$

$$T(c',\mp\mu',\phi':c,\pm\mu,\phi)=E(c',\mu':c,\mu)\times\exp\bigl\{-2c'cr(\mu')r(\mu)\bigl[|1-\alpha_t|-(1-\alpha_t)\cos(\phi'-\phi)\bigr]/\hat{\alpha}\bigr\}, \tag{2.23b}$$

and

$$E(c',\mu':c,\mu)=\exp\bigl\{-\bigl[|1-\alpha_t|cr(\mu)-c'r(\mu')\bigr]^2/\hat{\alpha}\bigr\}, \tag{2.23c}$$

with $r(x)=(1-x^2)^{1/2}$. In writing Eq. (2.23a), we have used $I_0(x)$ to label the lowest-order modified Bessel function of the first kind. Continuing, we see that each wall has two accommodation coefficients that should be used in Eq. (2.22) to yield the reflection functions required in Eqs. (2.21): $\{\alpha_{t,1},\alpha_{n,1}\}$ for the wall located at $z=-z_0$ and $\{\alpha_{t,2},\alpha_{n,2}\}$ for the wall located at $z=z_0$. It should be noted that we have derived the inhomogeneous terms in Eqs. (2.20) from Ref. [35] — while the notation is different, these results agree with those of Sharipov and Kalempa [36]; clearly these terms differ from the Maxwell case only in that $\alpha_{t,1}$ and $\alpha_{t,2}$ have replaced α_1 and α_2.

In this work, we compute the velocity, shear-stress (given by the xz-component of the stress tensor), and heat-flow profiles, which we express as

$$u(z)=\frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi}\mathrm{e}^{-c^2}h(z,\boldsymbol{c})c^3\bigl(1-\mu^2\bigr)^{1/2}\times\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.24}$$

$$P_{xz}(z)=\frac{2}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi}\mathrm{e}^{-c^2}h(z,\boldsymbol{c})c^4\mu\bigl(1-\mu^2\bigr)^{1/2}\times\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.25}$$

and

$$q(z)=\frac{1}{\pi^{3/2}}\int_0^\infty\int_{-1}^1\int_0^{2\pi}\mathrm{e}^{-c^2}h(z,\boldsymbol{c})\bigl(c^2-5/2\bigr)c^3\bigl(1-\mu^2\bigr)^{1/2}\times\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu\,\mathrm{d}c. \tag{2.26}$$

Now since the driving terms in Eqs. (2.11), (2.18), and (2.20) have a ϕ dependence given by $\cos\phi$, we can be sure that only one term is required in a Fourier expansion, in terms of the ϕ variable, of $h(z,\boldsymbol{c})$, and so we can write

$$h(\tau/\varepsilon_0,\boldsymbol{c})=\psi(\tau,c,\mu)\bigl(1-\mu^2\bigr)^{1/2}\cos\phi, \tag{2.27}$$

where we have introduced the dimensionless spatial variable $\tau=z\varepsilon_0$, and where $\psi(\tau,c,\mu)$ is the function to be determined. We now let $z=\tau/\varepsilon_0$ in Eqs. (2.24)–(2.26) and consider that

$$u(\tau)=\frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1\mathrm{e}^{-c^2}\psi(\tau,c,\mu)c^3\bigl(1-\mu^2\bigr)\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.28}$$

$$P_{xz}(\tau)=\frac{2}{\pi^{1/2}}\int_0^\infty\int_{-1}^1\mathrm{e}^{-c^2}\psi(\tau,c,\mu)c^4\bigl(1-\mu^2\bigr)\mu\,\mathrm{d}\mu\,\mathrm{d}c, \tag{2.29}$$

and

$$q(\tau)=\frac{1}{\pi^{1/2}}\int_0^\infty\int_{-1}^1\mathrm{e}^{-c^2}\psi(\tau,c,\mu)\bigl(c^2-5/2\bigr)c^3\bigl(1-\mu^2\bigr)\,\mathrm{d}\mu\,\mathrm{d}c \tag{2.30}$$

are the quantities to be computed. It should be noted that in order to avoid excessive notation, we have, in writing Eqs. (2.28)–(2.30), followed the (often-used) procedure of not always introducing new labels for dependent quantities (in this case u, P_{xz}, and q) when the independent variable is changed.

We can now use Eq. (2.27) in Eq. (2.11), multiply the resulting equation by $\cos\phi$, integrate over all ϕ, and use a Legendre expansion of the scattering kernel to find

$$\Upsilon(c)+c\mu\frac{\partial}{\partial\tau}\psi(\tau,c,\mu)+\nu(c)\psi(\tau,c,\mu)=\int_0^\infty\int_{-1}^1\mathrm{e}^{-c'^2}f(\mu',\mu)\mathcal{P}(c',\mu':c,\mu)\psi(\tau,c',\mu')c'^2\,\mathrm{d}\mu'\,\mathrm{d}c', \tag{2.31}$$

where

$$f(\mu',\mu)=\left(\frac{1-\mu'^2}{1-\mu^2}\right)^{1/2}. \tag{2.32}$$

In addition

$$\mathcal{P}(c',\mu':c,\mu)\cos\phi'=\int_0^{2\pi}\mathcal{P}(\boldsymbol{c}':\boldsymbol{c})\cos\phi\,\mathrm{d}\phi, \tag{2.33}$$

which we can express, in the notation of Ref. [37], as

$$\mathcal{P}(c',\mu':c,\mu)=\frac{1}{2}\sum_{n=1}^\infty(2n+1)P_n^1(\mu')P_n^1(\mu)\mathcal{P}_n(c',c), \tag{2.34}$$

where $P_n^1(x)$ is used to denote one of the normalized associated Legendre functions. More explicitly,

$$P_l^m(\mu)=\left[\frac{(l-m)!}{(l+m)!}\right]^{1/2}\bigl(1-\mu^2\bigr)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}\mu^m}P_l(\mu), \tag{2.35}$$

where $P_l(\mu)$ is the Legendre polynomial. We do not list here the definition of $\mathcal{P}_n(c',c)$ since it is given explicitly in Appendix A of Ref. [37]. To complete Eq. (2.31) we note that

$$\Upsilon(c)=(c/\varepsilon_0)\bigl[\bigl(c^2-5/2\bigr)K_T+K_P\bigr]. \tag{2.36}$$

