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A simplified implementation of the discrete-ordinates method for a class of problems in radiative transfer with polarization

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A B S T R A C T

A simplified implementation of the analytical discrete ordinates (ADO) method in radiative transfer with polarization is presented in this work. The class of problems that can be solved with the simplified ADO approach consists of problems defined in plane-parallel geometry and driven by external illumination in the form of obliquely incident parallel rays. Numerical results of benchmark quality are tabulated for the albedo problem with polarization and Lambert reflection. The new results improve on a tabulation made available in a previous work by the authors that was based on the (less accurate) spherical harmonics method.

1. Introduction

In a recent work [1], an alternative approach for solving the scalar albedo problem in radiative transfer with the ADO version [2] of the discrete-ordinates method was proposed. The approach is based on the use of a nascent delta function (rather than the Dirac delta distribution) to model the polar-angle dependence of the incident beam. Because a particular solution is not required when a nascent delta function is used, the implementation work was significantly simplified when compared to that of the classical approach [3], where both the polar-angle and the azimuthal-angle dependencies of the incident beam are modeled by Dirac delta distributions. Numerical results obtained with the new approach were found to be as accurate as those from the classical approach, when a sufficiently small "narrowness" parameter was used to define the nascent delta function.

And so, in this work, we extend the approach of Ref. [1] to the case where polarization effects are included in the equation of transfer. Since numerical results from three different methods are available for the albedo problem with polarization and Lambert ground reflection [4–7], we elected to use this problem here, to evaluate the merits of the idea of using a nascent delta function for modeling the polar-angle dependence of the incident beam in radiative-transfer problems with polarization.

2. Formulation

Following Ref. [8], we begin by writing the equation of transfer with polarization for a plane-parallel medium of optical thickness τ_0 in the form:

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}(\tau,\mu,\phi)+\mathbf{I}(\tau,\mu,\phi)=\frac{\varpi}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\mathbf{P}(\mu,\mu',\phi-\phi')\times\mathbf{I}(\tau,\mu',\phi')\,\mathrm{d}\phi'\,\mathrm{d}\mu', \quad (1)$$

where the optical variable $\tau\in(0,\tau_0)$ is used to define the position in the medium and $\mu\in[-1,1]$, with $\mu=\cos\theta$, and $\phi\in[0,2\pi]$ are, respectively, the polar and azimuthal variables that specify the direction of

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propagation. Here, μ is measured with respect to the *positive* τ-axis. Furthermore, the Stokes vector $\mathbf{I}(\tau,\mu,\phi)$ has the four Stokes parameters $I(\tau,\mu,\phi)$, $Q(\tau,\mu,\phi)$, $U(\tau,\mu,\phi)$, and $V(\tau,\mu,\phi)$ as components [9–11], ϖ is the single-scattering albedo, and $\mathbf{P}(\mu,\mu',\phi-\phi')$ is the phase matrix [12] that describes scattering from an initial direction (μ',ϕ') to a final direction (μ,ϕ). As in other works [4–7], we consider here a phase matrix of the form:

$$\mathbf{P}(\mu,\mu',\phi-\phi')=\sum_{m=0}^{L}\sum_{\alpha=1}^{2}\mathbf{\Phi}_\alpha^m(\phi-\phi')\mathbf{A}^m(\mu,\mu')\mathbf{D}_\alpha, \tag{2}$$

where

$$\mathbf{\Phi}_1^m(\phi)=(2-\delta_{0,m})\,\mathrm{diag}\{\cos m\phi,\,\cos m\phi,\,\sin m\phi,\,\sin m\phi\}, \tag{3a}$$

$$\mathbf{\Phi}_2^m(\phi)=(2-\delta_{0,m})\,\mathrm{diag}\{-\sin m\phi,-\sin m\phi,\cos m\phi,\cos m\phi\}, \tag{3b}$$

$$\mathbf{D}_1=\mathrm{diag}\{1,1,0,0\}, \tag{4a}$$

$$\mathbf{D}_2=\mathrm{diag}\{0,0,1,1\}, \tag{4b}$$

and

$$\mathbf{A}^m(\mu,\mu')=\sum_{l=m}^{L}\mathbf{P}_l^m(\mu)\mathbf{B}_l\mathbf{P}_l^m(\mu'). \tag{5}$$

Here,

$$\mathbf{P}_l^m(\mu)=\begin{pmatrix} P_l^m(\mu) & 0 & 0 & 0\\ 0 & R_l^m(\mu) & -T_l^m(\mu) & 0\\ 0 & -T_l^m(\mu) & R_l^m(\mu) & 0\\ 0 & 0 & 0 & P_l^m(\mu)\end{pmatrix}, \tag{6}$$

where $P_l^m(\mu)$ is a normalized version [7] of the associated Legendre function of the first kind and the normalized $R_l^m(\mu)$ and $T_l^m(\mu)$ functions can be expressed [7] in terms of the generalized spherical functions $P_{m,-2}^l(\mu)$ and $P_{m,2}^l(\mu)$ discussed by Gel'fand and Šapiro [13]. In addition, the $\mathbf{B}_l$ matrix in Eq. (5) is given by [12]

$$\mathbf{B}_l=\begin{pmatrix} \beta_l & \gamma_l & 0 & 0\\ \gamma_l & \alpha_l & 0 & 0\\ 0 & 0 & \zeta_l & -\epsilon_l\\ 0 & 0 & \epsilon_l & \delta_l\end{pmatrix}, \tag{7}$$

where the Greek constants $\{\alpha_l,\beta_l,\gamma_l,\delta_l,\epsilon_l,\zeta_l\}$, along with the truncation parameter L introduced in Eq. (2), are the basic parameters that characterize the scattering law.

As mentioned in Introduction, we seek in this work a solution of Eq. (1) subject to the boundary conditions [4–7]

$$\mathbf{I}(0,\mu,\phi)=\pi\delta(\mu-\mu_0)\delta(\phi-\phi_0)\mathbf{F} \tag{8a}$$

and

$$\mathbf{I}(\tau_0,-\mu,\phi)=\left(\frac{\lambda_0}{\pi}\right)\mathbf{L}\int_0^1\int_0^{2\pi}\mathbf{I}(\tau_0,\mu',\phi')\,\mathrm{d}\phi'\mu'\,\mathrm{d}\mu' \tag{8b}$$

for $\mu\in(0,1]$ and $\phi\in[0,2\pi]$. Eq. (8a) represents a beam characterized by the Stokes parameters:

$$\mathbf{F}=\begin{pmatrix}F_I\\F_Q\\F_U\\F_V\end{pmatrix} \tag{9}$$

that is incident externally on the surface $\tau=0$ with a direction specified by (μ_0,ϕ_0), while Eq. (8b) expresses Lambert reflection with a coefficient $\lambda_0\in(0,1]$ at the surface $\tau=\tau_0$. Note that $\mathbf{L}=\mathrm{diag}\{1,0,0,0\}$.

3. Fourier decomposition

In the manner of Ref. [7], we split the μ-range into positive and negative half-ranges and write the Stokes vector for each of these half-ranges as

$$\mathbf{I}(\tau,\mu,\phi)=\mathbf{I}_0(\tau,\mu,\phi)+\sum_{m=0}^{L}\sum_{\alpha=1}^{2}\mathbf{\Phi}_\alpha^m(\phi-\phi_0)\times[\mathbf{I}_\alpha^m(\tau,\mu)-\mathbf{I}_\alpha^m(0,\mu)\,\mathrm{e}^{-\tau/\mu}] \tag{10a}$$

and

$$\mathbf{I}(\tau,-\mu,\phi)=\sum_{m=0}^{L}\sum_{\alpha=1}^{2}\mathbf{\Phi}_\alpha^m(\phi-\phi_0)\mathbf{I}_\alpha^m(\tau,-\mu), \tag{10b}$$

for $\mu\in(0,1]$ and $\phi\in[0,2\pi]$. In Eq. (10a), $\mathbf{I}_0(\tau,\mu,\phi)$ represents the uncollided component of the Stokes vector, which is given by

$$\mathbf{I}_0(\tau,\mu,\phi)=\pi\delta(\mu-\mu_0)\delta(\phi-\phi_0)\,\mathrm{e}^{-\tau/\mu_0}\mathbf{F}. \tag{11}$$

Substituting the Fourier expansions expressed by Eqs. (10) in Eq. (1), we obtain, for $m=0,1,\ldots,L$ and $\alpha=1$ and 2:

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}_\alpha^m(\tau,\mu)+\mathbf{I}_\alpha^m(\tau,\mu)=\frac{\varpi}{2}\int_{-1}^{1}\mathbf{A}^m(\mu,\mu')\mathbf{I}_\alpha^m(\tau,\mu')\,\mathrm{d}\mu' \tag{12}$$

and the boundary conditions:

$$\mathbf{I}_\alpha^m(0,\mu)=\frac{1}{2}\delta(\mu-\mu_0)\mathbf{D}_\alpha\mathbf{F} \tag{13a}$$

and

$$\mathbf{I}_\alpha^m(\tau_0,-\mu)=2\lambda_0\delta_{0,m}\delta_{1,\alpha}\mathbf{L}\int_0^1\mathbf{I}_\alpha^m(\tau_0,\mu')\,\mu'\,\mathrm{d}\mu' \tag{13b}$$

for $\mu\in(0,1]$. Eqs. (12) and (13) constitute a set of $2(L+1)$ azimuthally independent problems to be solved. Now, in the spirit of Ref. [1], we consider in this work the set of problems that is obtained after we replace the Dirac delta distribution in Eq. (13a) by the rectangular nascent delta function:

$$\delta_\epsilon(\mu-\mu_0)=\begin{cases}(\mu_{\max}-\mu_{\min})^{-1}, & \mu_{\min}\le\mu\le\mu_{\max},\\ 0, & \text{otherwise}.\end{cases} \tag{14}$$

Here

$$\mu_{\min}=\max\{0,\mu_0-\epsilon/2\} \tag{15a}$$

and

$$\mu_{\max}=\min\{\mu_0+\epsilon/2,1\}, \tag{15b}$$

where ϵ is the "narrowness" parameter. Once the resulting set of problems is solved for $m=0,1,\ldots,L$ and $\alpha=1$ and 2, the complete solution of the original problem is available from Eqs. (10).

To close this section, we note that the $m=0$ problem can be reduced to two 2-vector problems (one for the I and Q components and another for the U and V components of the Stokes vector). This has been explored, for example, in Ref. [4]. In this work, to facilitate our computational implementation, we chose not to use such a reduction.

