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Technical note

Some evaluations of basic nodal-like schemes for a selection of classical problems in particle transport theory

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ABSTRACT

In order to provide some basic evaluations of elementary nodal techniques, as used in the general area of particle transport theory, critical and albedo problems for cylinders, spheres, and slabs are solved approximately in terms of averaged quantities and compared to exact results.

1. Introduction

In a recent work, Williams (2007) used a nodal-like approximation to study the problem of radiative transfer in infinite and semi-infinite cylinders subject to the mathematically challenging Fresnel boundary condition(s) on the curved (infinite length case) surface and both the curved and plane surfaces for the semi-infinite case. Fundamental background material for the nodal-like approximation used by Williams (2007) can be found in the works of Prinja and Pomraning (1984), Larsen (1984) and Larsen et al. (1986). In this brief communication we make use of the nodal-like approach to establish approximate solutions to two classical problems (the critical problem and the albedo problem) for an infinite cylinder. Since essentially exact results for these two problems are available (Thomas et al., 1983; Siewert and Thomas, 1985), we are able to evaluate exactly the merits of the nodal-like approximation used by Williams for these two elementary problems. These same two problems are investigated also for the case of spheres and slabs (Siewert and Thomas, 1985; Siewert and Grandjean, 1979; Grandjean and Siewert, 1979).

2. The cylindrical case

We consider the transport equation and the surface boundary condition, for the case of an infinitely long cylinder (with no variations in the axial direction and rotational symmetry about the axis of the cylinder) of radius R written (Bell and Glasstone, 1979) as

$$\left[(1-\mu^2)^{1/2}\left(\cos\phi\frac{\partial}{\partial r}-\frac{1}{r}\sin\phi\frac{\partial}{\partial\phi}\right)+1\right]\Psi(r,\mu,\phi) = \frac{c}{4\pi}\int_0^{2\pi}\int_{-1}^{1}\Psi(r,\mu',\phi')\mathrm{d}\mu'\,\mathrm{d}\phi', \tag{1}$$

for $r\in(0,R)$, $\mu\in[-1,1]$, $\phi\in(0,2\pi)$, and

$$\Psi(R,\mu,\phi)=F,\quad \mu\in[-1,1]\quad\text{and}\quad\phi\in[\pi/2,3\pi/2]. \tag{2}$$

For the critical problem we have $c > 1$ and $F = 0$, and we seek, for a given value of c, the critical radius R. For the albedo problem we have $c < 1$ and $F = 1$, and we seek the albedo, which after we take into account some symmetries in ϕ and μ, can be expressed as

$$A^* = \frac{4}{\pi}\int_0^1\int_0^{\pi/2}\Psi(R,\mu,\phi)(1-\mu^2)^{1/2}\cos\phi\,\mathrm{d}\phi\,\mathrm{d}\mu. \tag{3}$$

In order to work with a simpler version of the given problem we choose to eliminate the differential operators from Eq. (1), and so we integrate Eq. (1) and introduce the "average"

$$\Psi_{\mathrm{avg}}(\mu)=\frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R\Psi(r,\mu,\phi)r\,\mathrm{d}r\,\mathrm{d}\phi, \tag{4}$$

to find

$$\Psi_{\mathrm{avg}}(\mu)=\frac{c}{2}\int_{-1}^{1}\Psi_{\mathrm{avg}}(\mu')\mathrm{d}\mu'-\frac{1}{\pi R}(1-\mu^2)^{1/2}S(\mu), \tag{5}$$

where after we note the symmetry in ϕ, we write

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$$S(\mu) = 2\int_0^{\pi/2} \Psi(R,\mu,\phi)\cos\phi\,\mathrm{d}\phi - 2F, \quad \mu \in [-1,1]. \tag{6}$$

At this point we approximate, as have Williams (2007) and others (Prinja and Pomraning, 1984; Larsen, 1984; Larsen et al., 1986), the exiting (over half the symmetric ϕ range) radiation:

$$\Psi(R,\mu,\phi) = \Psi_{\rm avg}(\mu), \quad \mu \in [-1,1], \quad \phi \in [0,\pi/2]. \tag{7}$$