The boundary conditions that accompany Eq. (2.31) follow from Eqs. (2.18) and (2.20) once we make use of Eq. (2.27). For the Maxwell case, we find

$$\psi(-a,c,\mu)-(1-\alpha_1)\psi(-a,c,-\mu)=2\alpha_1u_{w,1}c \tag{2.37a}$$

and

$$\psi(a, c, -\mu) - (1-\alpha_2)\psi(a, c, \mu) = 2\alpha_2 u_{w,2} c, \tag{2.37b}$$

for $c \in [0, \infty)$ and $\mu \in (0, 1]$, where we have used $a = z_0\varepsilon_0$. Continuing, we deduce that the C-L conditions can be written as

$$\psi(-a, c, \mu) - \int_0^\infty \int_0^1 f(\mu', \mu)\psi(-a, c', -\mu')B_1(c', -\mu' : c, \mu) \times c'^2 \,\mathrm{d}\mu' \,\mathrm{d}c' = 2\alpha_{t,1} u_{w,1} c \tag{2.38a}$$

and

$$\psi(a, c, -\mu) - \int_0^\infty \int_0^1 f(\mu', \mu)\psi(a, c', \mu')B_2(c', \mu' : c, -\mu) \times c'^2 \,\mathrm{d}\mu' \,\mathrm{d}c' = 2\alpha_{t,2} u_{w,2} c, \tag{2.38b}$$

for $c \in [0, \infty)$ and $\mu \in (0, 1]$. Here

$$B(c', \mp\mu' : c, \pm\mu) = \operatorname{sgn}(1-\alpha_t)\frac{4c'\mu'}{\hat{\alpha}\alpha_n} S(c', \mp\mu' : c, \pm\mu) \times U(c', \mp\mu' : c, \pm\mu), \tag{2.39}$$

where $S(c', \mp\mu' : c, \pm\mu)$ is given by Eq. (2.23a) and

$$U(c', \mp\mu' : c, \pm\mu) = \exp\bigl\{-\bigl[c'^2 r^2(\mu') + (1-\alpha_t)^2 c^2 r^2(\mu)\bigr]/\hat{\alpha}\bigr\} \times I_1\bigl[2|1-\alpha_t| c' c r(\mu') r(\mu)/\hat{\alpha}\bigr]. \tag{2.40}$$

The reflection functions $B_1(c', -\mu' : c, \mu)$ and $B_2(c', \mu' : c, -\mu)$ that appear in Eqs. (2.38) are versions of Eqs. (2.39) based, respectively, on the two pairs of accommodation coefficients $\{\alpha_{t,1}, \alpha_{n,1}\}$ and $\{\alpha_{t,2}, \alpha_{n,2}\}$. Note that in Eq. (2.40) we have used $I_1(x)$ to denote the first-order modified Bessel function of the first kind. To conclude this section we list our expressions, deduced from Eqs. (2.23a), (2.39) and (2.40), for some special cases:

$$B(c', \mp\mu' : c, \pm\mu) = 0, \quad \alpha_t = 1, \tag{2.41a}$$

$$\lim_{\alpha_n \to 0} B(c', \mp\mu' : c, \pm\mu) = \frac{2}{\hat{\alpha}} \operatorname{sgn}(1-\alpha_t) \times U(c', \mp\mu' : c, \pm\mu)\delta(c'\mu' - c\mu), \tag{2.41b}$$

$$\lim_{\alpha_n \to 1} B(c', \mp\mu' : c, \pm\mu) = \frac{4c'\mu'}{\hat{\alpha}} \operatorname{sgn}(1-\alpha_t) \times U(c', \mp\mu' : c, \pm\mu)\exp\bigl\{-c'^2\mu'^2\bigr\}, \tag{2.41c}$$

$$\lim_{\hat{\alpha} \to 0} B(c', \mp\mu' : c, \pm\mu) = \frac{2\mu'}{\alpha_n r(\mu')} \operatorname{sgn}(1-\alpha_t) \times S(c', \mp\mu' : c, \pm\mu)\delta\bigl[c' r(\mu') - c r(\mu)\bigr], \tag{2.41d}$$

and

$$\lim_{\hat{\alpha} \to 0} \lim_{\alpha_n \to 0} B(c', \mp\mu' : c, \pm\mu) = \frac{1}{c^2} \operatorname{sgn}(1-\alpha_t)\delta(\mu' - \mu)\delta(c' - c). \tag{2.41e}$$

In regard to the flow problems solved in this work, we see that when $\alpha_t = 1$, for any value of α_n, the C-L boundary condition reduces to the case of diffuse reflection ($\alpha = 1$) — a special case of the Maxwell boundary condition.

3. The ADO solution

At the start of this section, we note that we will rely heavily on Ref. [2] since that work reports explicitly the solutions for the special case of Maxwell boundary conditions characterized by the same accommodation coefficient for each of the two walls. While it is considered important to extend Ref. [2] to the general case (different reflection properties on the two walls) of the Maxwell and the Cercignani–Lampis boundary conditions, this task is relatively straightforward in terms of our ADO method of solution.

We start with our solution to Eq. (2.31) expressed as

$$\psi(\tau, c, \mu) = \psi_*(\tau, c, \mu) + \psi_{ps}(\tau, c, \mu), \tag{3.1}$$

where the term $\psi_{ps}(\tau, c, \mu)$ represents a particular solution of Eq. (2.31) and the term $\psi_*(\tau, c, \mu)$ represents a (sufficiently general) solution of the homogeneous version of Eq. (2.31). Noting that in this work we use a dimensionless spatial variable expressed in terms of the rigid-sphere mean-free path $1/\varepsilon_0$, we use here the particular solution given in Refs. [2,22] written as

$$\psi_{ps}(\tau, c, \mu) = (K_P/\varepsilon_0)\bigl[c\bigl(\tau^2 - a^2\bigr) - 2B(c)\tau\mu + D(c)/5 + E(c)\bigl(5\mu^2 - 1\bigr)/5\bigr]/\varepsilon_p - (K_T/\varepsilon_0)A(c). \tag{3.2}$$