4. An ADO solution of the Fourier component problems

To solve the set of Fourier component problems defined by Eqs. (12) and (13) for $m=0,1,\ldots,L$ and $\alpha=1$ and 2, with the nascent delta function $\delta_\epsilon(\mu-\mu_0)$ defined by Eq. (14) replacing the Dirac delta distribution $\delta(\mu-\mu_0)$ in Eq. (13a), we use the ADO version [2,7] of the discrete-ordinates method. As the details of the application of the ADO method to radiative transfer problems with polarization are given in Ref. [7], our presentation here is brief.

Discrete-ordinates forms of Eq. (12) are obtained by splitting the integration interval in that equation into two half-range intervals and using the transformation $\mu\to-\mu$ in the negative interval, so that the integral in Eq. (12) can be written as a sum of two integrals over the positive interval [0, 1]. Next, to take care of the discontinuities introduced by the nascent delta function $\delta_\epsilon(\mu-\mu_0)$, the integration over [0, 1] is performed with a composite quadrature defined over three subintervals [only two, if $\delta_\epsilon(\mu-\mu_0)\neq 0$ for $\mu=0$ or $\mu=1$] and the resulting equation is split into two equations, one valid for $\mu\in(0,1]$ and another for $-\mu\in(0,1]$. Using a standard Gauss–Legendre quadrature with N_1 nodes mapped onto the subinterval(s) where $\delta_\epsilon(\mu-\mu_0)=0$ and a standard Gauss–Legendre quadrature with N_2 nodes mapped onto the subinterval where $\delta_\epsilon(\mu-\mu_0)\neq 0$, we obtain a composite quadrature with a total of $2N_1+N_2$ nodes [only N_1+N_2 nodes, if $\delta_\epsilon(\mu-\mu_0)\neq 0$ for $\mu=0$ or $\mu=1$] for integration over [0, 1]. As the interval of support of the nascent delta function is always very small for practical choices of the "narrowness" parameter ϵ, we usually take $N_2 \ll N_1$. We denote the nodes and weights of the composite quadrature so obtained as μ_n and w_n, $n=1,2,\ldots,N$, where N is the quadrature order. Although not made explicit in our notation here, we allow N to vary with m in our computational implementation. The reason for this is that the numerical results for the Fourier component problems for $m\to L$ converge faster with an increase in the quadrature order than those for $m\to 0$. Thus, it is more efficient to use discrete-ordinates schemes based on quadrature orders that decrease as m is increased.

We now discuss briefly our implementation of the ADO method. Note that, in order to keep our notation as simple as possible, we suppress the indices m and α that should be affixed to many quantities that are defined and/or used in this section. First, setting $\mu=\mu_n$, $n=1,2,\ldots,N$, in the discrete-ordinates forms of Eq. (12) that are obtained as discussed in the previous paragraph, we can write, for a given pair of Fourier indices (m,α):

$$\mu_n\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{I}(\tau,\mu_n)+\mathbf{I}(\tau,\mu_n)=\frac{\varpi}{2}\sum_{l=m}^{L}\mathbf{P}_l^m(\mu_n)\mathbf{B}_l\sum_{k=1}^{N}w_k\mathbf{I}_{l,k}(\tau) \tag{16a}$$

and

$$-\mu_n\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{I}(\tau,-\mu_n)+\mathbf{I}(\tau,-\mu_n)=\frac{\varpi}{2}\sum_{l=m}^{L}\mathbf{P}_l^m(-\mu_n)\mathbf{B}_l\sum_{k=1}^{N}w_k\mathbf{I}_{l,k}(\tau), \tag{16b}$$

where [6]

$$\mathbf{P}_l^m(-\mu_n)=(-1)^{l-m}\mathbf{D}\mathbf{P}_l^m(\mu_n)\mathbf{D} \tag{17}$$

and [7]

$$\mathbf{I}_{l,k}(\tau)=\mathbf{P}_l^m(\mu_k)\mathbf{I}(\tau,\mu_k)+\mathbf{P}_l^m(-\mu_k)\mathbf{I}(\tau,-\mu_k). \tag{18}$$

Eqs. (16) are taken to satisfy discrete-ordinates versions of the boundary conditions expressed by Eqs. (13) with the nascent delta replacing the Dirac delta in Eq. (13a), viz.

$$\mathbf{I}(0,\mu_n)=\frac{1}{2}\delta_\epsilon(\mu_n-\mu_0)\mathbf{D}_\alpha\mathbf{F} \tag{19a}$$

and

$$\mathbf{I}(\tau_0,-\mu_n)=2\lambda_0\delta_{0,m}\delta_{1,\alpha}\mathbf{L}\sum_{k=1}^{N}w_k\mu_k\mathbf{I}(\tau_0,\mu_k). \tag{19b}$$

Looking for solutions of the type [7]

$$\mathbf{I}(\tau,\pm\mu_n)=\mathbf{\Phi}(\nu,\pm\mu_n)\,\mathrm{e}^{-\tau/\nu}, \tag{20}$$

where ν is a parameter, we find that there are $4N$ pairs of positive/negative values of ν (called separation constants) that make Eq. (20) satisfy Eqs. (16). The separation constants are denoted as $\pm\nu_j$, $j=1,2,\ldots,4N$. To each separation constant ν_j, it corresponds an elementary solution $\mathbf{\Phi}(\nu_j,\pm\mu_n)$ in Eq. (20). Both the separation constants and the elementary solutions are obtained from the solution of an eigensystem of order $4N$, as discussed in detail in Ref. [7].

Defining the $4N$-vectors

$$\mathbf{I}_\pm(\tau)=\begin{pmatrix}\mathbf{I}(\tau,\pm\mu_1)\\ \mathbf{I}(\tau,\pm\mu_2)\\ \vdots\\ \mathbf{I}(\tau,\pm\mu_N)\end{pmatrix} \tag{21}$$

and

$$\mathbf{\Phi}_\pm(\nu_j)=\begin{pmatrix}\mathbf{\Phi}(\nu_j,\pm\mu_1)\\ \mathbf{\Phi}(\nu_j,\pm\mu_2)\\ \vdots\\ \mathbf{\Phi}(\nu_j,\pm\mu_N)\end{pmatrix}, \tag{22}$$

we can write our ADO solution as [7]

$$\mathbf{I}_+(\tau)=\sum_{j=1}^{4N}[A_j\mathbf{\Phi}_+(\nu_j)\,\mathrm{e}^{-\tau/\nu_j}+B_j\mathbf{\Phi}_-(\nu_j)\,\mathrm{e}^{-(\tau_0-\tau)/\nu_j}] \tag{23a}$$

and

$$\mathbf{I}_{-}(\tau)=\boldsymbol{\Delta}\sum_{j=1}^{4N}[A_j\boldsymbol{\Phi}_{-}(\nu_j)\,\mathrm{e}^{-\tau/\nu_j}+B_j\boldsymbol{\Phi}_{+}(\nu_j)\,\mathrm{e}^{-(\tau_0-\tau)/\nu_j}], \tag{23b}$$

where the coefficients A_j and B_j, $j=1,2,\ldots,4N$, are to be determined with the use of the boundary conditions and where the $4N\times4N$ diagonal matrix $\boldsymbol{\Delta}$ is given by

$$\boldsymbol{\Delta}=\mathrm{diag}\{\mathbf{D},\mathbf{D},\ldots,\mathbf{D}\}, \tag{24}$$

with

$$\mathbf{D}=\mathrm{diag}\{1,1,-1,-1\}. \tag{25}$$

Using the boundary condition expressed by Eq. (19a), we get, for the $\tau=0$ boundary:

$$\sum_{j=1}^{4N}[A_j\boldsymbol{\Phi}_{+}(\nu_j)+B_j\boldsymbol{\Phi}_{-}(\nu_j)\,\mathrm{e}^{-\tau_0/\nu_j}]=\mathbf{K}, \tag{26}$$

where

$$\mathbf{K}=\begin{pmatrix}\mathbf{N}(\mu_1)\\ \mathbf{N}(\mu_2)\\ \vdots\\ \mathbf{N}(\mu_N)\end{pmatrix} \tag{27}$$

with

$$\mathbf{N}(\mu_n)=\frac{1}{2}\delta_{\epsilon}(\mu_n-\mu_0)\mathbf{D}_{\alpha}\mathbf{F}. \tag{28}$$

Similarly, using the boundary condition expressed by Eq. (19b), we get, for the $\tau=\tau_0$ boundary:

$$\sum_{j=1}^{4N}\{A_j[\boldsymbol{\Phi}_{-}(\nu_j)-\mathbf{R}_{+}(\nu_j)]\,\mathrm{e}^{-\tau_0/\nu_j}+B_j[\boldsymbol{\Phi}_{+}(\nu_j)-\mathbf{R}_{-}(\nu_j)]\}=\mathbf{0}, \tag{29}$$

where

$$\mathbf{R}_{\pm}(\nu_j)=2\lambda_0\delta_{0,m}\delta_{1,\alpha}\boldsymbol{\Lambda}\boldsymbol{\Phi}_{\pm}(\nu_j) \tag{30}$$

with

$$\boldsymbol{\Lambda}=\begin{pmatrix}w_1\mu_1\mathbf{L} & w_2\mu_2\mathbf{L} & \ldots & w_N\mu_N\mathbf{L}\\ w_1\mu_1\mathbf{L} & w_2\mu_2\mathbf{L} & \ldots & w_N\mu_N\mathbf{L}\\ \vdots & \vdots & \ddots & \vdots\\ w_1\mu_1\mathbf{L} & w_2\mu_2\mathbf{L} & \ldots & w_N\mu_N\mathbf{L}\end{pmatrix}. \tag{31}$$

Eqs. (26) and (29) constitute a system of $8N$ linear algebraic equations for the $8N$ unknowns A_j and B_j, $j=1,2,\ldots,4N$. Once this system is solved, the Fourier component for a given pair (m,α) becomes available from Eqs. (23) for any $\tau\in[0,\tau_0]$ but only at the ordinates $\pm\mu_n$, $n=1,2,\ldots,N$. As discussed before, we allow quadrature orders that vary with m for improved computational efficiency, and so we see that, in general, Eqs. (23) will not provide all the Fourier components at a common set of ordinates. Consequently, Eq. (10) cannot be used to compute the complete solution. To overcome this difficulty, we use postprocessing, in the manner of Ref. [7], to obtain expressions for the Fourier components that are valid for any value of μ.

