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On the use of a nascent delta function in radiative-transfer calculations for multi-layer media subject to Fresnel boundary and interface conditions

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ABSTRACT

The "pre-processing" procedure and the "break-point" analysis developed in a previous work based on the ADO (analytical discrete ordinates) method are used, along with a nascent delta function to describe the polar-angle dependence of an incident beam, to solve the classical albedo problem for radiative transfer in a plane-parallel, multi-layer medium subject to Fresnel boundary and interface conditions. As a result of the use of a nascent delta function, rather than the Dirac distribution, to model the polar-angle dependence of the incident beam, the computational work is significantly simplified (since a particular solution is not required) in comparison to an approach where both the polar-angle and the azimuthal-angle dependence of the incident beam are formulated in terms of Dirac delta distributions. The numerical results from this approach are (when a sufficiently small "narrowness" parameter is used to define the nascent delta) found to be in complete agreement with already reported (high-quality) results for a set of challenging multi-layer problems.

1. Introduction

The classical albedo problem for radiative transfer in a plane-parallel medium is basic to many fields of study [1–5]. In a recent paper [6], we reported a general solution for the case of an arbitrary number of dissimilar media (with Fresnel boundary and interface conditions) illuminated by an incident beam modeled by Dirac delta distributions in both the polar and azimuthal angles. The solution developed in Ref. [6] required, because of the way the intensity was decomposed, a particular solution that was expressed (in a not so simple way) in terms of our ADO (analytical discrete ordinates) method [7,8]. While the particular solution used in Ref. [6] was not simple, it was general in that a possible singularity (that can occur when Chandrasekhar's particular solution [1] is used) was avoided. Here, in order to simplify the work reported in Ref. [6], we make use of a nascent delta function [9] instead of the Dirac distribution, to describe the polar-angle dependence of the incident beam.

As our numerical methods allow the "narrowness" parameter ε in the definition (to be given later in this work) of the nascent delta function to be an input parameter, we can simulate the effect of the Dirac delta distribution by using smaller and smaller values of ε.

Before continuing, we note that the way [6,10] in which we define the quadrature scheme in the ADO method (initially to take into account discontinuities in the derivative of the radiation intensity with respect to the cosine of the polar angle as radiation passes between media with different indices of refraction) allows us easily to take into account the discontinuity introduced by the discontinuous nascent delta function we use in this work.

2. Formulation of the problem

As in our previous work [6], we consider that the multi-layer medium is composed of K distinct layers

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which are labeled with subscript k, for $k = 1, 2, \ldots, K$, and we take the radiation field in layer k to be described by the monochromatic or grey equation of transfer [1]

$$\mu \frac{\partial}{\partial \tau} I_k(\tau, \mu, \phi) + I_k(\tau, \mu, \phi) = \frac{\varpi_k}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p_k(\mathbf{\Omega}' \cdot \mathbf{\Omega}) I_k(\tau, \mu', \phi') \, \mathrm{d}\phi' \, \mathrm{d}\mu', \tag{1}$$

for $\tau \in (a_{k-1}, a_k)$, $\mu \in [-1, 1]$ and $\phi \in [0, 2\pi]$. In our notation, a_0 defines the location of the first surface, a_k for $k = 1, 2, \ldots, K-1$ the locations of the interfaces between the layers, and a_K the location of the last surface. In addition, τ denotes the optical variable defined for each layer as in Ref. [10], $\varpi_k \in [0, 1]$ is the single-scattering albedo for layer k and $p_k(\mathbf{\Omega}' \cdot \mathbf{\Omega})$ is the phase function for layer k, which we assume depends only on the scattering angle defined by

$$\cos \Theta = \mathbf{\Omega}' \cdot \mathbf{\Omega}, \tag{2}$$

where $\mathbf{\Omega}'$ and $\mathbf{\Omega}$ are unit vectors that define the directions of propagation of the radiation before and after a scattering event. We consider in this work phase functions that can be represented by truncated Legendre-polynomial expansions of the form

$$p_k(\mathbf{\Omega}' \cdot \mathbf{\Omega}) = \sum_{l=0}^{L_k} \beta_{k,l} P_l(\cos \Theta), \tag{3}$$