The functions $A(c)$ and $B(c)$ in Eq. (3.2) are the solutions of the Chapman–Enskog equations for heat conduction and viscosity [38], while the functions $D(c)$ and $E(c)$ are solutions of the (so-called) Burnett equations [22,39]. A method for computing these four functions (and numerical results) have been reported (for example and in this notation) in Ref. [39]. In addition,

$$\varepsilon_p = \frac{16}{15\pi^{1/2}} \int_0^\infty \mathrm{e}^{-c^2} B(c) c^4 \,\mathrm{d}c. \tag{3.3}$$

Continuing to follow Ref. [2], we now write our ADO solution relevant to the homogeneous version of Eq. (2.31) as

$$\psi_*(\tau, c, \pm\mu_i) = A_1 c + B_1\bigl[c\tau \mp \mu_i B(c)\bigr] + \boldsymbol{P}(c)\boldsymbol{G}_*(\tau, \pm\mu_i), \tag{3.4}$$

for $i = 1, 2, \ldots, N$. Here, we quote from Ref. [2] and write

$$\boldsymbol{P}(c) = \bigl[P_0\bigl(2\mathrm{e}^{-c} - 1\bigr) \quad P_1\bigl(2\mathrm{e}^{-c} - 1\bigr) \cdots P_K\bigl(2\mathrm{e}^{-c} - 1\bigr)\bigr] \tag{3.5}$$

and

$$\boldsymbol{G}_*(\tau, \pm\mu_i) = \sum_{j=2}^{J} \bigl[A_j \boldsymbol{\Phi}(\nu_j, \pm\mu_i)\mathrm{e}^{-(a+\tau)/\nu_j} + B_j \boldsymbol{\Phi}(\nu_j, \mp\mu_i)\mathrm{e}^{-(a-\tau)/\nu_j}\bigr]. \tag{3.6}$$

Some explanations: N is the number of (Gaussian) quadrature points $\{\mu_i\}$ used for integrals defined on the half-range [0,1] and $J = N(K+1)$, where $K+1$ is the number of basis functions used to model the speed dependence. In addition, the separation constants $\{\nu_j\}$ and the eigenvectors $\{\boldsymbol{\Phi}(\nu_j, \pm\mu_i)\}$ can be found by solving an eigensystem of order J, as explained in detail in Ref. [2].

In regard to Ref. [2], we comment here on an error (of no consequence) in that work. In Eqs. (104), (107), (109), (112), and (116) of Ref. [2], the factor $(-1)^l$ should be replaced with $(-1)^{l+1}$. The net effect of this error is simply an interchange in the definitions of the matrices $\boldsymbol{E}$ and $\boldsymbol{H}$ that are given by Eqs. (116) of Ref. [2]. Since the separation constants $\{\nu_j\}$ and eigenvectors $\{\boldsymbol{\Phi}(\nu_j, \pm\mu_i)\}$ can be found, as shown by Eqs. (118) of Ref. [2], from the eigenvalues and eigenvectors of either $\boldsymbol{HE}$ or $\boldsymbol{EH}$, it is clear that the use of the incorrect factor $(-1)^l$ had no effect in the calculation of these quantities in Ref. [2].

Continuing, we note that the constants $\{A_j, B_j\}$ in Eq. (3.6) are arbitrary in regard to defining our solution to the homogeneous version of Eq. (2.31). And so now, to establish these constants, we substitute Eq. (3.1) into Eqs. (2.38), multiply the resulting equations by

$$W_k(c) = c^2 e^{-c^2} P_k(2e^{-c} - 1), \quad k = 0, 1, 2, \ldots, K, \tag{3.7}$$

and integrate over all c to obtain a $2J \times 2J$ system of linear algebraic equations for the required constants $\{A_j, B_j\}$. We then solve the linear system to complete our ADO solution.

4. Quantities of interest

Having established our ADO solution, we now use that result to find the quantities we wish to evaluate numerically. We can use Eq. (3.1) in analytical/discrete-ordinates versions of Eq. (2.28) and (2.30) to find

$$u(\tau) = (K_P/\varepsilon_0)\frac{\tau^2 - a^2}{2\varepsilon_p} + \frac{1}{2}(A_1 + B_1\tau) + \sum_{j=2}^{J}\left[A_j e^{-(a+\tau)/\nu_j} + B_j e^{-(a-\tau)/\nu_j}\right]N_j, \tag{4.1}$$

and

$$q(\tau) = (K_P/\varepsilon_0)\frac{4d_5}{15\varepsilon_p} - (K_T/\varepsilon_0)\frac{5\varepsilon_t}{4} + \sum_{j=2}^{J}\left[A_j e^{-(a+\tau)/\nu_j} + B_j e^{-(a-\tau)/\nu_j}\right]M_j. \tag{4.2}$$

Here, continuing to quote from Ref. [2], we write

$$N_j = \pi^{-1/2}\boldsymbol{P}_1\boldsymbol{N}(\nu_j) \tag{4.3a}$$

and

$$M_j = \pi^{-1/2}\left[\boldsymbol{P}_3 - (5/2)\boldsymbol{P}_1\right]\boldsymbol{N}(\nu_j), \tag{4.3b}$$

where

$$\boldsymbol{N}(\nu_j) = \sum_{n=1}^{N} w_n\left(1 - \mu_n^2\right)\left[\boldsymbol{\Phi}(\nu_j, \mu_n) + \boldsymbol{\Phi}(\nu_j, -\mu_n)\right], \tag{4.4}$$

with $\{w_n\}$ denoting the Gaussian weights. To complete Eqs. (4.3) we also have the definition [2]

$$\boldsymbol{P}_n = \int_0^\infty e^{-c^2}\boldsymbol{P}(c)c^{n+2}\,dc. \tag{4.5}$$

In addition,

$$\varepsilon_t = \frac{16}{15\pi^{1/2}}\int_0^\infty e^{-c^2}A(c)c^5\,dc \tag{4.6}$$

and

$$d_5 = \frac{1}{\pi^{1/2}}\int_0^\infty e^{-c^2}D(c)c^5\,dc. \tag{4.7}$$

For the shear-stress profile, we find

$$P_{xz}(\tau) = -(K_P/\varepsilon_0)\tau + P_{xz}^*, \tag{4.8}$$

where the (problem-dependent) constant component P_{xz}^* is

$$P_{xz}^* = -\frac{1}{2}\varepsilon_p B_1 \tag{4.9}$$

and we have neglected the contribution of the exponential terms, since it can be shown that the exact result for $P_{xz}(\tau)$ is of the form of Eq. (4.8).