Our postprocessed expressions for the Stokes vector are obtained by approximating the integrals in Eqs. (12) and (13b) with our composite quadrature scheme and using the ADO solutions expressed by Eqs. (23) as an approximation for the Stokes vector in those integrals. The equation that results from Eq. (12) is then split into two forms, one valid for $\mu\in(0,1]$ and the other for $-\mu\in(0,1]$, and integrated over the spatial variable τ, from one of the boundaries to an interior point. We can write the equations so obtained as

$$\mathbf{I}(\tau,\mu)=\mathbf{I}(0,\mu)\,\mathrm{e}^{-\tau/\mu}+\frac{\varpi}{2}\sum_{j=1}^{4N}\nu_j[A_j\mathbf{T}_{+}(\nu_j,\mu)C(\tau:\nu_j,\mu)+B_j\mathbf{T}_{-}(\nu_j,\mu)\,\mathrm{e}^{-(\tau_0-\tau)/\nu_j}S(\tau:\nu_j,\mu)] \tag{32a}$$

and

$$\mathbf{I}(\tau,-\mu)=\mathbf{I}(\tau_0,-\mu)\,\mathrm{e}^{-(\tau_0-\tau)/\mu}+\frac{\varpi}{2}\mathbf{D}\sum_{j=1}^{4N}\nu_j[A_j\mathbf{T}_{-}(\nu_j,\mu)\,\mathrm{e}^{-\tau/\nu_j}S(\tau:\nu_j,\mu)+B_j\mathbf{T}_{+}(\nu_j,\mu)C(\tau:\nu_j,\mu)], \tag{32b}$$

for $\mu\in[0,1]$. The boundary terms in these equations are given by

$$\mathbf{I}(0,\mu)=\frac{1}{2}\delta(\mu-\mu_0)\mathbf{D}_{\alpha}\mathbf{F} \tag{33a}$$

and

$$\mathbf{I}(\tau_0,-\mu)=2\lambda_0\delta_{0,m}\delta_{1,\alpha}\mathbf{L}\sum_{j=1}^{4N}[A_j\mathbf{S}_{+}(\nu_j)\,\mathrm{e}^{-\tau_0/\nu_j}+B_j\mathbf{S}_{-}(\nu_j)], \tag{33b}$$

with

$$\mathbf{S}_{\pm}(\nu_j)=\sum_{n=1}^{N}w_n\mu_n\boldsymbol{\Phi}(\nu_j,\pm\mu_n). \tag{34}$$

In addition, we define

$$C(a:x,y)=\frac{\mathrm{e}^{-a/x}-\mathrm{e}^{-a/y}}{x-y}, \tag{35a}$$

$$S(a:x,y)=\frac{1-\mathrm{e}^{-a/x}\,\mathrm{e}^{-a/y}}{x+y}, \tag{35b}$$

and

$$\mathbf{T}_{\pm}(\nu_j,\mu)=\sum_{l=m}^{L}\mathbf{P}_l^m(\mu)\mathbf{B}_l[\mathbf{G}_{\pm}(\nu_j)+(-1)^{l-m}\mathbf{D}\mathbf{G}_{\mp}(\nu_j)], \tag{36}$$

where

$$\mathbf{G}_{\pm}(\nu_j)=\sum_{n=1}^{N}w_n\mathbf{P}_l^m(\mu_n)\boldsymbol{\Phi}(\nu_j,\pm\mu_n). \tag{37}$$

5. Computational implementation and numerical results

Before reporting our numerical results for some test cases, we would like to comment briefly on a few important aspects of our FORTRAN implementation of the reported solution. First of all, to find the separation constants $\{\nu_j\}$ and the vectors $\{\boldsymbol{\Phi}_{\pm}(\nu_j)\}$ that appear in Eqs. (23), we have used the sequence of subroutines BALANC, ELMHES, ELTRAN, HQR2, and BALBAK from the

Table 1
Basic data for the test problem of Ref. [4].

Case	ϖ	L	τ_0	μ_0	λ_0	F_I	F_Q	F_U	F_V
1	0.99	13	1.0	1.0	0.1	1.0	0.0	0.0	0.0
2	0.99	13	10.0	1.0	0.1	1.0	0.0	0.0	0.0
3	0.99	60	1.0	1.0	0.1	1.0	0.0	0.0	0.0
4	0.99	60	10.0	1.0	0.1	1.0	0.0	0.0	0.0
5	0.99	13	1.0	1.0	0.1	0.0	0.0	0.0	1.0
6	0.99	13	10.0	1.0	0.1	0.0	0.0	0.0	1.0
7	0.99	60	1.0	1.0	0.1	0.0	0.0	0.0	1.0
8	0.99	60	10.0	1.0	0.1	0.0	0.0	0.0	1.0

EISPACK collection [14]. As noted in Ref. [7], some separation constants (and the corresponding elementary solutions) can appear as complex conjugate pairs. And so, to avoid having to use complex mode when solving the linear system defined by Eqs. (26) and (29), we have worked with the real and imaginary parts of these quantities, as done in Ref. [7], to be able to express the elements of the matrix of coefficients of the linear system in terms of real quantities only. With this simplification, we were able to use the real subroutines DGECO and DGESL from the LINPACK collection [15] to find the coefficients $\{A_j\}$ and $\{B_j\}$ that are required in Eqs. (32) and (33b).

Table 2
The Stokes parameter $I_*(\tau,\mu,\phi)$ for Case 1.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	2.02565(−1)	1.98207(−1)	1.93258(−1)	1.82186(−1)	1.44089(−1)	1.11594(−1)	8.37897(−2)
−0.9	2.08359(−1)	2.04507(−1)	1.99929(−1)	1.89270(−1)	1.50323(−1)	1.15216(−1)	8.37897(−2)
−0.8	2.15454(−1)	2.12242(−1)	2.08139(−1)	1.98046(−1)	1.58249(−1)	1.19959(−1)	8.37897(−2)
−0.7	2.23992(−1)	2.21610(−1)	2.18144(−1)	2.08863(−1)	1.68373(−1)	1.26237(−1)	8.37897(−2)
−0.6	2.34050(−1)	2.32781(−1)	2.30195(−1)	2.22125(−1)	1.81403(−1)	1.34688(−1)	8.37897(−2)
−0.5	2.45504(−1)	2.45769(−1)	2.44438(−1)	2.38226(−1)	1.98341(−1)	1.46345(−1)	8.37897(−2)
−0.4	2.57710(−1)	2.60134(−1)	2.60630(−1)	2.57324(−1)	2.20547(−1)	1.62971(−1)	8.37897(−2)
−0.3	2.68826(−1)	2.74302(−1)	2.77458(−1)	2.78685(−1)	2.49598(−1)	1.87749(−1)	8.37897(−2)
−0.2	2.74783(−1)	2.84413(−1)	2.91238(−1)	2.99078(−1)	2.85870(−1)	2.26537(−1)	8.37897(−2)
−0.1	2.69718(−1)	2.84664(−1)	2.95898(−1)	3.11561(−1)	3.21977(−1)	2.86477(−1)	8.37897(−2)
−0.0	2.43633(−1)	2.70739(−1)	2.88347(−1)	3.13166(−1)	3.43359(−1)	3.34366(−1)	8.37897(−2)
0.0		2.70739(−1)	2.88347(−1)	3.13166(−1)	3.43359(−1)	3.34366(−1)	2.84619(−1)
0.1		1.08989(−1)	1.83439(−1)	2.70981(−1)	3.54487(−1)	3.60598(−1)	3.32544(−1)
0.2		6.65179(−2)	1.23081(−1)	2.10614(−1)	3.43327(−1)	3.75204(−1)	3.65122(−1)
0.3		5.12244(−2)	9.78519(−2)	1.77185(−1)	3.25687(−1)	3.78750(−1)	3.87218(−1)
0.4		4.43703(−2)	8.59367(−2)	1.59777(−1)	3.13643(−1)	3.80426(−1)	4.03884(−1)
0.5		4.13904(−2)	8.06590(−2)	1.51912(−1)	3.09335(−1)	3.85498(−1)	4.20371(−1)
0.6		4.06693(−2)	7.94391(−2)	1.50496(−1)	3.12603(−1)	3.96202(−1)	4.40046(−1)
0.7		4.15250(−2)	8.11383(−2)	1.54025(−1)	3.23077(−1)	4.13601(−1)	4.65011(−1)
0.8		4.36522(−2)	8.52376(−2)	1.61790(−1)	3.40613(−1)	4.38428(−1)	4.96775(−1)
0.9		4.69281(−2)	9.15311(−2)	1.73528(−1)	3.65355(−1)	4.71419(−1)	5.36649(−1)
1.0		5.13327(−2)	9.99950(−2)	1.89258(−1)	3.97718(−1)	5.13448(−1)	5.85958(−1)