where $\{\beta_{k,l}\}$ are the expansion coefficients for the kth layer. For phase functions expressed as in Eq. (3), we can use the addition theorem for the Legendre polynomials to write

$$p_k(\mathbf{\Omega}' \cdot \mathbf{\Omega}) = \sum_{m=0}^{L_k} (2 - \delta_{0,m}) \sum_{l=m}^{L_k} \beta_{k,l} P_l^m(\mu') P_l^m(\mu) \cos[m(\phi' - \phi)], \tag{4}$$

where $\delta_{i,j}$ denotes the Kronecker delta and $P_l^m(\mu)$ denotes a *normalized* Legendre function of the first kind. To complete the specification of Eq. (1), we note that the polar and azimuthal angles, $\theta = \arccos \mu$ and ϕ, define the direction $\mathbf{\Omega}$, whereas $\theta' = \arccos \mu'$ and ϕ' define the direction $\mathbf{\Omega}'$.

With regard to the boundary (surface) conditions, we assume there is incident on the surface located at $\tau = a_0$, from an external medium characterized by an index of refraction n_0, a distribution of radiation described by Dirac distributions in μ and ϕ, i.e.

$$\psi_0(\mu, \phi) = \delta(\mu - \mu_0) \delta(\phi - \phi_0), \tag{5}$$

for $\mu, \mu_0 \in (0, 1]$ and $\phi, \phi_0 \in [0, 2\pi]$. Here $\theta_0 = \arccos \mu_0$ and ϕ_0 are the polar and azimuthal angles that define the direction of the incoming beam. In addition, we assume that there is no radiation incident on the surface located at $\tau = a_K$ from an external medium characterized by index of refraction n_{K+1}. And so, following previous works [11–13], we can use Snell's law and the Fresnel formulas for the reflection and transmission coefficients [14,15], to write the surface conditions as

$$I_1(a_0, \mu, \phi) - X(n_{1,0}, \mu) I_1(a_0, -\mu, \phi) = Y(n_{1,0}, \mu) \delta[f(n_{1,0}, \mu) - \mu_0] \delta(\phi - \phi_0) \tag{6a}$$

and

$$I_K(a_K, -\mu, \phi) - X(n_{K,K+1}, \mu) I_K(a_K, \mu, \phi) = 0, \tag{6b}$$

for $\mu \in (0, 1]$ and $\phi \in [0, 2\pi]$. Here, X and Y are the Fresnel reflection and transmission coefficients expressed as in Ref. [6],

$$n_{\alpha,\beta} = n_\alpha / n_\beta, \tag{7}$$

with n_α and n_β denoting the indices of refraction for layers α and β, respectively, and, in general,

$$f(n, \mu) = [1 - n^2(1 - \mu^2)]^{1/2}. \tag{8}$$

As mentioned in Ref. [6], should $n_{1,0} < 1$ we must apply the restriction

$$\mu_0 > (1 - n_{1,0}^2)^{1/2}, \tag{9}$$

for the driving term in Eq. (6a) not to be zero.

At each of the interfaces we have conditions similar to Eqs. (6) except that now there are no known terms, i.e.

$$I_k(a_k, -\mu, \phi) - X(n_{k,k+1}, \mu) I_k(a_k, \mu, \phi) = Y(n_{k,k+1}, \mu) I_{k+1}[a_k, -f(n_{k,k+1}, \mu), \phi] \tag{10a}$$

and

$$I_{k+1}(a_k, \mu, \phi) - X(n_{k+1,k}, \mu) I_{k+1}(a_k, -\mu, \phi) = Y(n_{k+1,k}, \mu) I_k[a_k, f(n_{k+1,k}, \mu), \phi], \tag{10b}$$

for $\mu \in (0, 1]$, $\phi \in [0, 2\pi]$ and $k = 1, 2, \ldots, K-1$.