Using Eq. (7) in Eq. (6) and then the resulting equation in Eq. (5), we find, after making use of the symmetry in μ

$$\left[1 + \frac{2}{\pi R}(1-\mu^2)^{1/2}\right]\Psi_{\rm avg}(\mu) = c\int_0^1 \Psi_{\rm avg}(\mu')\mathrm{d}\mu' + \frac{2}{\pi R}(1-\mu^2)^{1/2}F,$$
$$\mu \in [0,1]. \tag{8}$$

For the critical problem, we set $F = 0$ and integrate Eq. (8) to find the critical condition that relates c and R, viz.

$$1 = c\pi R\int_0^1 \frac{\mathrm{d}\mu}{\pi R + 2(1-\mu^2)^{1/2}}. \tag{9}$$

In a similar way, we can put $F = 1$ in Eq. (8) and use Eqs. (3) and (7) to find the albedo:

$$A^* = 1 - 2(1-c)I(c,R), \tag{10}$$

where

$$I(c,R) = \frac{2R}{\Lambda(c,R)}\int_0^1 \frac{(1-\mu^2)^{1/2}}{\pi R + 2(1-\mu^2)^{1/2}}\mathrm{d}\mu \tag{11}$$

and

$$\Lambda(c,R) = 1 - c\pi R\int_0^1 \frac{1}{\pi R + 2(1-\mu^2)^{1/2}}\mathrm{d}\mu. \tag{12}$$

Some numerical results that can be used to compare exact (Thomas et al., 1983; Siewert and Thomas, 1985) and approximate results, Eqs. (9) and (10), are given in Tables 1 and 2. While we have explicitly evaluated the integrals in Eqs. (9), (11) and (12), we do not list the resulting expressions here, in order to keep our presentation concise.

3. The spherical case

For the case of a homogeneous sphere of radius R, we consider

$$\mu\frac{\partial}{\partial r}\Psi(r,\mu) + \frac{1}{r}(1-\mu^2)\frac{\partial}{\partial \mu}\Psi(r,\mu) + \Psi(r,\mu)$$
$$= \frac{c}{2}\int_{-1}^1 \Psi(r,\mu')\mathrm{d}\mu' \tag{13}$$

and

$$\Psi(R,-\mu) = F, \quad \mu \in (0,1], \tag{14}$$

where again, for the critical problem we have $F = 0$, and for the albedo problem, we put $F = 1$ and compute the albedo

$$A^* = 2\int_0^1 \Psi(R,\mu)\mu\,\mathrm{d}\mu. \tag{15}$$

Continuing, we integrate Eq. (13) and introduce

$$\Psi_{\rm avg} = \frac{3}{2R^3}\int_{-1}^1\int_0^R \Psi(r,\mu)r^2\,\mathrm{d}r\,\mathrm{d}\mu \tag{16}$$

to find the critical condition

$$1 + (1-c)\frac{4R}{3} = 0. \tag{17}$$

For the albedo we find

Table 1
Approximate ($\widehat{R}$) and exact (R) values for the critical radius of an infinite cylinder.

c	$\widehat{R}$	R	$\lvert 1-\widehat{R}/R\rvert \times 100$
1.01	4.996(1)	1.313(1)	280
1.02	2.496(1)	9.043	176
1.05	9.961	5.411	84
1.1	4.962	3.577	39
1.2	2.464	2.287	8
1.3	1.632	1.725	5
1.4	1.217	1.397	13
1.5	9.686(−1)	1.178	18
1.6	8.032(−1)	1.021	21
1.8	5.972(−1)	8.074(−1)	26
2.0	4.741(−1)	6.686(−1)	29

Table 2
Approximate ($\widehat{A}^*$) and exact (A^*) values of the albedo for an infinite cylinder.

R	c	$\widehat{A}^*$	A^*	$\lvert 1-\widehat{A}^*/A^*\rvert \times 100$
1	0.7	6.294(−1)	5.865(−1)	7
1	0.8	7.168(−1)	6.929(−1)	3
1	0.9	8.342(−1)	8.264(−1)	1
1	0.99	9.804(−1)	9.803(−1)	0.01
10	0.7	1.457(−1)	2.753(−1)	47
10	0.8	2.025(−1)	3.665(−1)	45
10	0.9	3.351(−1)	5.128(−1)	35
10	0.99	8.334(−1)	8.614(−1)	3

Table 3
Approximate ($\widehat{R}$) and exact (R) values for the critical radius for a sphere.

c	$\widehat{R}$	R	$\lvert 1-\widehat{R}/R\rvert \times 100$
1.05	1.5(1)	7.277	106
1.07	1.071(1)	6.007	78
1.09	8.333	5.187	61
1.1	7.5	4.873	54
1.3	2.5	2.425	3
1.5	1.5	1.690	11
1.7	1.071	1.313	18
1.9	8.333(−1)	1.078	23

Table 4
Approximate ($\widehat{A}^*$) and exact (A^*) values for the albedo of a sphere.