Finally, since we have Eqs. (4.1) and (4.2), we can simply integrate those expressions to obtain the (normalized) mass- and heat-flow rates:

$$U = \frac{1}{2a}\int_{-a}^{a} u(\tau)\,d\tau \tag{4.10}$$

and

$$Q = \frac{1}{2a}\int_{-a}^{a} q(\tau)\,d\tau. \tag{4.11}$$

And so

$$U = \frac{1}{2a}\left[-(K_p/\varepsilon_0)\frac{2a^3}{3\varepsilon_p} + aA_1 + \sum_{j=2}^{J}\nu_j(A_j + B_j)\left(1 - e^{-2a/\nu_j}\right)N_j\right] \tag{4.12}$$

and

$$Q = \frac{1}{2a}\left[(K_P/\varepsilon_0)\frac{8ad_5}{15\varepsilon_p} - (K_T/\varepsilon_0)\frac{5a\varepsilon_t}{2} + \sum_{j=2}^{J}\nu_j(A_j + B_j)\left(1 - e^{-2a/\nu_j}\right)M_j\right]. \tag{4.13}$$

5. Kinetic models

Our solutions for the LBE can be easily modified to yield solutions for a class of kinetic models. And so we include in this work, in addition to our numerical results for the LBE, similar results for five kinetic models: the BGK model [18], the S model [16], the GJ model [40], the MRS model [5], and the CES model [41]. The required expressions, relevant to these models, for the scattering kernels, collision frequencies and the Chapman–Enskog functions $A(c)$ and $B(c)$ are all given in Ref. [5], where the MRS model was first introduced as a special case of the McCormack model [42]. However, here we also require the functions $D(c)$ and $E(c)$ for each of the considered kinetic models. And so we supplement the expressions given in Ref. [5] with some additional results.

We note that for the BGK, S, GJ, and MRS models the functions $D(c)$ and $E(c)$ can be written as

$$D(c) = 2\varepsilon_p\varepsilon_t c\left(c^2 - 5/2\right) \tag{5.1a}$$

and

$$E(c) = 2\varepsilon_p c^3, \tag{5.1b}$$

where the numerical values of ε_p and ε_t for these models are listed in Ref. [5].

The CES model is based [41] on the use of the Chapman–Enskog functions $A(c)$ and $B(c)$ to define synthetic scattering kernels that can be used to approximate the true rigid-sphere kernels. The required results for the functions $D(c)$ and $E(c)$, for the CES model, were reported in Ref. [43], and so we can write

$$D(c) = \nu^{-1}(c)\left[2cB(c) - 5c\varepsilon_p + \varpi_{11}c_{11}\Delta_{11}(c) + \varpi_{12}c_{12}\Delta_{12}(c)\right] \tag{5.2}$$

and

$$E(c) = 2cB(c)/\nu(c), \tag{5.3}$$

where all quantities required here (and not defined in this work) are given by Eqs. (8), (12), (62), and (63) of Ref. [43].

Table 1
Defining data for various cases.

Case	$u_{w,1}$	$u_{w,2}$	$\alpha_{t,1}$	$\alpha_{n,1}$	$\alpha_{t,2}$	$\alpha_{n,2}$	$2a$
1	1.0	−2.0	0.25	0.50	0.75	0.25	1.0
2	0.0	1.0	0.50	0.75	0.25	0.25	2.0
3	1.0	−2.0	0.75	0.25	0.50	0.25	0.1
4	0.0	1.0	1.50	0.00	1.25	0.25	1.0
5	1.0	−2.0	0.75	0.50	0.00	0.25	0.5
6	1.0	−2.0	0.25	0.75	2.00	0.00	3.0

6. Numerical results

As we have recently discussed [7] in considerable detail all the numerical procedures we use to implement and evaluate our ADO solution, our discussion here is very brief. While Ref. [7] deals with a binary mixture of rigid spheres (it also includes the special case of a single-species gas) and while Ref. [7] considers only the case of Maxwell (specular/diffuse) boundary conditions, we can deduce from that work all of the numerical approximations we require here to evaluate our solutions for the case of the Cercignani–Lampis boundary conditions.

In Table 1, we list six data sets that define the physical parameters of the specific problems we consider. It can be noted that we have defined our input data for these test problems to be as general as possible (the first three cases) and also to cover the special cases of the C-L kernel listed as Eqs. (2.41) (the last three cases). In this way the full merits of our solutions can be appreciated. To be clear, we note that the values listed for $u_{w,1}$ and $u_{w,2}$ are relevant only to the problem of Couette flow. In addition, for the problems of Poiseuille and thermal-creep flow, we have used the normalizations $K_p = \varepsilon_0$ and $K_T = \varepsilon_0$, respectively.

In regard to the accuracy of our results, we can report that we have varied all of the approximation parameters in order to have some confidence that all of the data listed in Tables 2–10 are correct to within ±1 on the last digit given. We note that in Tables 2–10 we have listed some exact results [reported as 1.0, 0.0, −2.0 and/or 2.5(−1)] that we were able to establish using the expressions

$$\int_0^\infty \int_0^1 f(\mu', \mu) B(c', \mp\mu' : c, \pm\mu) c'^3 \, d\mu' \, dc' = (1-\alpha_t) c \tag{6.1}$$

and

$$\int_0^\infty \int_0^1 e^{-c^2} c^4 (1-\mu^2) \mu f(\mu', \mu) B(c', \mp\mu' : c, \pm\mu) \, d\mu \, dc = (1-\alpha_t) c'^2 \mu' (1-\mu'^2) e^{-c'^2}, \tag{6.2}$$

which can be found by carrying out the indicated integrations. We also note that in the process of testing our solution we have considered other cases in addition to those defined in Table 1, in order to cover all flow regimes. The degree of the accuracy of the numerical results in all tests was found to be the same as that of the results reported in Tables 2–10, except in the case of the near free-molecular-flow regime (say, $2a < 10^{-3}$). We have found

Table 2
Couette flow: the mass-flow rate, heat-flow rate, and shear stress.