Table 3
The Stokes parameter $Q_*(\tau,\mu,\phi)$ for Case 1.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	0.0	0.0	0.0	0.0	0.0	0.0	
−0.9	−1.06671(−2)	−1.03468(−2)	−9.96430(−3)	−9.08192(−3)	−5.91249(−3)	−2.96970(−3)	
−0.8	−2.24128(−2)	−2.17908(−2)	−2.10331(−2)	−1.92582(−2)	−1.27283(−2)	−6.49434(−3)	
−0.7	−3.53995(−2)	−3.45114(−2)	−3.34004(−2)	−3.07464(−2)	−2.06908(−2)	−1.07589(−2)	
−0.6	−4.97781(−2)	−4.86888(−2)	−4.72721(−2)	−4.37980(−2)	−3.01321(−2)	−1.60432(−2)	
−0.5	−6.56311(−2)	−6.44544(−2)	−6.28246(−2)	−5.86724(−2)	−4.15070(−2)	−2.27909(−2)	
−0.4	−8.28391(−2)	−8.17666(−2)	−8.00888(−2)	−7.55448(−2)	−5.54191(−2)	−3.17446(−2)	
−0.3	−1.00794(−1)	−1.00127(−1)	−9.86688(−2)	−9.42367(−2)	−7.25546(−2)	−4.42202(−2)	
−0.2	−1.17946(−1)	−1.18092(−1)	−1.17203(−1)	−1.13569(−1)	−9.30946(−2)	−6.25964(−2)	
−0.1	−1.31819(−1)	−1.33305(−1)	−1.33290(−1)	−1.30864(−1)	−1.14070(−1)	−8.96141(−2)	
−0.0	−1.37168(−1)	−1.43241(−1)	−1.45149(−1)	−1.44739(−1)	−1.29933(−1)	−1.11340(−1)	
0.0		−1.43241(−1)	−1.45149(−1)	−1.44739(−1)	−1.29933(−1)	−1.11340(−1)	−8.59289(−2)
0.1		−5.93704(−2)	−9.65306(−2)	−1.32905(−1)	−1.41833(−1)	−1.24976(−1)	−1.02435(−1)
0.2		−3.50695(−2)	−6.29389(−2)	−1.01443(−1)	−1.38660(−1)	−1.32064(−1)	−1.13641(−1)
0.3		−2.51153(−2)	−4.66756(−2)	−8.01213(−2)	−1.25990(−1)	−1.29307(−1)	−1.17594(−1)
0.4		−1.93952(−2)	−3.66657(−2)	−6.50150(−2)	−1.11028(−1)	−1.20269(−1)	−1.14545(−1)
0.5		−1.53877(−2)	−2.93717(−2)	−5.30842(−2)	−9.55401(−2)	−1.07523(−1)	−1.06110(−1)
0.6		−1.21482(−2)	−2.33248(−2)	−4.26676(−2)	−7.95580(−2)	−9.20474(−2)	−9.33371(−2)
0.7		−9.22149(−3)	−1.77716(−2)	−3.27700(−2)	−6.26386(−2)	−7.39749(−2)	−7.66019(−2)
0.8		−6.34038(−3)	−1.22485(−2)	−2.27087(−2)	−4.41917(−2)	−5.30089(−2)	−5.58028(−2)
0.9		−3.31764(−3)	−6.41905(−3)	−1.19456(−2)	−2.35555(−2)	−2.85976(−2)	−3.05026(−2)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 4
The Stokes parameter $I_*(\tau,\mu,\phi)$ for Case 2.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	6.79736(−1)	6.80784(−1)	6.55293(−1)	5.77979(−1)	3.29767(−1)	1.52077(−1)	1.94323(−2)
−0.9	6.71605(−1)	6.83338(−1)	6.63669(−1)	5.91472(−1)	3.43697(−1)	1.63156(−1)	1.94323(−2)
−0.8	6.61346(−1)	6.84883(−1)	6.71599(−1)	6.05025(−1)	3.57928(−1)	1.74945(−1)	1.94323(−2)
−0.7	6.48587(−1)	6.85242(−1)	6.78989(−1)	6.18614(−1)	3.72431(−1)	1.87390(−1)	1.94323(−2)
−0.6	6.32880(−1)	6.84210(−1)	6.85731(−1)	6.32213(−1)	3.87189(−1)	2.00392(−1)	1.94323(−2)
−0.5	6.13670(−1)	6.81551(−1)	6.91706(−1)	6.45796(−1)	4.02197(−1)	2.13815(−1)	1.94323(−2)
−0.4	5.90231(−1)	6.76985(−1)	6.96775(−1)	6.59333(−1)	4.17465(−1)	2.27495(−1)	1.94323(−2)
−0.3	5.61542(−1)	6.70171(−1)	7.00777(−1)	6.72788(−1)	4.33011(−1)	2.41307(−1)	1.94323(−2)
−0.2	5.25956(−1)	6.60679(−1)	7.03520(−1)	6.86124(−1)	4.48854(−1)	2.55229(−1)	1.94323(−2)
−0.1	4.80066(−1)	6.47904(−1)	7.04766(−1)	6.99297(−1)	4.65014(−1)	2.69312(−l)	1.94323(−2)
−0.0	4.06402(−1)	6.30845(−1)	7.04187(−1)	7.12257(−1)	4.81513(−1)	2.83595(−1)	1.94323(−2)
0.0		6.30845(−1)	7.04187(−1)	7.12257(−1)	4.81513(−1)	2.83595(−1)	8.79570(−2)
0.1		6.04386(−1)	7.01185(−1)	7.24941(−1)	4.98372(−1)	2.98102(−1)	1.07734(−1)
0.2		5.48620(−1)	6.92281(−1)	7.37219(−1)	5.15617(−1)	3.12856(−1)	1.23934(−1)
0.3		4.90204(−1)	6.74002(−1)	7.48509(−1)	5.33277(−1)	3.27879(−1)	1.39345(−1)
0.4		4.45886(−1)	6.52905(−1)	7.58122(−1)	5.51386(−1)	3.43196(−1)	1.54420(−1)
0.5		4.16637(−1)	6.35747(−1)	7.66788(−1)	5.69975(−1)	3.58830(−1)	1.69352(−1)
0.6		4.00504(−1)	6.26198(−1)	7.76462(−1)	5.89091(−1)	3.74811(−1)	1.84248(−1)
0.7		3.95690(−1)	6.26047(−1)	7.89453(−1)	6.08841(−1)	3.91170(−1)	1.99182(−1)
0.8		4.01026(−1)	6.36302(−1)	8.07993(−1)	6.29465(−1)	4.07956(−1)	2.14210(−1)
0.9		4.15926(−1)	6.57756(−1)	8.34159(−1)	6.51389(−1)	4.25244(−1)	2.29383(−1)
1.0		4.40283(−1)	6.91274(−1)	8.69948(−1)	6.75248(−1)	4.43165(−1)	2.44755(−1)

Table 5
The Stokes parameter $Q_*(\tau,\mu,\phi)$ for Case 2.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	0.0	0.0	0.0	0.0	0.0	0.0	
−0.9	−1.46683(−2)	−1.17821(−2)	−8.69552(−3)	−4.46006(−3)	−8.44952(−4)	−5.36078(−4)	
−0.8	−2.98438(−2)	−2.40613(−2)	−1.77625(−2)	−9.07718(−3)	−1.66200(−3)	−1.01109(−3)	
−0.7	−4.55361(−2)	−3.68728(−2)	−2.72319(−2)	−1.38667(−2)	−2.45329(−3)	−1.42272(−3)	
−0.6	−6.17308(−2)	−5.02456(−2)	−3.71333(−2)	−1.88440(−2)	−3.22068(−3)	−1.77000(−3)	
−0.5	−7.83696(−2)	−6.41970(−2)	−4.74920(−2)	−2.40242(−2)	−3.96524(−3)	−2.05471(−3)	
−0.4	−9.53158(−2)	−7.87217(−2)	−5.83247(−2)	−2.94205(−2)	−4.68682(−3)	−2.28319(−3)	
−0.3	−1.12287(−1)	−9.37729(−2)	−6.96310(−2)	−3.50427(−2)	−5.38386(−3)	−2.46705(−3)	
−0.2	−1.28705(−1)	−1.09228(−1)	−8.13788(−2)	−4.08930(−2)	−6.05370(−3)	−2.61915(−3)	
−0.1	−1.43269(−1)	−1.24820(−1)	−9.34772(−2)	−4.69600(−2)	−6.69298(−3)	−2.74446(−3)	
−0.0	−1.50740(−1)	−1.39973(−1)	−1.05722(−1)	−5.32059(−2)	−7.29746(−3)	−2.83991(−3)	
0.0		−1.39973(−1)	−1.05722(−1)	−5.32059(−2)	−7.29746(−3)	−2.83991(−3)	−5.79692(−3)
0.1		−1.52101(−1)	−1.17648(−1)	−5.95427(−2)	−7.86137(−3)	−2.90090(−3)	−4.84545(−3)
0.2		−1.48358(−1)	−1.27003(−1)	−6.57652(−2)	−8.37610(−3)	−2.92255(−3)	−4.21466(−3)
0.3		−1.34462(−1)	−1.29449(−1)	−7.12125(−2)	−8.82746(−3)	−2.89951(−3)	−3.65742(−3)
0.4		−1.18155(−1)	−1.24849(−1)	−7.45216(−2)	−9.18844(−3)	−2.82557(−3)	−3.13758(−3)
0.5		−1.01356(−1)	−1.14788(−1)	−7.44061(−2)	−9.39764(−3)	−2.69259(−3)	−2.63805(−3)
0.6		−8.41225(−2)	−1.00334(−1)	−7.00568(−2)	−9.32174(−3)	−2.48741(−3)	−2.14693(−3)
0.7		−6.60067(−2)	−8.18819(−2)	−6.09555(−2)	−8.72762(−3)	−2.18443(−3)	−1.65342(−3)
0.8		−4.64066(−2)	−5.93410(−2)	−4.66403(−2)	−7.27871(−3)	−1.73421(−3)	−1.14500(−3)
0.9		−2.46502(−2)	−3.22796(−2)	−2.65611(−2)	−4.54336(−3)	−1.05074(−3)	−6.03533(−4)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Now, with regard to the "narrowness" parameter ϵ that is used to define the nascent delta function in Eq. (14), we have concluded from numerical experiments that a good choice is $10^{-8} \leq \epsilon \leq 10^{-6}$, when calculations are performed in double (64-bit) precision and six significant figures of accuracy in the numerical results for the Stokes parameters are sought. With a choice of ϵ in this range, we have found that a very low quadrature order can be used in the interval of support of the nascent delta, and so we have fixed $N_2=2$. The order of the quadrature applied to the subintervals where the nascent delta vanishes was initially chosen as $N_1=10$ and subsequently increased in steps of 10 in our calculations, until convergence to the desired accuracy in the numerical results was attained.

We have considered all of the problems solved in Refs. [4–7] to test our solution. We do not tabulate our

Table 6
The Stokes parameter $I_*(\tau,\mu,\phi)$ for Case 3.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	1.91132(−1)	1.86169(−1)	1.81136(−1)	1.70944(−1)	1.39952(−1)	1.14341(−1)	8.97139(−2)
−0.9	1.93042(−1)	1.88297(−1)	1.83428(−1)	1.73428(−1)	1.42170(−1)	1.15618(−1)	8.97139(−2)
−0.8	1.66339(−1)	1.63474(−1)	1.60389(−1)	1.53719(−1)	1.30935(−1)	1.10223(−1)	8.97139(−2)
−0.7	1.61591(−1)	1.59481(−1)	1.57058(−1)	1.51495(−1)	1.30654(−1)	1.10351(−1)	8.97139(−2)
−0.6	1.63466(−1)	1.61978(−1)	1.60057(−1)	1.55216(−1)	1.34686(−1)	1.12870(−1)	8.97139(−2)
−0.5	1.69736(−1)	1.68981(−1)	1.67629(−1)	1.63546(−1)	1.42694(−1)	1.17859(−1)	8.97139(−2)
−0.4	1.78937(−1)	1.79293(−1)	1.78832(−1)	1.76031(−1)	1.55372(−1)	1.26218(−1)	8.97139(−2)
−0.3	1.88810(−1)	1.91042(−1)	1.92170(−1)	1.91932(−1)	1.74244(−1)	1.40260(−1)	8.97139(−2)
−0.2	1.93917(−1)	1.99372(−1)	2.03258(−1)	2.07855(−1)	2.00306(−1)	1.64990(−1)	8.97139(−2)
−0.1	1.83929(−1)	1.94882(−1)	2.03004(−1)	2.14832(−1)	2.27375(−1)	2.07199(−1)	8.97139(−2)
−0.0	1.38745(−1)	1.66288(−1)	1.82148(−1)	2.04350(−1)	2.38557(−1)	2.43263(−1)	8.97139(−2)
0.0		1.66288(−1)	1.82148(−1)	2.04350(−1)	2.38557(−1)	2.43263(−1)	2.07562(−1)
0.1		5.64708(−2)	9.93784(−2)	1.56750(−1)	2.32076(−1)	2.54388(−1)	2.51089(−1)
0.2		3.08370(−2)	5.95828(−2)	1.09311(−1)	2.07112(−1)	2.49306(−1)	2.65663(−1)
0.3		2.26536(−2)	4.47665(−2)	8.61210(−2)	1.83729(−1)	2.37119(−1)	2.67786(−1)
0.4		1.96550(−2)	3.90326(−2)	7.61769(−2)	1.70846(−1)	2.29252(−1)	2.68790(−1)
0.5		1.91023(−2)	3.79176(−2)	7.41743(−2)	1.69143(−1)	2.31051(−1)	2.76574(−1)
0.6		2.04165(−2)	4.04020(−2)	7.87574(−2)	1.79336(−1)	2.46001(−1)	2.96932(−1)
0.7		2.38609(−2)	4.70152(−2)	9.10462(−2)	2.04805(−1)	2.79535(−1)	3.37101(−1)
0.8		3.03993(−2)	5.96288(−2)	1.14588(−1)	2.53238(−1)	3.41961(−1)	4.09385(−1)
0.9		4.09481(−2)	8.01824(−2)	1.53629(−1)	3.36931(−1)	4.52405(−1)	5.39043(−1)
1.0		5.63046(−1)	1.07789	1.97545	3.80489	4.59973	4.95021