R	c	$\widehat{A}^*$	A^*	$\lvert 1-\widehat{A}^*/A^*\rvert \times 100$
1	0.7	7.143(−1)	6.866(−1)	4
1	0.8	7.895(−1)	7.750(−1)	2
1	0.9	8.824(−1)	8.780(−1)	0.5
1	0.99	9.868(−1)	9.868(−1)	0.0
10	0.7	2.000(−1)	2.948(−1)	32
10	0.8	2.727(−1)	3.916(−1)	30
10	0.9	4.286(−1)	5.463(−1)	22
10	0.99	8.824(−1)	8.938(−1)	1.3

Table 5
Approximate ($\hat{a}$) and exact (a) values for the critical half-thickness for a slab.

c	$\hat{a}$	a	$\lvert 1-\hat{a}/a\rvert \times 100$
1.1	2.422	2.113	15
1.3	7.637(−1)	9.377(−1)	19
1.5	4.371(−1)	6.051(−1)	28
1.7	2.996(−1)	4.425(−1)	32
1.9	2.248(−1)	3.459(−1)	35

Table 6
Approximate ($\widehat{A}^*$ and $\widehat{B}^*$) and exact (A^* and B^*) values for the albedo and transmission factor for a slab.

$2a$	c	$\widehat{A}^*$	$\widehat{B}^*$	A^*	B^*	$\lvert 1-\widehat{A}^*/A^*\rvert \times 100$	$\lvert 1-\widehat{B}^*/B^*\rvert \times 100$
0.1	0.8	5.721(−2)	9.052(−1)	6.493(−2)	8.966(−1)	12	1
0.5	0.8	1.449(−1)	6.943(−1)	2.056(−1)	6.220(−1)	30	12
1.0	0.8	1.691(−1)	5.554(−1)	2.802(−1)	4.162(−1)	40	33
2.0	0.8	1.628(−1)	4.066(−1)	3.280(−1)	1.973(−1)	50	106
5.0	0.8	1.154(−1)	2.315(−1)	3.417(−1)	2.292(−2)	66	910

$$A^* = \frac{3}{3+4R(1-c)}. \tag{18}$$

Some numerical results that can be used to compare exact (Siewert and Thomas, 1985; Siewert and Grandjean, 1979) and approximate results, Eqs. (17) and (18), are given in Tables 3 and 4.

4. The slab case

For the case of a homogeneous slab of half-thickness a, we consider

$$\mu\frac{\partial}{\partial x}\Psi(x,\mu)+\Psi(x,\mu)=\frac{c}{2}\int_{-1}^{1}\Psi(x,\mu')\mathrm{d}\mu', \tag{19}$$

$$\Psi(-a,\mu)=F, \quad \mu\in(0,1], \tag{20a}$$

and

$$\Psi(a,-\mu)=0, \quad \mu\in(0,1]. \tag{20b}$$

For the critical problem we have $F = 0$ and seek, for a given value of c the critical half-thickness a. For the albedo problem, we put $F = 1$ and compute the albedo

$$A^* = 2\int_0^1 \Psi(-a,-\mu)\mu\,\mathrm{d}\mu \tag{21a}$$

and the transmission factor

$$B^* = 2\int_0^1 \Psi(a,\mu)\mu\,\mathrm{d}\mu. \tag{21b}$$

Following the procedure of Sections 2 and 3, we integrate Eq. (19) and introduce "the average"

$$\Psi_{\mathrm{avg}}(\mu)=\frac{1}{2a}\int_{-a}^{a}\Psi(x,\mu)\mathrm{d}x \tag{22}$$

to find

$$\frac{\mu}{2a}[\Psi(a,\mu)-\Psi(-a,\mu)]+\Psi_{\mathrm{avg}}(\mu)=\frac{c}{2}\int_{-1}^{1}\Psi_{\mathrm{avg}}(\mu')\mathrm{d}\mu'. \tag{23}$$