At this point, we can use the pre-processing procedure introduced in Ref. [10] and extended for azimuthally dependent problems in Ref. [6] to express all of the boundary/interface conditions as

$$I_k(a_{k-1}, \mu, \phi) - Z_k^-(\mu) I_k(a_{k-1}, -\mu, \phi) = F_k \delta(\mu - \mu_k) \delta(\phi - \phi_0) + W_k^-(\mu, \phi) \tag{11a}$$

and

$$I_k(a_k, -\mu, \phi) - Z_k^+(\mu) I_k(a_k, \mu, \phi) = W_k^+(\mu, \phi), \tag{11b}$$

for $\mu \in (0, 1]$, $\phi \in [0, 2\pi]$ and $k = 1, 2, \ldots, K$. Here

$$\mu_k = f(n_{0,k}, \mu_0), \tag{12}$$

the functions $Z_k^\pm(\mu)$ and $W_k^\pm(\mu, \phi)$ are defined by recurrence relations which involve a change of argument at each evaluation step, and the constants F_k are also defined by a recurrence relation. Details are given in Ref. [6].

3. Fourier decomposition

Our problem, as defined in Section 2, can be setup more conveniently by using a (finite) Fourier representation of the intensity. And so, using $L_{\max}$ to denote the largest scattering order in the multi-layer medium, i.e. $L_{\max} = \max\{L_1, L_2, \ldots, L_K\}$, we can write

$$I_k(\tau, \mu, \phi) = I_k^{(0)}(\tau, \mu, \phi) + \frac{1}{2\pi} \sum_{m=0}^{L_{\max}} (2 - \delta_{0,m}) \cos[m(\phi - \phi_0)] \times \{I_{k,m}(\tau, \mu) - S_k^+ \exp[-(\tau - a_{k-1})/\mu] \delta(\mu - \mu_k)\} \tag{13a}$$

and

$$I_k(\tau,-\mu,\phi)=I_k^{(0)}(\tau,-\mu,\phi)+\frac{1}{2\pi}\sum_{m=0}^{L_{\max}}(2-\delta_{0,m})\cos[m(\phi-\phi_0)]\times\{I_{k,m}(\tau,-\mu)-S_k^-\exp[-(a_k-\tau)/\mu]\delta(\mu-\mu_k)\},\quad(13b)$$

for $\tau\in(a_{k-1},a_k)$, $\mu\in(0,1]$ and $\phi\in[0,2\pi]$. We note that the special result for the case of no scattering, i.e. $\varpi_k=0$, $k=1,2,\ldots,K$, is a component of Eqs. (13) and is given by [6]

$$I_k^{(0)}(\tau,\mu,\phi)=S_k^+\exp[-(\tau-a_{k-1})/\mu_k]\delta(\mu-\mu_k)\delta(\phi-\phi_0)\quad(14a)$$

and

$$I_k^{(0)}(\tau,-\mu,\phi)=S_k^-\exp[-(a_k-\tau)/\mu_k]\delta(\mu-\mu_k)\delta(\phi-\phi_0).\quad(14b)$$

Here and in Eqs. (13),

$$S_k^+=F_k\,\Upsilon_k(\mu_k)\quad(15a)$$

and

$$S_k^-=F_k\,\Upsilon_k(\mu_k)Z_k^+(\mu_k)\exp(-\Delta_k/\mu_k),\quad(15b)$$

where $\Delta_k=a_k-a_{k-1}$ and

$$\Upsilon_k(\mu)=[1-Z_k^+(\mu)Z_k^-(\mu)\exp(-2\Delta_k/\mu)]^{-1}.\quad(16)$$

Now, making use of Eq. (4), we find that Eqs. (13) will be a solution of Eq. (1) provided $I_{k,m}(\tau,\mu)$ satisfies the reduced equation

$$\mu\frac{\partial}{\partial\tau}I_{k,m}(\tau,\mu)+I_{k,m}(\tau,\mu)=\frac{\varpi_k}{2}\sum_{l=m}^{L_k}\beta_{k,l}P_l^m(\mu)\int_{-1}^{1}P_l^m(\mu')I_{k,m}(\tau,\mu')\mathrm{d}\mu',\quad(17)$$