Case	Quantity	BGK	S	GJ	MRS	CES	LBE
1	U	−1.29540	−1.29510	−1.32576	−1.19671	−1.19838	−1.19842
	Q	−2.86897(−3)	−3.90158(−3)	−5.20447(−3)	−1.73468(−3)	−1.47320(−3)	−7.52959(−4)
	P_{xz}	3.37454(−1)	3.37405(−1)	3.50504(−1)	2.95529(−1)	2.95493(−1)	2.95571(−1)
2	U	3.41735(−1)	3.41928(−1)	3.32125(−1)	3.70840(−1)	3.68740(−1)	3.68858(−1)
	Q	−1.68772(−3)	−2.37941(−3)	−3.43802(−3)	−9.42740(−4)	−9.02373(−4)	−8.86343(−4)
	P_{xz}	−8.96764(−2)	−8.96460(−2)	−9.53125(−2)	−7.31381(−2)	−7.36032(−2)	−7.35957(−2)
3	U	−8.89955(−2)	−8.90325(−2)	−8.58338(−2)	−1.00727(−1)	−1.06966(−1)	−1.06439(−1)
	Q	2.78044(−3)	2.95224(−3)	3.10294(−3)	2.47985(−3)	4.61321(−3)	3.74207(−3)
	P_{xz}	7.01001(−1)	7.00998(−1)	7.06499(−1)	6.80914(−1)	6.73609(−1)	6.74405(−1)
4	U	4.45971(−1)	4.46332(−1)	4.35727(−1)	4.66960(−1)	4.62568(−1)	4.63007(−1)
	Q	−7.93607(−3)	−9.70441(−3)	−1.37377(−2)	−3.77342(−3)	−4.30030(−3)	−4.34793(−3)
	P_{xz}	−5.11351(−1)	−5.11210(−1)	−6.16051(−1)	−3.10781(−1)	−3.14162(−1)	−3.14151(−1)
5	U	1.0	1.0	1.0	1.0	1.0	1.0
	Q	0.0	0.0	0.0	0.0	0.0	0.0
	P_{xz}	0.0	0.0	0.0	0.0	0.0	0.0
6	U	−1.49499	−1.49376	−1.61093	−1.20885	−1.22457	−1.22394
	Q	−1.19968(−2)	−1.62866(−2)	−2.39937(−2)	−6.10638(−3)	−3.41900(−3)	−3.59362(−3)
	P_{xz}	3.20675(−1)	3.20577(−1)	3.58798(−1)	2.28294(−1)	2.30718(−1)	2.30693(−1)

Table 3
Couette flow: velocity and heat flow profiles based on the LBE.

η	Case 1		Case 2		Case 3	
	$u(-a+2a\eta)$	$q(-a+2a\eta)$	$u(-a+2a\eta)$	$q(-a+2a\eta)$	$u(-a+2a\eta)$	$q(-a+2a\eta)$
0.0	−7.7030(−1)	−3.3402(−2)	1.8206(−1)	8.1009(−3)	7.7244(−2)	−2.5328(−2)
0.1	−8.8870(−1)	−2.3114(−2)	2.2961(−1)	4.3638(−3)	2.9633(−2)	−1.7541(−2)
0.2	−9.7373(−1)	−1.6319(−2)	2.6601(−1)	2.5670(−3)	−7.3803(−3)	−1.1651(−2)
0.3	−1.0515	−1.0703(−2)	3.0057(−1)	1.3515(−3)	−4.1407(−2)	−6.2973(−3)
0.4	−1.1261	−5.7162(−3)	3.3446(−1)	4.0016(−4)	−7.4040(−2)	−1.2005(−3)
0.5	−1.1993	−1.0070(−3)	3.6815(−1)	−4.7282(−4)	−1.0611(−1)	3.7836(−3)
0.6	−1.2724	3.7257(−3)	4.0193(−1)	−1.4207(−3)	−1.3825(−1)	8.7604(−3)
0.7	−1.3465	8.8020(−3)	4.3613(−1)	−2.6134(−3)	−1.7109(−1)	1.3834(−2)
0.8	−1.4233	1.4647(−2)	4.7135(−1)	−4.2942(−3)	−2.0548(−1)	1.9143(−2)
0.9	−1.5065	2.2024(−2)	5.0923(−1)	−6.9070(−3)	−2.4309(−1)	2.4951(−2)
1.0	−1.6204	3.4485(−2)	5.6248(−1)	−1.2114(−2)	−2.9191(−1)	3.2544(−2)

Table 4
Couette flow: velocity and heat flow profiles based on the LBE.

η	Case 4		Case 5		Case 6	
	$u(-a+2a\eta)$	$q(-a+2a\eta)$	$u(-a+2a\eta)$	$q(-a+2a\eta)$	$u(-a+2a\eta)$	$q(-a+2a\eta)$
0.0	6.8048(−2)	8.7816(−3)	1.0	0.0	−3.8648(−1)	−2.3596(−2)
0.1	1.5950(−1)	4.7822(−3)	1.0	0.0	−5.9585(−1)	−1.1433(−2)
0.2	2.3757(−1)	2.5414(−3)	1.0	0.0	−7.5985(−1)	−6.3089(−3)
0.3	3.1281(−1)	5.9499(−4)	1.0	0.0	−9.1773(−1)	−3.5808(−3)
0.4	3.8694(−1)	−1.2886(−3)	1.0	0.0	−1.0735	−2.0630(−3)
0.5	4.6079(−1)	−3.2460(−3)	1.0	0.0	−1.2285	−1.1984(−3)
0.6	5.3496(−1)	−5.4070(−3)	1.0	0.0	−1.3830	−6.9676(−4)
0.7	6.1016(−1)	−7.9362(−3)	1.0	0.0	−1.5374	−3.9957(−4)
0.8	6.8760(−1)	−1.1096(−2)	1.0	0.0	−1.6916	−2.1683(−4)
0.9	7.7016(−1)	−1.5449(−2)	1.0	0.0	−1.8458	−9.5158(−5)
1.0	8.7745(−1)	−2.4190(−2)	1.0	0.0	−2.0	0.0

Table 5
Poiseuille flow: the mass-flow rate, heat-flow rate, and the constant component of the shear stress.