Table 7
The Stokes parameter $Q_*(\tau,\mu,\phi)$ for Case 3.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	0.0	0.0	0.0	0.0	0.0	0.0	
−0.9	3.57005(−3)	3.41397(−3)	3.25196(−3)	2.91516(−3)	1.83997(−3)	9.10614(−4)	
−0.8	9.84840(−3)	9.36334(−3)	8.87381(−3)	7.88931(−3)	4.92406(−3)	2.45111(−3)	
−0.7	1.12134(−2)	1.07181(−2)	1.02084(−2)	9.15953(−3)	5.85060(−3)	2.95477(−3)	
−0.6	1.16945(−2)	1.12525(−2)	1.07839(−2)	9.78714(−3)	6.43526(−3)	3.31147(−3)	
−0.5	1.25140(−2)	1.21340(−2)	1.17129(−2)	1.07754(−2)	7.34607(−3)	3.88063(−3)	
−0.4	1.37876(−2)	1.34955(−2)	1.31440(−2)	1.22995(−2)	8.80141(−3)	4.83508(−3)	
−0.3	1.53953(−2)	1.52546(−2)	1.50290(−2)	1.43809(−2)	1.10319(−2)	6.46802(−3)	
−0.2	1.68880(−2)	1.70192(−2)	1.70223(−2)	1.67686(−2)	1.42192(−2)	9.37869(−3)	
−0.1	1.73198(−2)	1.79412(−2)	1.83118(−2)	1.86666(−2)	1.78171(−2)	1.44002(−2)	
−0.0	1.47216(−2)	1.69030(−2)	1.80106(−2)	1.93135(−2)	2.02178(−2)	1.90143(−2)	
0.0		1.69030(−2)	1.80106(−2)	1.93135(−2)	2.02178(−2)	1.90143(−2)	1.43146(−2)
0.1		6.25545(−3)	1.07363(−2)	1.61940(−2)	2.13278(−2)	2.14880(−2)	1.95581(−2)
0.2		3.67119(−3)	6.90966(−3)	1.21146(−2)	2.04363(−2)	2.25776(−2)	2.22469(−2)
0.3		2.84718(−3)	5.48078(−3)	1.00714(−2)	1.91086(−2)	2.26157(−2)	2.35979(−2)
0.4		2.52125(−3)	4.88464(−3)	9.11989(−3)	1.82152(−2)	2.24151(−2)	2.42632(−2)
0.5		2.37042(−3)	4.60319(−3)	8.64713(−3)	1.76689(−2)	2.21934(−2)	2.45533(−2)
0.6		2.26223(−3)	4.39611(−3)	8.27706(−3)	1.70869(−2)	2.16916(−2)	2.43011(−2)
0.7		2.08313(−3)	4.04774(−3)	7.62566(−3)	1.58137(−2)	2.01889(−2)	2.27851(−2)
0.8		1.62818(−3)	3.16606(−3)	5.97612(−3)	1.24844(−2)	1.60532(−2)	1.82666(−2)
0.9		7.39513(−4)	1.45257(−3)	2.79492(−3)	6.14543(−3)	8.19845(−3)	9.64026(−3)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

numerical results for the problems solved in Refs. [5–7] here, since we were able to reproduce the highly accurate results reported for these problems in Refs. [6,7] with a tolerance of ± 1 in the last (sixth) significant figure, except for two entries in Table 16 of Ref. [6] which displayed differences of 2 and 3 units in the sixth figure. However, with regard to the azimuthally independent problem solved in Ref. [4], we note that the spherical harmonics method used in that work could not provide accurate results at the boundaries as $|\mu|\to 0$. As the ADO method used in this work does not suffer from that limitation, we believed it useful to report improved numerical results for the problem of Ref. [4]. And so, in Table 1 we report the basic parameters that define the eight cases of the problem introduced in Ref. [4] and reconsidered in this work. We note that the Greek constants for the $L=13$ and $L=60$ phase matrices used to define these test cases were computed by de Rooij and van der Stap with the methods described in Ref. [16] and tabulated by Vestrucci and Siewert [17].

Table 8
The Stokes parameter $I_*(\tau,\mu,\phi)$ for Case 4.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	6.08045(−1)	5.88511(−1)	5.62618(−1)	5.01210(−1)	2.99056(−1)	1.44219(−1)	2.98843(−2)
−0.9	6.03519(−1)	5.91338(−1)	5.70313(−1)	5.14655(−1)	3.16061(−1)	1.56519(−1)	2.98843(−2)
−0.8	5.60424(−1)	5.67648(−1)	5.59948(−1)	5.20054(−1)	3.33226(−1)	1.70446(−1)	2.98843(−2)
−0.7	5.41003(−1)	5.60449(−1)	5.60999(−1)	5.31012(−1)	3.51666(−1)	1.86228(−1)	2.98843(−2)
−0.6	5.25185(−1)	5.56476(−1)	5.64677(−1)	5.43669(−1)	3.70917(−1)	2.03781(−1)	2.98843(−2)
−0.5	5.08556(−1)	5.52526(−1)	5.68633(−1)	5.56793(−1)	3.90703(−1)	2.22857(−1)	2.98843(−2)
−0.4	4.88173(−1)	5.46589(−1)	5.71440(−1)	5.69659(−1)	4.10856(−1)	2.43002(−1)	2.98843(−2)
−0.3	4.61362(−1)	5.37055(−1)	5.71987(−1)	5.81705(−1)	4.31286(−1)	2.63662(−1)	2.98843(−2)
−0.2	4.24284(−1)	5.22361(−1)	5.69239(−1)	5.92417(−1)	4.51948(−1)	2.84466(−1)	2.98843(−2)
−0.1	3.68572(−1)	5.00606(−1)	5.62081(−1)	6.01265(−1)	4.72804(−1)	3.05376(−1)	2.98843(−2)
−0.0	2.61379(−1)	4.68882(−1)	5.49143(−1)	6.07676(−1)	4.93817(−1)	3.26476(−1)	2.98843(−2)
0.0		4.68882(−1)	5.49143(−1)	6.07676(−1)	4.93817(−1)	3.26476(−1)	1.14236(−1)
0.1		4.20368(−1)	5.28544(−1)	6.11029(−1)	5.14951(−1)	3.47836(−1)	1.55633(−1)
0.2		3.49514(−1)	4.97168(−1)	6.10696(−1)	5.36178(−1)	3.69515(−1)	1.84038(−1)
0.3		2.88943(−1)	4.58340(−1)	6.06358(−1)	5.57516(−1)	3.91572(−1)	2.09212(−1)
0.4		2.50017(−1)	4.24340(−1)	5.99863(−1)	5.79102(−1)	4.14084(−1)	2.33295(−1)
0.5		2.30299(−1)	4.03738(−1)	5.96452(−1)	6.01390(−1)	4.37185(−1)	2.57037(−1)
0.6		2.27705(−1)	4.01583(−1)	6.03545(−1)	6.25554(−1)	4.61148(−1)	2.80783(−1)
0.7		2.43846(−1)	4.23961(−1)	6.30725(−1)	6.54194(−1)	4.86583(−1)	3.04826(−1)
0.8		2.85327(−1)	4.82139(−1)	6.92571(−1)	6.92795(−1)	5.14939(−1)	3.29614(−1)
0.9		3.63744(−1)	6.00542(−1)	8.26548(−1)	7.59019(−1)	5.51112(−1)	3.56347(−1)
1.0		3.82769	5.00279	4.40287	1.40335	6.83487(−1)	3.98260(−1)

Table 9
The Stokes parameter $Q_*(\tau,\mu,\phi)$ for Case 4.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	0.0	0.0	0.0	0.0	0.0	0.0	
−0.9	6.38581(−3)	5.46015(−3)	4.49551(−3)	2.88556(−3)	7.24980(−4)	3.46293(−4)	
−0.8	1.47773(−2)	1.16782(−2)	9.11464(−3)	5.49008(−3)	1.31616(−3)	6.45905(−4)	
−0.7	1.70385(−2)	1.40066(−2)	1.12531(−2)	7.06484(−3)	1.80115(−3)	9.06684(−4)	
−0.6	1.78360(−2)	1.52929(−2)	1.26538(−2)	8.27116(−3)	2.22677(−3)	1.13120(−3)	
−0.5	1.86226(−2)	1.65724(−2)	1.40451(−2)	9.46639(−3)	2.63761(−3)	1.32291(−3)	
−0.4	1.95143(−2)	1.79878(−2)	1.55623(−2)	1.07490(−2)	3.05703(−3)	1.48463(−3)	
−0.3	2.04179(−2)	1.95292(−2)	1.72290(−2)	1.21575(−2)	3.49950(−3)	1.62335(−3)	
−0.2	2.11013(−2)	2.11544(−2)	1.90508(−2)	1.37189(−2)	3.97585(−3)	1.75200(−3)	
−0.1	2.09696(−2)	2.27814(−2)	2.10190(−2)	1.54555(−2)	4.49551(−3)	1.88296(−3)	
−0.0	1.78800(−2)	2.42071(−2)	2.30805(−2)	1.73789(−2)	5.06696(−3)	2.02237(−3)	
0.0		2.42071(−2)	2.30805(−2)	1.73789(−2)	5.06696(−3)	2.02237(−3)	1.85206(−3)
0.1		2.47890(−2)	2.50721(−2)	1.94722(−2)	5.69655(−3)	2.17229(−3)	1.88786(−3)
0.2		2.32683(−2)	2.65334(−2)	2.16550(−2)	6.38513(−3)	2.33239(−3)	1.79535(−3)
0.3		2.13065(−2)	2.70930(−2)	2.37258(−2)	7.12097(−3)	2.49945(−3)	1.67235(−3)
0.4		1.98924(−2)	2.70949(−2)	2.54106(−2)	7.86652(−3)	2.66483(−3)	1.55308(−3)
0.5		1.89366(−2)	2.67947(−2)	2.64781(−2)	8.53609(−3)	2.80899(−3)	1.44115(−3)
0.6		1.80224(−2)	2.60164(−2)	2.66233(−2)	8.96325(−3)	2.89105(−3)	1.32258(−3)
0.7		1.64681(−2)	2.40264(−2)	2.51570(−2)	8.84908(−3)	2.83090(−3)	1.17312(−3)
0.8		1.28946(−2)	1.90720(−2)	2.05335(−2)	7.69113(−3)	2.48644(−3)	9.53702(−4)
0.9		6.33707(−3)	1.00293(−2)	1.18747(−2)	5.08435(−3)	1.68806(−3)	6.07368(−4)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