for $\tau\in(a_{k-1},a_k)$ and $\mu\in[-1,1]$. In regard to the boundary and interface conditions subject to which we must solve Eq. (17), we find that Eqs. (11) can be decomposed so as to yield

$$I_{k,m}(a_{k-1},\mu)-Z_k^-(\mu)I_{k,m}(a_{k-1},-\mu)=F_k\delta(\mu-\mu_k)+W_{k,m}^-(\mu)\quad(18a)$$

and

$$I_{k,m}(a_k,-\mu)-Z_k^+(\mu)I_{k,m}(a_k,\mu)=W_{k,m}^+(\mu),\quad(18b)$$

for $\mu\in(0,1]$. Here, the functions

$$W_{k,m}^{\pm}(\mu)=\pi(1+\delta_{0,m})\int_0^{2\pi}W_k^{\pm}(\mu,\phi)\cos[m(\phi-\phi_0)]\,\mathrm{d}\phi\quad(19)$$

are explicitly given by Eqs. (3.12)–(3.16) of Ref. [6], with only a minor modification in Eq. (3.16) of Ref. [6]: the source term $\Psi_k^m(\tau,\mu)$ should be disregarded, as it has been avoided in the current formulation.

At this point, we replace the Dirac distribution $\delta(\mu-\mu_k)$ on the right-hand side of Eq. (18a) by the nascent delta function

$$\delta_\varepsilon(\mu-\mu_k)=\begin{cases}(\mu_{\max}-\mu_{\min})^{-1}, & \mu_{\min}\le\mu\le\mu_{\max},\\ 0, & \text{otherwise.}\end{cases}\quad(20)$$

Here

$$\mu_{\min}=\max\{0,\mu_k-\varepsilon/2\}\quad(21a)$$

and

$$\mu_{\max}=\min\{\mu_k+\varepsilon/2,1\},\quad(21b)$$

where ε is our "narrowness" parameter. We thus consider that the boundary and interface conditions subject to which we must solve Eq. (17) are

$$I_{k,m}(a_{k-1},\mu)-Z_k^-(\mu)I_{k,m}(a_{k-1},-\mu)=F_k\delta_\varepsilon(\mu-\mu_k)+W_{k,m}^-(\mu)\quad(22a)$$

and

$$I_{k,m}(a_k,-\mu)-Z_k^+(\mu)I_{k,m}(a_k,\mu)=W_{k,m}^+(\mu),\quad(22b)$$

for $\mu\in(0,1]$.

In the end, we are left with $K(L_{\max}+1)$ azimuthally independent problems defined by Eqs. (17) and (22) for $k=1,2,\ldots,K$ and $m=0,1,\ldots,L_{\max}$. For a given value of the Fourier index, m, we note that the resulting K problems for $I_{k,m}(\tau,\mu)$, $k=1,2,\ldots,K$, are coupled by way of the terms $W_{k,m}^{\pm}(\mu)$ in Eqs. (22). For this reason, the approach we use to solve these problems is the same iterative procedure defined and used in Ref. [6].

4. An ADO solution for the Fourier component problems

In order to solve the set of Fourier component problems just defined so as to be able to complete the solution given by Eqs. (13), we use the ADO version [16] of the discrete-ordinates method. Since the ADO method has been discussed extensively in previous works [7,16], our presentation in this work is brief.

Before starting on the ADO solution, we wish to add a few words on our way of obtaining a discrete-ordinates version of Eq. (17). First, we split the integration interval $[-1,1]$ in that equation into two half-range intervals and apply the transformation $\mu\to-\mu$ to the negative interval, so that the integral in Eq. (17) can be expressed as a sum of two integrals over the positive half-range interval [0,1]. Next, we approximate the integral over [0,1] by a composite quadrature of order N_k with nodes and weights η_i and w_i, $i=1,2,\ldots,N_k$. As mentioned in Section 1, such a quadrature scheme is obtained in this work by adding the points of discontinuity $\mu_{\min}$ and $\mu_{\max}$ of the nascent delta function defined by Eq. (20) to the set of break points obtained from the break-point analysis reported in Ref. [10]. Using a standard Gauss-Legendre quadrature of order M mapped linearly onto each of the subintervals of [0,1] defined by the enlarged set of break points, we obtain a composite quadrature of order $N_k=s_kM$, where s_k denotes the number of subintervals into which [0,1] is subdivided. Note that we end up using, as in Refs. [6,10], a different quadrature for each layer (the number and location of the break points are layer-dependent), and so, to be rigorous in our notation, we should have affixed an index k to the quadrature nodes and weights. However, for simplicity, we prefer not to do so.