Case	Quantity	BGK	S	GJ	MRS	CES	LBE
1	U	−1.63462	−1.64451	−1.60269	−1.80867	−1.71391	−1.72168
	Q	1.67243(−1)	2.13317(−1)	2.61337(−1)	1.25515(−1)	1.48136(−1)	1.41313(−1)
	P^*_{xz}	−2.65133(−1)	−2.65034(−1)	−2.75253(−1)	−2.32235(−1)	−2.32795(−1)	−2.32805(−1)
2	U	−4.58634	−4.60590	−4.48284	−5.09494	−4.94816	−4.95873
	Q	2.23824(−1)	3.06359(−1)	4.06377(−1)	1.57647(−1)	1.80625(−1)	1.73775(−1)
	P^*_{xz}	3.16530(−1)	3.16145(−1)	3.35751(−1)	2.58320(−1)	2.62520(−1)	2.62284(−1)
3	U	−1.62401(−1)	−1.62937(−1)	−1.62991(−1)	−1.62973(−1)	−1.41552(−1)	−1.43420(−1)
	Q	4.34626(−2)	4.60676(−2)	4.79853(−2)	4.00097(−2)	3.96778(−2)	3.99780(−2)
	P^*_{xz}	1.37001(−2)	1.36989(−2)	1.38055(−2)	1.33091(−2)	1.31011(−2)	1.31187(−2)
4	U	−5.13081(−1)	−5.18960(−1)	−4.95645(−1)	−6.14811(−1)	−5.44160(−1)	−5.51298(−1)
	Q	1.30240(−1)	1.59326(−1)	1.87362(−1)	1.01539(−1)	1.38766(−1)	1.28653(−1)
	P^*_{xz}	5.40288(−2)	5.36678(−2)	6.42729(−2)	3.30405(−2)	3.74316(−2)	3.69930(−2)
5	U	−1.04116	−1.04747	−1.02583	−1.14032	−1.07145	−1.07758
	Q	1.36438(−1)	1.68492(−1)	1.99733(−1)	1.05707(−1)	1.29676(−1)	1.22122(−1)
	P^*_{xz}	2.5(−1)	2.5(−1)	2.5(−1)	2.5(−1)	2.5(−1)	2.5(−1)
6	U	−2.79683	−2.82654	−2.24433	−4.61142	−4.31590	−4.33686
	Q	2.77167(−1)	3.74316(−1)	4.97558(−1)	1.91315(−1)	2.15938(−1)	2.07122(−1)
	P^*_{xz}	−9.94994(−1)	−9.93759(−1)	−1.11093	−7.08846(−1)	−7.24570(−1)	−7.23941(−1)

Table 6
Poiseuille flow: velocity and heat flow profiles based on the LBE.

η	Case 1		Case 2		Case 3	
	$u(-a+2a\eta)$	$q(-a+2a\eta)$	$u(-a+2a\eta)$	$q(-a+2a\eta)$	$u(-a+2a\eta)$	$q(-a+2a\eta)$
0.0	−1.6963	9.5650(−2)	−3.2072	−1.9451(−2)	−1.2100(−1)	3.0563(−2)
0.1	−1.8392	1.3997(−1)	−4.0688	1.1373(−1)	−1.3378(−1)	3.6314(−2)
0.2	−1.8981	1.5862(−1)	−4.5847	1.6753(−1)	−1.4086(−1)	3.9376(−2)
0.3	−1.9177	1.6819(−1)	−4.9639	1.9740(−1)	−1.4559(−1)	4.1350(−2)
0.4	−1.9051	1.7148(−1)	−5.2334	2.1371(−1)	−1.4860(−1)	4.2541(−2)
0.5	−1.8624	1.6941(−1)	−5.4020	2.2063(−1)	−1.5013(−1)	4.3064(−2)
0.6	−1.7894	1.6197(−1)	−5.4725	2.1962(−1)	−1.5027(−1)	4.2954(−2)
0.7	−1.6845	1.4839(−1)	−5.4439	2.1026(−1)	−1.4896(−1)	4.2185(−2)
0.8	−1.5427	1.2678(−1)	−5.3099	1.8979(−1)	−1.4603(−1)	4.0651(−2)
0.9	−1.3508	9.2457(−2)	−5.0507	1.5061(−1)	−1.4093(−1)	3.8070(−2)
1.0	−1.0242	1.9727(−2)	−4.5117	5.2614(−2)	−1.3078(−1)	3.2938(−2)

that while the ADO method could require the post-processing step reported, for example, in Refs. [5] and [10] in order to yield numerical results good to 5 or 6 figures of accuracy for the cases of thin channels, but not so thin that the free-molecular result can be used with great accuracy, the method does yield (for these cases) without the post-processing step results good enough for graphical presentation.

As an aid to check the accuracy of our results, we have followed a procedure similar to that reported in Section 6 of Ref. [9] for channel-flow of a binary mixture subject to Maxwell boundary conditions to deduce the following Onsager relations:

$$K_T Q_P = K_P U_T, \tag{6.3a}$$

$$K_T Q_C = K_C P^*_{xz,T} \tag{6.3b}$$

and

$$K_P(U_C - u_{w,av}) = K_C P^*_{xz,P}, \tag{6.3c}$$

where

$$K_C = \frac{u_{w,1} - u_{w,2}}{2z_0}, \tag{6.4}$$

$$u_{w,av} = \frac{u_{w,1} + u_{w,2}}{2}, \tag{6.5}$$

and P, T, and C subscripts were used to indicate for which problem (Poiseuille, thermal-creep, or Couette flow) the U, Q, and P^*_{xz}

Table 7
Poiseuille flow: velocity and heat flow profiles based on the LBE.