In Tables 2–17, we report our numerical results for the diffuse component of the Stokes vector which we write, following Eqs. (10), as

$$\mathbf{I}_*(\tau,\mu,\phi)=\sum_{m=0}^{L}\sum_{\alpha=1}^{2}\mathbf{\Phi}_\alpha^m(\phi-\phi_0)[\mathbf{I}_\alpha^m(\tau,\mu)-\mathbf{I}_\alpha^m(0,\mu)\,\mathrm{e}^{-\tau/\mu}] \tag{38a}$$

and

$$\mathbf{I}_*(\tau,-\mu,\phi)=\sum_{m=0}^{L}\sum_{\alpha=1}^{2}\mathbf{\Phi}_\alpha^m(\phi-\phi_0)\mathbf{I}_\alpha^m(\tau,-\mu), \tag{38b}$$

for $\mu\in[0,1]$ and $\phi\in[0,2\pi]$. Only the nonzero Stokes parameters for each case are shown in these tables. The converged numerical results reported in Tables 2–17 were

Table 10
The Stokes parameter $U_*(\tau,\mu,\phi)$ for Case 5.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	0.0	0.0	0.0	0.0	0.0	0.0	
−0.9	1.01903(−3)	9.25123(−4)	8.39427(−4)	6.87888(−4)	3.48178(−4)	1.52233(−4)	
−0.8	1.75675(−3)	1.58513(−3)	1.42975(−3)	1.15792(−3)	5.65803(−4)	2.40863(−4)	
−0.7	2.18846(−3)	1.95526(−3)	1.74630(−3)	1.38593(−3)	6.32897(−4)	2.53317(−4)	
−0.6	2.28398(−3)	2.00480(−3)	1.75801(−3)	1.34063(−3)	5.21118(−4)	1.70281(−4)	
−0.5	2.00929(−3)	1.69814(−3)	1.42798(−3)	9.82998(−4)	1.90035(−4)	−3.91008(−5)	
−0.4	1.33232(−3)	9.99840(−4)	7.17652(−4)	2.68746(−4)	−4.17738(−4)	−4.26834(−4)	
−0.3	2.39167(−4)	−1.10051(−4)	−3.98584(−4)	−8.39364(−4)	−1.37901(−3)	−1.08628(−3)	
−0.2	−1.23319(−3)	−1.60194(−3)	−1.89770(−3)	−2.33283(−3)	−2.76705(−3)	−2.19273(−3)	
−0.1	−2.97646(−3)	−3.37419(−3)	−3.67828(−3)	−4.10375(−3)	−4.50210(−3)	−4.00190(−3)	
−0.0	−4.75009(−3)	−5.29302(−3)	−5.63854(−3)	−6.06594(−3)	−6.29029(−3)	−5.88105(−3)	
0.0		−5.29302(−3)	−5.63854(−3)	−6.06594(−3)	−6.29029(−3)	−5.88105(−3)	−4.99868(−3)
0.1		−3.07772(−3)	−5.05776(−3)	−7.13040(−3)	−8.15801(−3)	−7.57956(−3)	−6.61020(−3)
0.2		−2.32352(−3)	−4.16677(−3)	−6.73024(−3)	−9.36144(−3)	−9.11858(−3)	−8.12633(−3)
0.3		−1.98891(−3)	−3.67419(−3)	−6.26028(−3)	−9.76892(−3)	−1.00638(−2)	−9.29945(−3)
0.4		−1.75801(−3)	−3.29482(−3)	−5.77067(−3)	−9.65155(−3)	−1.03848(−2)	−9.93311(−3)
0.5		−1.54860(−3)	−2.92672(−3)	−5.21152(−3)	−9.12101(−3)	−1.01356(−2)	−9.97469(−3)
0.6		−1.32705(−3)	−2.52157(−3)	−4.53950(−3)	−8.20085(−3)	−9.33706(−3)	−9.40082(−3)
0.7		−1.07442(−3)	−2.04919(−3)	−3.71771(−3)	−6.87450(−3)	−7.97500(−3)	−8.17806(−3)
0.8		−7.76990(−4)	−1.48596(−3)	−2.71135(−3)	−5.10352(−3)	−6.00890(−3)	−6.25455(−3)
0.9		−4.22774(−4)	−8.10209(−4)	−1.48486(−3)	−2.83425(−3)	−3.37724(−3)	−3.55887(−3)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 11
The Stokes parameter $V_*(\tau,\mu,\phi)$ for Case 5.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	−4.40011(−2)	−4.16854(−2)	−3.94272(−2)	−3.50329(−2)	−2.23385(−2)	−1.16580(−2)	
−0.9	−3.97350(−2)	−3.74123(−2)	−3.51949(−2)	−3.09931(−2)	−1.95011(−2)	−1.02459(−2)	
−0.8	−3.44658(−2)	−3.20948(−2)	−2.98897(−2)	−2.58561(−2)	−1.57211(−2)	−8.26369(−3)	
−0.7	−2.79277(−2)	−2.54476(−2)	−2.32098(−2)	−1.92947(−2)	−1.06639(−2)	−5.47642(−3)	
−0.6	−1.98037(−2)	−1.71212(−2)	−1.47767(−2)	−1.08817(−2)	−3.84599(−3)	−1.51696(−3)	
−0.5	−9.75899(−3)	−6.72623(−3)	−4.15012(−3)	−8.41158(−5)	5.43962(−3)	4.21506(−3)	
−0.4	2.44343(−3)	6.06376(−3)	9.08298(−3)	1.36812(−2)	1.82193(−2)	1.27561(−2)	
−0.3	1.66377(−2)	2.12116(−2)	2.50154(−2)	3.07878(−2)	3.58776(−2)	2.60065(−2)	
−0.2	3.16846(−2)	3.76916(−2)	4.27357(−2)	5.06129(−2)	5.96398(−2)	4.75857(−2)	
−0.1	4.54561(−2)	5.33534(−2)	5.99861(−2)	7.06002(−2)	8.73437(−2)	8.27468(−2)	
−0.0	5.52840(−2)	6.68269(−2)	7.56921(−2)	8.95627(−2)	1.12494(−1)	1.17302(−1)	
0.0		6.68269(−2)	7.56921(−2)	8.95627(−2)	1.12494(−1)	1.17302(−1)	1.06140(−1)
0.1		3.75868(−2)	6.42147(−2)	9.78150(−2)	1.37525(−1)	1.46051(−1)	1.41096(−1)
0.2		3.01055(−2)	5.56196(−2)	9.53114(−2)	1.57294(−1)	1.74383(−1)	1.74044(−1)
0.3		2.85021(−2)	5.39337(−2)	9.64212(−2)	1.73525(−1)	2.00445(−1)	2.06365(−1)
0.4		2.89054(−2)	5.52510(−2)	1.00706(−1)	1.90041(−1)	2.25975(−1)	2.38491(−1)
0.5		3.03912(−2)	5.83755(−2)	1.07470(−1)	2.08487(−1)	2.52796(−1)	2.71731(−1)
0.6		3.26567(−2)	6.28866(−2)	1.16417(−1)	2.29681(−1)	2.82196(−1)	3.07383(−1)
0.7		3.56000(−2)	6.86481(−2)	1.27487(−1)	2.54203(−1)	3.15162(−1)	3.46596(−1)
0.8		3.92039(−2)	7.56538(−2)	1.40761(−1)	2.82594(−1)	3.52566(−1)	3.90438(−1)
0.9		4.34963(−2)	8.39713(−2)	1.56412(−1)	3.15429(−1)	3.95273(−1)	4.39967(−1)
1.0		4.85343(−2)	9.37184(−2)	1.74689(−1)	3.53353(−1)	4.44201(−1)	4.96293(−1)

obtained by using our FORTRAN code in double precision with $\epsilon=10^{-8}$ and are thought to be accurate to within ± 1 in the sixth digit, with the exception of only one entry: $V_*(\tau,\mu,\phi)$ of Case 5 for $\tau=\tau_0/5$ and $\mu=-0.5$. As can be seen from Table 11, this entry has a magnitude much smaller than those of its neighbor entries and, for this reason, it is very difficult to compute accurately when double precision is used. We show in Table 18 a comparison of numerical results from our code used in double and quadruple (128-bit) precisions for this entry, with $N_1=40$ and $N_2=2$. A progressive deterioration in the accuracy of the double-precision results as ϵ is reduced can be clearly seen. Also, it is apparent that the quadruple-precision results converge to a stable value (-8.41165×10^{-5}).