Considering Eq. (23) for $\mu > 0$ and $\mu < 0$ separately, we write

$$\frac{\mu}{2a}[\Psi(a,\mu)-F]+\Psi_{\mathrm{avg}}(\mu)=\frac{c}{2}\int_{-1}^{1}\Psi_{\mathrm{avg}}(\mu')\mathrm{d}\mu', \quad \mu\in(0,1], \tag{24a}$$

and

$$\frac{\mu}{2a}\Psi(-a,-\mu)+\Psi_{\mathrm{avg}}(-\mu)=\frac{c}{2}\int_{-1}^{1}\Psi_{\mathrm{avg}}(\mu')\mathrm{d}\mu', \quad \mu\in(0,1]. \tag{24b}$$

Now, using the approximations

$$\Psi(a,\mu)=\Psi_{\mathrm{avg}}(\mu) \tag{25a}$$

and

$$\Psi(-a,-\mu)=\Psi_{\mathrm{avg}}(-\mu), \tag{25b}$$

we can then solve Eqs. (24) to find

$$\Lambda(c,a)\int_{-1}^{1}\Psi_{\mathrm{avg}}(\mu')\mathrm{d}\mu' = F\Lambda(1,a), \tag{26}$$

where

$$\Lambda(c,a)=1-2ca\ln[1+1/(2a)]. \tag{27}$$

When $F = 0$ we find the critical condition for the slab:

$$1=2ca\ln[1+1/(2a)]. \tag{28}$$

For $F = 1$ we find the albedo can be expressed as

$$A^* = 2ca\frac{\Lambda^2(1,a)}{\Lambda(c,a)} \tag{29}$$

and the transmission factor as

$$B^* = 1-A^*-4a(1-c)\frac{\Lambda(1,a)}{\Lambda(c,a)}. \tag{30}$$

Some numerical results that can be used to compare exact (Grandjean and Siewert, 1979) and approximate results, Eqs. (28)–(30), are given in Tables 5 and 6.

5. Concluding remarks

In this work, we have used the idea of integration over differential operators and the introduction of average quantities (nodal-like approximations) to solve six basic problems in particle transport theory. We note that the purpose of this work is not to endorse the approach we use to provide numerical results for the considered problems, but rather we selected these simple problems as test cases since they can be solved essentially exactly, and so we were able to compare in a good way the simplifying nodal-like approximations used. As can be seen from the six tables of results reported in this work, the (single-node) nodal-like approximations we used led sometimes to reasonable results, but often the results are not all good. While results for problems that cannot be solved well by rigorous methods can be obtained using the approximations discussed here, it is clear that this approach should be used with a great deal of care.

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References

Bell, G.I., Glasstone, S., 1979. Nuclear Reactor Theory. Robert E. Krieger Pub. Co., Huntington.

Grandjean, P., Siewert, C.E., 1979. The F_N method in neutron-transport theory. Part II: Applications and numerical results. Nuclear Science and Engineering 69, 161–168.

Larsen, E.W., 1984. A one-dimensional model for three-dimensional transport in a pipe. Transport Theory and Statistical Physics 13, 599–614.

Larsen, E.W., Malvagi, F., Pomraning, G.C., 1986. One-dimensional models for neutral particle transport in ducts. Nuclear Science and Engineering 93, 13–30.

Prinja, A.K., Pomraning, G.C., 1984. A statistical model of transport in a vacuum. Transport Theory and Statistical Physics 13, 567–598.
Siewert, C.E., Grandjean, P., 1979. Three basic neutron-transport problems in spherical geometry. Nuclear Science and Engineering 70, 96–98.
Siewert, C.E., Thomas Jr., J.R., 1985. Radiative transfer calculations in spheres and cylinders. Journal of Quantitative Spectroscopy and Radiative Transfer 34, 59–64.
Thomas Jr., J.R., Southres, J.D., Siewert, C.E., 1983. The critical problem for an infinite cylinder. Nuclear Science and Engineering 84, 79–82.
Williams, M.M.R., 2007. Radiation transport in a light duct using a one-dimensional model. Physica Scripta 76, 306–313.