We begin our description of the ADO method with the case of a non-conservative layer ($\varpi_k\neq1$). Following previous works [6,7,16], we write our ADO solution to the

discrete-ordinates version of Eq. (17) as

$$I_{k,m}(\tau,\pm\eta_i)=\sum_{j=1}^{N_k}\{A^m_{k,j}\phi^m_k(\nu_{k,j},\pm\eta_i)\exp[-(\tau-a_{k-1})/\nu_{k,j}] + B^m_{k,j}\phi^m_k(\nu_{k,j},\mp\eta_i)\exp[-(a_k-\tau)/\nu_{k,j}]\}, \quad (23)$$

for $\tau\in[a_{k-1},a_k]$. In this expression, the separation constants $\{\nu_{k,j}\}$ and the elementary solutions $\{\phi^m_k(\nu_{k,j},\pm\eta_i)\}$ are determined by solving an eigensystem of order N_k, as discussed in Ref. [7], and the coefficients $\{A^m_{k,j}\}$ and $\{B^m_{k,j}\}$ are to be determined from a discrete-ordinates version of the Fresnel boundary and interface conditions expressed by Eqs. (22). Again, in order to avoid complicated notation, we have suppressed the index m that should, in principle, be affixed to the separation constants $\{\nu_{k,j}\}$. In addition, as we usually employ, to increase the computational efficiency of our solution, quadrature orders that decrease as the Fourier index of the component problem is increased, we should keep in mind that the number of quadrature points per subinterval M (and so the quadrature order N_k) may depend on m.

To find the coefficients $\{A^m_{k,j}\}$ and $\{B^m_{k,j}\}$ that are required for completing our ADO solution listed as Eq. (23), we substitute Eq. (23) into discrete-ordinates versions of Eqs. (22) evaluated at $\mu=\eta_i$, $i=1,2,\ldots,N_k$. We thus obtain, for layer k, a system of $2N_k$ linear equations to be solved for $\{A^m_{k,j}\}$ and $\{B^m_{k,j}\}$. However, since for a given Fourier index m the systems of linear equations that are obtained for $k=1,2,\ldots,K$ are coupled by the (at this point) unknown right-hand side terms $W^{\pm}_{k,m}(\eta_i)$, our way of solving these systems is, as in Ref. [6], iterative. Since the procedure used for this purpose is exactly the same procedure that has been used and explained in detail in Ref. [6], we do not elaborate more on this point here.

Finally, we would like to comment briefly on the modifications required by our ADO solution for the case of a conservative layer, $\varpi_k=1$. As discussed in previous works [7,10], the difficulty with the conservative case is that one of the separation constants becomes unbounded when $\varpi_k=1$ and $m=0$. To overcome this difficulty, we follow Refs. [7,10] and simply replace the solutions associated with the unbounded separation constant, say $\nu_{k,1}$, by the exact solutions 1 and $\tau-3\mu/h_{k,1}$, where

$$h_{k,1}=3-\beta_{k,1}. \quad (24)$$

We thus rewrite Eq. (23) for the case $\varpi_k=1$ and $m=0$ as

$$I_k(\tau,\pm\eta_i)=I^*_k(\tau,\pm\eta_i)+\sum_{j=2}^{N_k}\{A_{k,j}\phi_k(\nu_{k,j},\pm\eta_i)\exp[-(\tau-a_{k-1})/\nu_{k,j}] + B_{k,j}\phi_k(\nu_{k,j},\mp\eta_i)\exp[-(a_k-\tau)/\nu_{k,j}]\}, \quad (25)$$

where

$$I^*_k(\tau,\mu)=A_{k,1}+B_{k,1}(\tau-3\mu/h_{k,1}). \quad (26)$$

Note that, to simplify our notation, we have omitted the index m in some quantities of Eq. (25), with the understanding that the absence of index m implies that these quantities are being considered for $m=0$.