The CPU time for running a typical case on a notebook computer equipped with a 2.1-GHz Intel Core 2 Duo processor varied between 0.06 s (Case 6) and 0.44 s

Table 12
The Stokes parameter $U_*(\tau,\mu,\phi)$ for Case 6.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	0.0	0.0	0.0	0.0	0.0	0.0	
−0.9	1.10058(−3)	4.67872(−4)	1.81520(−4)	−9.28361(−6)	−2.21412(−5)	−6.65485(−6)	
−0.8	1.85552(−3)	7.06210(−4)	2.11901(−4)	−8.51058(−5)	−5.17537(−5)	−1.49308(−5)	
−0.7	2.25842(−3)	7.10093(−4)	8.81225(−5)	−2.28264(−4)	−8.86613(−5)	−2.47728(−5)	
−0.6	2.30183(−3)	4.73507(−4)	−1.93516(−4)	−4.39715(−4)	−1.32635(−4)	−3.60935(−5)	
−0.5	1.97821(−3)	−1.03540(−5)	−6.37262(−4)	−7.20517(−4)	−1.83377(−4)	−4.87675(−5)	
−0.4	1.28187(−3)	−7.48535(−4)	−1.24766(−3)	−1.07174(−3)	−2.40502(−4)	−6.26307(−5)	
−0.3	2.12918(−4)	−1.74726(−3)	−2.02911(−3)	−1.49433(−3)	−3.03510(−4)	−7.74882(−5)	
−0.2	−1.21400(−3)	−3.00988(−3)	−2.98501(−3)	−1.98884(−3)	−3.71746(−4)	−9.31183(−5)	
−0.1	−2.94531(−3)	−4.53262(−3)	−4.11620(−3)	−2.55507(−3)	−4.44354(−4)	−1.09244(−4)	
−0.0	−4.72955(−3)	−6.29405(−3)	−5.41749(−3)	−3.19129(−3)	−5.20209(−4)	−1.25493(−4)	
0.0		−6.29405(−3)	−5.41749(−3)	−3.19129(−3)	−5.20209(−4)	−1.25493(−4)	−2.98122(−5)
0.1		−8.16950(−3)	−6.86845(−3)	−3.89281(−3)	−5.97819(−4)	−1.41384(−4)	−3.38023(−5)
0.2		−9.38106(−3)	−8.34213(−3)	−4.64777(−3)	−6.75185(−4)	−1.56299(−4)	−3.72847(−5)
0.3		−9.79404(−3)	−9.49060(−3)	−5.41303(−3)	−7.49566(−4)	−1.69435(−4)	−4.02410(−5)
0.4		−9.67911(−3)	−1.01037(−2)	−6.07519(−3)	−8.16957(−4)	−1.79735(−4)	−4.24122(−5)
0.5		−9.14839(−3)	−1.01241(−2)	−6.48223(−3)	−8.70357(−4)	−1.85768(−4)	−4.34241(−5)
0.6		−8.22589(−3)	−9.52675(−3)	−6.48884(−3)	−8.95887(−4)	−1.85437(−4)	−4.27620(−5)
0.7		−6.89540(−3)	−8.27743(−3)	−5.96559(−3)	−8.68322(−4)	−1.75256(−4)	−3.96919(−5)
0.8		−5.11874(−3)	−6.32406(−3)	−4.79176(−3)	−7.48359(−4)	−1.49039(−4)	−3.30704(−5)
0.9		−2.84245(−3)	−3.59523(−3)	−2.84624(−3)	−4.81857(−4)	−9.62012(−5)	−2.09593(−5)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 13
The Stokes parameter $V_*(\tau,\mu,\phi)$ for Case 6.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	−5.53582(−2)	−4.04641(−2)	−3.00816(−2)	−1.70352(−2)	−3.35730(−3)	−9.06632(−4)	
−0.9	−4.63502(−2)	−3.08442(−2)	−2.12589(−2)	−1.07402(−2)	−1.81396(−3)	−4.92407(−4)	
−0.8	−3.67491(−2)	−2.02445(−2)	−1.14078(−2)	−3.63053(−3)	−5.53592(−5)	−1.31397(−5)	
−0.7	−2.65222(−2)	−8.55368(−3)	−3.98874(−4)	4.40347(−3)	1.94822(−3)	5.39569(−4)	
−0.6	−1.56428(−2)	4.35381(−3)	1.19174(−2)	1.34894(−2)	4.23160(−3)	1.17489(−3)	
−0.5	−4.09910(−3)	1.86189(−2)	2.57132(−2)	2.37761(−2)	6.83583(−3)	1.90290(−3)	
−0.4	8.08932(−3)	3.43969(−2)	4.11871(−2)	3.54382(−2)	9.80950(−3)	2.73511(−3)	
−0.3	2.08344(−2)	5.18536(−2)	5.85662(−2)	4.86811(−2)	1.32105(−2)	3.68581(−3)	
−0.2	3.38943(−2)	7.11521(−2)	7.81074(−2)	6.37463(−2)	1.71078(−2)	4.77401(−3)	
−0.1	4.66205(−2)	9.24204(−2)	1.00092(−1)	8.09182(−2)	2.15845(−2)	6.02392(−3)	
−0.0	5.60610(−2)	1.15659(−1)	1.24804(−1)	1.00529(−1)	2.67411(−2)	7.46448(−3)	
0.0		1.15659(−1)	1.24804(−1)	1.00529(−1)	2.67411(−2)	7.46448(−3)	1.68478(−3)
0.1		1.39911(−1)	1.52456(−1)	1.22959(−1)	3.27004(−2)	9.13074(−3)	2.25923(−3)
0.2		1.59097(−1)	1.82408(−1)	1.48616(−1)	3.96126(−2)	1.10659(−2)	2.83743(−3)
0.3		1.74885(−1)	2.12741(−1)	1.77751(−1)	4.76632(−2)	1.33241(−2)	3.48649(−3)
0.4		1.91067(−1)	2.43394(−1)	2.10270(−1)	5.70796(−2)	1.59733(−2)	4.23319(−3)
0.5		2.09257(−1)	2.75487(−1)	2.46100(−1)	6.81315(−2)	1.91001(−2)	5.10475(−3)
0.6		2.30251(−1)	3.10220(−1)	2.85491(−1)	8.11209(−2)	2.28144(−2)	6.13410(−3)
0.7		2.54614(−1)	3.48686(−1)	3.29004(−1)	9.63736(−2)	2.72532(−2)	7.36327(−3)
0.8		2.82878(−1)	3.91913(−1)	3.77433(−1)	1.14246(−1)	3.25831(−2)	8.84670(−3)
0.9		3.15611(−1)	4.40936(−1)	4.31756(−1)	1.35145(−1)	3.90033(−2)	1.06546(−2)
1.0		3.53454(−1)	4.96846(−1)	4.93120(−1)	1.59551(−1)	4.67497(−2)	1.28763(−2)

(Case 3). To establish confidence in the accuracy of the numerical results reported in Tables 2–17, we have compared them with similar results obtained by running our code in quadruple precision and high order ($N_1=200$, $N_2=4$, and $\epsilon=10^{-12}$). With regard to the quadrature order N_1 required for six-digit convergence, we have found that the following values of N_1 were sufficient: $N_1=20$ for Cases 2 and 6, $N_1=30$ for Cases 1 and 8, $N_1=40$ for Cases 5 and 7, $N_1=50$ for Case 4, and $N_1=60$ for Case 3.

6. Concluding remarks

We have described in this work an alternative (and we believe much simpler) implementation of the ADO version of the discrete-ordinates method for solving the albedo problem with polarization and Lambert ground reflection. The ease of implementation characteristic of the new approach is due, in great part, to the fact that it avoids the use of particular solutions that can be very complicated if all

Table 14
The Stokes parameter $U_*(\tau,\mu,\phi)$ for Case 7.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	0.0	0.0	0.0	0.0	0.0	0.0	
−0.9	5.74154(−2)	5.41439(−2)	5.09361(−2)	4.46984(−2)	2.71434(−2)	1.34025(−2)	
−0.8	3.35261(−2)	3.17352(−2)	2.99598(−2)	2.64589(−2)	1.63008(−2)	8.09941(−3)	
−0.7	2.58526(−2)	2.45489(−2)	2.32414(−2)	2.06293(−2)	1.28511(−2)	6.41201(−3)	
−0.6	2.26491(−2)	2.15990(−2)	2.05277(−2)	1.83477(−2)	1.16216(−2)	5.84946(−3)	
−0.5	2.19766(−2)	2.10710(−2)	2.01250(−2)	1.81542(−2)	1.17888(−2)	6.04153(−3)	
−0.4	2.27917(−2)	2.20027(−2)	2.11509(−2)	1.93198(−2)	1.30254(−2)	6.89310(−3)	
−0.3	2.44369(−2)	2.38061(−2)	2.30874(−2)	2.14672(−2)	1.53478(−2)	8.59448(−3)	
−0.2	2.60966(−2)	2.57631(−2)	2.52958(−2)	2.41132(−2)	1.88873(−2)	1.18054(−2)	
−0.1	2.62908(−2)	2.65823(−2)	2.65598(−2)	2.60970(−2)	2.28953(−2)	1.75221(−2)	
−0.0	2.16735(−2)	2.43486(−2)	2.53584(−2)	2.61487(−2)	2.51542(−2)	2.25324(−2)	
0.0		2.43486(−2)	2.53584(−2)	2.61487(−2)	2.51542(−2)	2.25324(−2)	1.63907(−2)
0.1		8.09757(−3)	1.37951(−2)	2.04043(−2)	2.52466(−2)	2.44146(−2)	2.17674(−2)
0.2		4.12362(−3)	7.75906(−3)	1.35330(−2)	2.22034(−2)	2.39533(−2)	2.32782(−2)
0.3		2.76791(−3)	5.34713(−3)	9.85164(−3)	1.86035(−2)	2.18489(−2)	2.26795(−2)
0.4		2.13071(−3)	4.15350(−3)	7.81465(−3)	1.57703(−2)	1.94772(−2)	2.11253(−2)
0.5		1.76889(−3)	3.46082(−3)	6.57102(−3)	1.36924(−2)	1.73813(−2)	1.93681(−2)
0.6		1.55413(−3)	3.04078(−3)	5.78582(−3)	1.22059(−2)	1.56964(−2)	1.77533(−2)
0.7		1.44684(−3)	2.82354(−3)	5.35621(−3)	1.12742(−2)	1.45249(−2)	1.65052(−2)
0.8		1.48891(−3)	2.88787(−3)	5.42206(−3)	1.11425(−2)	1.41338(−2)	1.58695(−2)
0.9		9.83806(−4)	1.86850(−3)	3.36743(−3)	6.14990(−3)	7.09817(−3)	7.27340(−3)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 15
The Stokes parameter $V_*(\tau,\mu,\phi)$ for Case 7.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	−4.51545(−2)	−4.20410(−2)	−3.90846(−2)	−3.35852(−1)	−1.95960(−2)	−9.64789(−3)	
−0.9	2.60070(−2)	2.48166(−2)	2.35856(−2)	2.10368(−2)	1.30025(−2)	6.25990(−3)	
−0.8	2.02095(−2)	1.95168(−2)	1.87459(−2)	1.70233(−2)	1.08743(−2)	5.24220(−3)	
−0.7	2.46781(−2)	2.39901(−2)	2.31855(−2)	2.13018(−2)	1.40500(−2)	6.93976(−3)	
−0.6	3.04552(−2)	2.98503(−2)	2.90720(−2)	2.70998(−2)	1.86075(−2)	9.49404(−3)	
−0.5	3.80022(−2)	3.76048(−2)	3.69537(−2)	3.50352(−2)	2.52377(−2)	1.34090(−2)	
−0.4	4.75687(−2)	4.76207(−2)	4.73090(−2)	4.57940(−2)	3.50422(−2)	1.96211(−2)	
−0.3	5.85690(−2)	5.95311(−2)	5.99862(−2)	5.96685(−1)	4.96324(−2)	3.00493(−2)	
−0.2	6.83037(−2)	7.09390(−2)	7.28606(−2)	7.51460(−2)	7.04104(−2)	4.87172(−2)	
−0.1	7.11407(−2)	7.65261(−2)	8.06724(−2)	8.69098(−2)	9.39425(−2)	8.14479(−2)	
−0.0	5.79644(−2)	7.07520(−2)	7.86451(−2)	9.01711(−2)	1.09125(−1)	1.12390(−1)	
0.0		7.07520(−2)	7.86451(−2)	9.01711(−2)	1.09125(−1)	1.12390(−1)	8.99593(−2)
0.1		2.71184(−2)	4.78708(−2)	7.62047(−2)	1.15852(−1)	1.28830(−1)	1.27299(−1)
0.2		1.69878(−2)	3.26751(−2)	5.97619(−2)	1.13708(−1)	1.37800(−1)	1.47671(−1)
0.3		1.41663(−2)	2.77754(−2)	5.29116(−2)	1.11479(−1)	1.43480(−1)	1.62171(−1)
0.4		1.37145(−2)	2.70028(−2)	5.20492(−2)	1.14193(−1)	1.51716(−1)	1.77043(−1)
0.5		1.46193(−2)	2.87831(−2)	5.55938(−2)	1.23520(−1)	1.66343(−1)	1.97326(−1)
0.6		1.68434(−2)	3.31002(−2)	6.37882(−2)	1.41545(−1)	1.91093(−1)	2.27991(−1)
0.7		2.08731(−2)	4.09077(−2)	7.84860(−2)	1.72578(−1)	2.31945(−1)	2.76262(−1)
0.8		2.77566(−2)	5.42373(−2)	1.03511(−1)	2.24653(−1)	2.99361(−1)	3.54282(−1)
0.9		3.81777(−2)	7.45806(−2)	1.42274(−1)	3.08319(−1)	4.10187(−1)	4.84644(−1)
1.0		5.61898(−1)	1.07541	1.96986	3.78756	4.57128	4.91070