η	Case 4		Case 5		Case 6	
	$u(-a+2a\eta)$	$q(-a+2a\eta)$	$u(-a+2a\eta)$	$q(-a+2a\eta)$	$u(-a+2a\eta)$	$q(-a+2a\eta)$
0.0	−1.4863(−1)	3.7516(−2)	−7.2314(−1)	4.1421(−2)	−4.7562	6.1052(−2)
0.1	−3.8673(−1)	9.8831(−2)	−8.6863(−1)	8.0773(−2)	−5.4639	1.8080(−1)
0.2	−5.2220(−1)	1.2778(−1)	−9.5776(−1)	1.0187(−1)	−5.7388	2.2171(−1)
0.3	−6.1291(−1)	1.4527(−1)	−1.0266	1.1674(−1)	−5.7779	2.4118(−1)
0.4	−6.6864(−1)	1.5510(−1)	−1.0815	1.2759(−1)	−5.6037	2.4990(−1)
0.5	−6.9291(−1)	1.5871(−1)	−1.1250	1.3536(−1)	−5.2224	2.5171(−1)
0.6	−6.8673(−1)	1.5654(−1)	−1.1585	1.4050(−1)	−4.6357	2.4736(−1)
0.7	−6.4922(−1)	1.4818(−1)	−1.1825	1.4314(−1)	−3.8419	2.3513(−1)
0.8	−5.7700(−1)	1.3220(−1)	−1.1970	1.4312(−1)	−2.8353	2.0951(−1)
0.9	−4.6024(−1)	1.0468(−1)	−1.2009	1.3971(−1)	−1.5970	1.5594(−1)
1.0	−2.3629(−1)	4.3238(−2)	−1.1875	1.2831(−1)	0.0	0.0

Table 8
Thermal-creep flow: the mass-flow rate, heat-flow rate, and the shear stress.

Case	Quantity	BGK	S	GJ	MRS	CES	LBE
1	U	1.67243(−1)	2.13317(−1)	2.61337(−1)	1.25515(−1)	1.48136(−1)	1.41313(−1)
	Q	−7.96628(−1)	−1.01371	−1.23900	−6.00635(−1)	−6.33890(−1)	−6.24701(−1)
	P_{xz}	−9.56325(−4)	−1.30053(−3)	−1.73483(−3)	−5.78227(−4)	−4.91068(−4)	−2.50986(−4)
2	U	2.23824(−1)	3.06359(−1)	4.06377(−1)	1.57647(−1)	1.80625(−1)	1.73775(−1)
	Q	−9.88903(−1)	−1.34705	−1.77673	−7.03915(−1)	−7.42614(−1)	−7.35395(−1)
	P_{xz}	3.37543(−3)	4.75881(−3)	6.87604(−3)	1.88548(−3)	1.80475(−3)	1.77269(−3)
3	U	4.34626(−2)	4.60676(−2)	4.79853(−2)	4.00097(−2)	3.96778(−2)	3.99780(−2)
	Q	−2.11338(−1)	−2.23996(−1)	−2.33313(−1)	−1.94565(−1)	−1.72889(−1)	−1.74686(−1)
	P_{xz}	9.26814(−5)	9.84082(−5)	1.03431(−4)	8.26616(−5)	1.53774(−4)	1.24736(−4)
4	U	1.30240(−1)	1.59326(−1)	1.87362(−1)	1.01539(−1)	1.38766(−1)	1.28653(−1)
	Q	−6.64528(−1)	−8.10468(−1)	−9.49678(−1)	−5.21568(−1)	−5.53644(−1)	−5.38634(−1)
	P_{xz}	7.93607(−3)	9.70441(−3)	1.37377(−2)	3.77342(−3)	4.30030(−3)	4.34793(−3)
5	U	1.36438(−1)	1.68492(−1)	1.99733(−1)	1.05707(−1)	1.29676(−1)	1.22122(−1)
	Q	−7.02789(−1)	−8.66587(−1)	−1.02615	−5.45616(−1)	−5.77547(−1)	−5.67070(−1)
	P_{xz}	0.0	0.0	0.0	0.0	0.0	0.0
6	U	2.77167(−1)	3.74316(−1)	4.97558(−1)	1.91315(−1)	2.15938(−1)	2.07122(−1)
	Q	−9.16449(−1)	−1.23579	−1.61442	−6.59275(−1)	−7.34034(−1)	−7.17052(−1)
	P_{xz}	−1.19968(−2)	−1.62866(−2)	−2.39937(−2)	−6.10638(−3)	−3.41900(−3)	−3.59362(−3)

Table 9
Thermal-creep flow: velocity and heat flow profiles based on the LBE.

η	Case 1		Case 2		Case 3	
	$u(-a+2a\eta)$	$q(-a+2a\eta)$	$u(-a+2a\eta)$	$q(-a+2a\eta)$	$u(-a+2a\eta)$	$q(-a+2a\eta)$
0.0	9.7457(−2)	−4.9562(−1)	7.7233(−2)	−4.0853(−1)	3.1658(−2)	−1.3815(−1)
0.1	1.3162(−1)	−6.0885(−1)	1.4448(−1)	−6.4779(−1)	3.6691(−2)	−1.6105(−1)
0.2	1.4631(−1)	−6.5389(−1)	1.6979(−1)	−7.2735(−1)	3.9387(−2)	−1.7310(−1)
0.3	1.5444(−1)	−6.7729(−1)	1.8334(−1)	−7.6791(−1)	4.1134(−2)	−1.8075(−1)
0.4	1.5828(−1)	−6.8698(−1)	1.9054(−1)	−7.8928(−1)	4.2193(−2)	−1.8525(−1)
0.5	1.5863(−1)	−6.8559(−1)	1.9359(−1)	−7.9876(−1)	4.2668(−2)	−1.8708(−1)
0.6	1.5568(−1)	−6.7331(−1)	1.9335(−1)	−7.9889(−1)	4.2588(−2)	−1.8637(−1)
0.7	1.4914(−1)	−6.4832(−1)	1.8987(−1)	−7.8924(−1)	4.1934(−2)	−1.8302(−1)
0.8	1.3807(−1)	−6.0574(−1)	1.8233(−1)	−7.6564(−1)	4.0619(−2)	−1.7659(−1)
0.9	1.2008(−1)	−5.3296(−1)	1.6817(−1)	−7.1502(−1)	3.8407(−2)	−1.6594(−1)
1.0	8.1818(−2)	−3.5897(−1)	1.3289(−1)	−5.5350(−1)	3.4038(−2)	−1.4496(−1)

quantities should be considered in these expressions. It should be noted that Eqs. (6.3) are valid for both the cases of Maxwell and Cercignani–Lampis boundary conditions. In addition, while Eqs. (6.3b,c) can be used to eliminate, for example, the U and Q calculations for the Couette-flow problem, we note that we have used Eqs. (6.3) simply as one of the checks of our numerical work.

Even though several hours of computer time were required to establish the high-quality results we are reporting in Tables 2–10, a solution good enough for graphical presentation can be obtained with very modest computational expense. To give an idea of the CPU time for what we might consider "practical results," we found, for example, that the results reported in Tables 2–10 for the six considered LBE cases could be obtained with essentially three figures of accuracy in less than 8 minutes on an Intel Core 2 Duo machine running at 2.6 GHz.