From this point on, we proceed in exactly the same way as for the non-conservative case. We substitute Eq. (25) into discrete-ordinates versions of Eqs. (22) evaluated at $\mu=\eta_i$, for $i=1,2,\ldots,N_k$, to obtain a linear system of $2N_k$ equations that is solved iteratively for the $2N_k$ coefficients $A_{k,j}$ and $B_{k,j}$, $j=1,2,\ldots,N_k$.

5. Numerical results

As a test of our numerical implementation, we have considered a collection of five multi-layer problems which have been accurately solved in Ref. [6]. In Table 1, we list the basic data for the layers that are used to define these problems in Table 2. For all of the problems, it is assumed that the multi-layers are surrounded by vacuum, so that $n_0=n_{K+1}=1$. In addition, the cosine of the polar angle of incidence is specified as $\mu_0=\frac{1}{2}$, and the azimuthal angle of incidence ϕ_0 can be arbitrarily chosen. Scattering is described by the binomial law [17] with different degrees of anisotropy in the layers, as specified by parameter L in Table 1.

In Tables 3 and 4, we report, for four different values of the "narrowness" parameter ε in Eqs. (21), our converged numerical results for the reflectance

$$A=J^-_0/J^+_0 \quad (27)$$

and the transmittance

$$B=J^+_K/J^+_0. \quad (28)$$

As in Ref. [6], we use

$$J^+_0\equiv\int_0^1\int_0^{2\pi}\psi_0(\mu,\phi)\,\mathrm{d}\phi\mu\,\mathrm{d}\mu=\mu_0 \quad (29)$$

to denote the incoming flux at $\tau=a_0$ and

$$J^-_0\equiv\int_0^1\int_0^{2\pi}X(n_{0,1},\mu)\psi_0(\mu,\phi)\,\mathrm{d}\phi\mu\,\mathrm{d}\mu + \int_0^1\int_0^{2\pi}[1-X(n_{1,0},\mu)]I_1(a_0,-\mu,\phi)\,\mathrm{d}\phi\mu\,\mathrm{d}\mu \quad (30a)$$

and

$$J^+_K\equiv\int_0^1\int_0^{2\pi}[1-X(n_{K,K+1},\mu)]I_K(a_K,\mu,\phi)\,\mathrm{d}\phi\mu\,\mathrm{d}\mu \quad (30b)$$

to denote the fluxes exiting, respectively, the surfaces $\tau=a_0$ and $\tau=a_K$. Using Eqs. (5), (13), and (14) and approximating the μ-integration in the last integral of Eq. (30a) and in the integral of Eq. (30b) with our quadrature schemes defined, respectively, for the first

Table 1
Basic data for the layers.

Layer #	Δ	ϖ	L	n
1	1.0	0.95	40	1.65
2	1.2	0.94	60	2.00
3	1.3	0.93	30	1.70
4	0.6	0.96	70	1.60
5	1.9	0.90	20	1.80
6	1.4	0.92	50	1.85
7	0.5	0.97	80	1.55
8	0.3	0.98	90	1.50
9	1.6	0.91	10	1.75
10	5.2	1.00	100	1.30

Table 2
The problems.

Problem	Layers
I	1–3
II	6–10
III	4–10
IV	1–9
V	1–10

Table 3
The reflectance A.

Problem	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$
I	2.003203(−1)	2.005067(−1)	2.005097(−1)	2.005097(−1)
II	2.177122(−1)	2.179171(−1)	2.179189(−1)	2.179189(−1)
III	1.486410(−1)	1.486853(−1)	1.486857(−1)	1.486857(−1)
IV	1.442097(−1)	1.442142(−1)	1.442142(−1)	1.442142(−1)
V	1.443562(−1)	1.443597(−1)	1.443597(−1)	1.443597(−1)

Table 4
The transmittance B.