cases are to be considered. The approach follows an idea introduced in a previous work on the scalar case [1] and consists essentially in replacing the Dirac delta distribution used to model the polar-angle dependence of the incident beam by a rectangular nascent delta function defined in terms of a “narrowness” parameter ϵ.

Numerical results of benchmark quality that we believe useful for code verification and validation are provided for the problem introduced in Ref. [4]. From extensive numerical testing, we have concluded that a sufficiently small value of the “narrowness” parameter can always be found so that the highly accurate results obtained from the exact formulation based on the Dirac delta distribution are reproduced. And so, in our continuing work on radiative transfer with polarization, we plan to extend the methodology developed in this work to the case of a multilayer medium subject to Fresnel boundary and interface conditions.

Table 16
The Stokes parameter $U_*(\tau,\mu,\phi)$ for Case 8.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	0.0	0.0	0.0	0.0	0.0	0.0	
−0.9	7.50987(−2)	5.27714(−2)	3.71334(−2)	1.85435(−2)	2.63816(−3)	6.27991(−4)	
−0.8	4.72667(−2)	3.55077(−2)	2.66317(−2)	1.50593(−2)	3.06908(−3)	9.32642(−4)	
−0.7	3.79812(−2)	2.97482(−2)	2.31741(−2)	1.40899(−2)	3.50737(−3)	1.21074(−3)	
−0.6	3.38256(−2)	2.74880(−2)	2.20695(−2)	1.41463(−2)	3.99013(−3)	1.48555(−3)	
−0.5	3.22578(−2)	2.70140(−2)	2.21858(−2)	1.47599(−2)	4.50071(−3)	1.75764(−3)	
−0.4	3.19049(−2)	2.74348(−2)	2.29484(−2)	1.56922(−2)	5.02508(−3)	2.02271(−3)	
−0.3	3.19837(−2)	2.82709(−2)	2.40520(−2)	1.68138(−2)	5.55312(−3)	2.27472(−3)	
−0.2	3.18847(−2)	2.92340(−2)	2.53156(−2)	1.80462(−2)	6.07660(−3)	2.50931(−3)	
−0.1	3.06043(−2)	3.00671(−2)	2.65956(−2)	1.93295(−2)	6.58748(−3)	2.72505(−3)	
−0.0	2.45301(−2)	3.03673(−2)	2.77147(−2)	2.05990(−2)	7.07646(−3)	2.92010(−3)	
0.0		3.03673(−2)	2.77147(−2)	2.05990(−2)	7.07646(−3)	2.92010(−3)	9.92237(−4)
0.1		2.91236(−2)	2.83576(−2)	2.17611(−2)	7.53140(−3)	3.09034(−3)	1.27540(−3)
0.2		2.48698(−2)	2.78618(−2)	2.26574(−2)	7.93472(−3)	3.22928(−3)	1.37403(−3)
0.3		2.03509(−2)	2.59534(−2)	2.30335(−2)	8.25929(−3)	3.32756(−3)	1.41681(−3)
0.4		1.69005(−2)	2.34424(−2)	2.26950(−2)	8.46262(−3)	3.37149(−3)	1.42342(−3)
0.5		1.44189(−2)	2.09845(−2)	2.17072(−2)	8.48240(−3)	3.34076(−3)	1.39118(−3)
0.6		1.26697(−2)	1.88612(−2)	2.02981(−2)	8.24046(−3)	3.20443(−3)	1.31021(−3)
0.7		1.15680(−2)	1.72469(−2)	1.87286(−2)	7.63654(−3)	2.91068(−3)	1.16290(−3)
0.8		1.13269(−2)	1.63480(−2)	1.69301(−2)	6.34639(−3)	2.33253(−3)	9.12137(−4)
0.9		6.26146(−3)	7.55583(−3)	5.83080(−3)	1.85867(−3)	9.23082(−4)	4.48917(−4)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 17
The Stokes parameter $V_*(\tau,\mu,\phi)$ for Case 8.

μ	$\tau=0$	$\tau=\tau_0/20$	$\tau=\tau_0/10$	$\tau=\tau_0/5$	$\tau=\tau_0/2$	$\tau=3\tau_0/4$	$\tau=\tau_0$
−1.0	1.40319(−2)	3.76063(−2)	4.88559(−2)	5.22606(−2)	2.65666(−2)	9.36148(−3)	
−0.9	9.98858(−2)	9.47523(−2)	8.76314(−2)	7.15298(−2)	3.11860(−2)	1.13361(−2)	
−0.8	9.30948(−2)	9.32541(−2)	8.93932(−2)	7.62802(−2)	3.53098(−2)	1.35757(−2)	
−0.7	9.76801(−2)	1.00037(−1)	9.72120(−2)	8.43866(−2)	4.02636(−2)	1.62702(−2)	
−0.6	1.02087(−1)	1.07441(−1)	1.06026(−1)	9.36826(−2)	4.59246(−2)	1.94384(−2)	
−0.5	1.05974(−1)	1.15176(−1)	1.15603(−1)	1.04071(−1)	5.23371(−2)	2.31006(−2)	
−0.4	1.08936(−1)	1.23048(−1)	1.25868(−1)	1.15603(−1)	5.95918(−2)	2.72613(−2)	
−0.3	1.10250(−1)	1.30726(−1)	1.36679(−1)	1.28329(−1)	6.78052(−2)	3.19215(−2)	
−0.2	1.08601(−1)	1.37727(−1)	1.47817(−1)	1.42282(−1)	7.71165(−2)	3.71205(−2)	
−0.1	1.01043(−1)	1.43328(−1)	1.58961(−1)	1.57477(−1)	8.76869(−2)	4.29584(−2)	
−0.0	7.70467(−2)	1.46336(−1)	1.69623(−1)	1.73900(−1)	9.97034(−2)	4.95665(−2)	
0.0		1.46336(−1)	1.69623(−1)	1.73900(−1)	9.97034(−2)	4.95665(−2)	1.44547(−2)
0.1		1.44118(−1)	1.79053(−1)	1.91507(−1)	1.13385(−1)	5.70934(−2)	2.18075(−2)
0.2		1.33830(−1)	1.85971(−1)	2.10242(−1)	1.28996(−1)	6.57115(−2)	2.75238(−2)
0.3		1.25441(−1)	1.91105(−1)	2.30191(−1)	1.46871(−1)	7.56281(−2)	3.33080(−2)
0.4		1.23981(−1)	1.98878(−1)	2.52365(−1)	1.67470(−1)	8.71023(−2)	3.96322(−2)
0.5		1.30489(−1)	2.13745(−1)	2.79592(−1)	1.91520(−1)	1.00477(−1)	4.67681(−2)
0.6		1.46554(−1)	2.40295(−1)	3.16765(−1)	2.20303(−1)	1.16247(−1)	5.49692(−2)
0.7		1.76187(−1)	2.85435(−1)	3.71737(−1)	2.56247(−1)	1.35217(−1)	6.45617(−2)
0.8		2.27235(−1)	3.61051(−1)	4.57792(−1)	3.04280(−1)	1.58941(−1)	7.60666(−2)
0.9		3.10088(−1)	4.89446(−1)	6.09304(−1)	3.80842(−1)	1.92230(−1)	9.07559(−2)
1.0		3.78850	4.91348	4.21146	1.03805	3.23683(−1)	1.21978(−1)

Table 18
$V_*(\tau,\mu,\phi)$ of Case 5 for $\tau=\tau_0/5$ and $\mu=-0.5$.

ϵ	Double precision	Quadruple precision
10^{-5}	−8.37565(−5)	−8.37565(−5)
10^{-6}	−8.40805(−5)	−8.40805(−5)
10^{-7}	−8.41128(−5)	−8.41129(−5)
10^{-8}	−8.41158(−5)	−8.41162(−5)
10^{-9}	−8.41395(−5)	−8.41165(−5)
10^{-10}	−8.38776(−5)	−8.41165(−5)
10^{-11}	−8.92018(−5)	−8.41165(−5)

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