It can be noted that we have included in Tables 2, 5, and 8 the results we found for the five kinetic models listed in Section 5, as well as for the linearized Boltzmann equation (for rigid-sphere scattering). Of the five models, the CES looks the best; however, this model is somewhat more complicated than the other four. On the other hand, the MRS model looks very good, especially since it can be used with very little more effort than that required for the BGK model (which looks rather poor in this study).

We consider that the numerical results based on the LBE and the C-L boundary conditions are the most important contribution of this work, and so we would have liked to have given some

Table 10
Thermal-creep flow: velocity and heat flow profiles based on the LBE.

η	Case 4		Case 5		Case 6	
	$u(-a+2a\eta)$	$q(-a+2a\eta)$	$u(-a+2a\eta)$	$q(-a+2a\eta)$	$u(-a+2a\eta)$	$q(-a+2a\eta)$
0.0	2.8882(−2)	−2.5507(−1)	7.2301(−2)	−3.3979(−1)	1.0980(−1)	−5.1855(−1)
0.1	9.6653(−2)	−4.5086(−1)	9.8097(−2)	−4.5240(−1)	1.9730(−1)	−7.3083(−1)
0.2	1.2682(−1)	−5.3440(−1)	1.1180(−1)	−5.0974(−1)	2.2402(−1)	−7.8758(−1)
0.3	1.4439(−1)	−5.8265(−1)	1.2124(−1)	−5.4912(−1)	2.3675(−1)	−8.1087(−1)
0.4	1.5406(−1)	−6.0941(−1)	1.2787(−1)	−5.7766(−1)	2.4269(−1)	−8.1845(−1)
0.5	1.5768(−1)	−6.1989(−1)	1.3226(−1)	−5.9847(−1)	2.4391(−1)	−8.1515(−1)
0.6	1.5587(−1)	−6.1566(−1)	1.3464(−1)	−6.1315(−1)	2.4022(−1)	−8.0005(−1)
0.7	1.4840(−1)	−5.9568(−1)	1.3502(−1)	−6.2253(−1)	2.2934(−1)	−7.6653(−1)
0.8	1.3398(−1)	−5.5546(−1)	1.3309(−1)	−6.2684(−1)	2.0548(−1)	−6.9704(−1)
0.9	1.0891(−1)	−4.8246(−1)	1.2797(−1)	−6.2548(−1)	1.5388(−1)	−5.4189(−1)
1.0	5.1736(−2)	−3.0474(−1)	1.1443(−1)	−6.1337(−1)	0.0	0.0

comparisons with other works, but (since, to our knowledge, there exist no similar results) this has not been possible. However, as mentioned in the Introduction, there are some numerical results available in the literature for two of the kinetic models also considered in this work: the BGK model and the S model. We have used the S-model results of Ref. [14] which were reported with seven digits of accuracy to establish confidence that the procedures used in this work are able to yield results with the number of digits shown in our tables (five for the profiles and six for the flow rates). In regard to this point, it should be noted that, as reported, for example, in Ref. [14], the kinetic models can be solved in terms of a projection technique that has the potential to yield results even more accurate than the more general procedure we use here for the LBE (and the kinetic models). The main reason for this is that the projection technique used, for example, in Ref. [14] takes care of the dependence of the c variable in the kinetic equation exactly and in a very efficient way. Since the projection technique that can be used for simple kinetic models is more compact (than what is used in this work for the LBE and the kinetic models), it has the advantage of allowing (before computer-memory limitations of easily available machines are reached) the use of discrete-ordinates approximations of higher order than those allowed by the general (basis-function) procedure.

Continuing with our comparisons with kinetic models, we note that we found general agreement (but, in some cases, less than the reported digits) with the results of Refs. [15,19], and [20]. Specifically, we have observed that some results of Ref. [15] (for example, the values of G_T reported in Table 2 of Ref. [15] for the case $\delta = 10.0$, $\alpha_t = 0.25$, and $\alpha_n = 1.0$ and for the case $\delta = 100.0$, $\alpha_t = 0.25$, and $\alpha_n = 0.25$) deviate by more than the stated tolerance (0.1%) from what we consider to be the correct results (respectively 0.09834 and 0.009445) and that the first three rows of results for the Poiseuille flow rate reported in Table 5 of Ref. [19] for $\delta = 5.0$ and for $\delta = 9.0$ differ from again what we consider to be the correct results by amounts that may reach seven units in the last (fifth) figure. Here we should mention that the δ parameter in Refs. [15] and [19] is equivalent to $2a$ in our work. In addition, we found that the second column (of results) of Table 1 of Ref. [20] is correct to only two of the reported six digits and that the results based on $\alpha_n = 0.01$ in that work [20] are (except for those in Table 16) not as accurate as reported. Moreover, to confirm the shear stresses reported in Table 16 of Ref. [20] for Couette flow, we had to multiply our results by $\sqrt{\pi}/2$, the factor 2 being due to a difference in definition, as can be seen by comparing Eq. (2.25) of our work and Eq. (14) of Ref. [20]. To complete our comments on works based on kinetic models and C-L boundary conditions, we note that Cercignani, Lampis, and Lorenzani, whose work [19] is based on the BGK model, have included in their Table 2 Siewert's results [14], without noting that those results were based on the S model, not the BGK model.

Finally, we would like to comment on two different ways of comparing results, for rigid-sphere scattering, from the linearized Boltzmann equation and the five kinetic models listed in our tables. In this work, we have considered that the problems are defined in terms of the physical width of the channel (in cm, for example), the diameter of the rigid spheres (also in cm), the equilibrium number density (in cm^{-3}), and the pressure and temperature gradients (in cm^{-1}). We have further considered that these defining parameters are the same for the LBE and all five kinetic models. On the other hand, in a previous paper [5] we took a different approach when comparing the kinetic models and the LBE. In that work [5] we considered the basic parameters to be the same when defined in terms of dimensionless quantities which were based on one of many possible mean-free paths that could be different for some of the models and the LBE. While the merits of each of these two ways of comparing the kinetic models and the LBE can be seen, it is important to keep in mind in which way the comparisons have been done.

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