Problem	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$
I	3.245426(−1)	3.235017(−1)	3.234892(−1)	3.234892(−1)
II	2.453355(−1)	2.447075(−1)	2.447012(−1)	2.447012(−1)
III	1.483737(−1)	1.480923(−1)	1.480895(−1)	1.480895(−1)
IV	8.917569(−2)	8.902618(−2)	8.902468(−2)	8.902468(−2)
V	8.999551(−2)	8.985556(−2)	8.985415(−2)	8.985415(−2)

and last layers, we find

$$J_0^- = \mu_0 X(n_{1,0}, \mu_0) + \sum_{i=1}^{N_1} w_i \eta_i [1 - X(n_{1,0}, \eta_i)] I_1(a_0, -\eta_i) \tag{31a}$$

and

$$J_K^+ = \sum_{i=1}^{N_K} w_i \eta_i [1 - X(n_{K,K+1}, \eta_i)] I_K(a_K, \eta_i). \tag{31b}$$

Here, $I_1(a_0, -\eta_i)$ and $I_K(a_K, \eta_i)$ are given by either Eq. (23) for $m = 0$ or Eq. (25), whichever is applicable.

We note that the converged numerical results reported for A in Table 3 and for B in Table 4 were established by varying M, the number of quadrature points per subinterval, between 50 and 240, for each of the considered values of ε. On comparing these results with the numerical results given in Table 4 of Ref. [6], we can see that the results obtained in this work with $\varepsilon = 10^{-2}$ have at least four figures of accuracy and that the results obtained with the choices $\varepsilon = 10^{-4}$ and 10^{-6} are in complete agreement with those reported with seven digits of accuracy in Ref. [6]. We have also verified that our converged numerical results for the scattered components of the intensities exiting the surfaces $\tau = a_0$ and $\tau = a_K$ are, when the "narrowness" parameter ε is taken to be equal to 10^{-5}, in complete agreement with the 5-digit accurate numerical results reported in Tables 5–10 of Ref. [6]. The same kind of agreement was observed for intensities computed at interior points.

Computations were carried out on a machine equipped with an AMD Athlon 64 X2 4800+ dual core processor running at 2.4 GHz and 2 GB of RAM. While some hours of CPU time were required to generate our most accurate results, numerical results good enough for practical use can be generated in much less time. For example, we have found that we could use our developed code with the choices $\varepsilon = 10^{-5}$ and $M = 10$ (for all of the Fourier component problems) to obtain, in 175 s of CPU time on the Athlon machine, numerical results with at least three significant figures of accuracy for the reflectance, the transmittance, and the exiting intensities at the boundaries for the most challenging of our test problems (problem V). This should be compared with the 110 s of CPU time used by the code of Ref. [6] to compute similar results for problem V on the same machine. The observed increase in CPU time can be explained by the use of an additional subinterval in this work to define the layer-dependent composite quadrature scheme for the ADO method. As mentioned in the Introduction, the additional subinterval is needed in order to represent well the discontinuous nascent delta function that replaces the Dirac delta distribution in this work.

Finally, we note that we have been able to reproduce the numerical results for the reflectances and transmittances reported for a set of two-layer problems in Tables 1 and 2 of Ref. [6], using $\varepsilon = 10^{-5}$ in our formulation. With this same value of ε, we have also been able to reproduce the accurate fluxes reported for a set of atmospheric radiative-transfer problems in Table 22 of Ref. [18].

6. Concluding remarks

We have developed in this work an alternative and (we believe) much simpler way of solving the classical albedo problem for radiative transfer in a plane-parallel, multi-layer medium subject to Fresnel boundary and interface conditions. Our approach is based on approximating the Dirac delta distribution that describes the polar-angle dependence of the incident beam by a nascent delta function defined in terms of a "narrowness" parameter ε. In particular, it has been shown that numerical results of an accuracy comparable to that of the exact formulation based on the Dirac delta distribution can be obtained when sufficiently small values of ε are used. We expect the proposed approach to be useful in a planned extension of our work [6,10], i.e. radiative transfer with polarization in multi-layers subject to Fresnel boundary and interface conditions.

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