Annals of Nuclear Energy 69 (2014) 203–204

Contents lists available at ScienceDirect

Annals of Nuclear Energy

journal homepage: www.elsevier.com/locate/anucene

Technical note

On the dispersion function for complex values of the parameter c

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ARTICLE INFO

Article history:
Received 26 October 2013
Received in revised form 12 February 2014
Accepted 14 February 2014

Keywords:
Transport theory
Dispersion function
Discrete eigenvalue
Complex c

ABSTRACT

The dispersion function relevant to one-speed transport theory with isotropic scattering is analyzed for the case of complex values of c, and an explicit expression is given for the discrete eigenvalue ν_0.

1. Introduction

One of us (CES) was recently asked (Prinja, 2013) about extending to the case of complex values of c a previous result (Siewert, 1999) that gives an analytical expression for the *discrete eigenvalue* ν_0 relevant to steady-state, one-speed neutron transport equation (for isotropic scattering). While c is normally used to denote the mean number of secondary particles per collision and as such is contained in the interval $[0, \infty)$, it can be considered an ordinary complex variable for transformed versions of the time-dependent transport equation (Case, 1960; Bowden and Williams, 1964; Kuščer and Zweifel, 1965). Here we give a brief discussion of the *dispersion function* $\Lambda(z)$ for the case of complex c. We also report a generalization of the mentioned analytical expression (Siewert, 1999) for ν_0 appropriate to the case of complex values of c.

2. Preliminary analysis

We start with the dispersion function written as

$$\Lambda(z) = 1 + \frac{cz}{2}\int_{-1}^{1}\frac{d\mu}{\mu - z}, \tag{1}$$

where

$$c = \alpha + i\beta \tag{2}$$

for real α and β. We note that the zeros of $\Lambda(z)$ must occur in $\pm$ pairs. Furthermore, a zero of $\Lambda(z)$ for a negative value of β is just the complex conjugate of a zero for the same positive value of β. For this reason, we can consider only positive values of β in our analysis.

Clearly $\Lambda(z)$ as a function of the complex variable z has a branch cut along the real axis from -1 to 1. We can use the Plemelj formulas (Muskhelishvili, 1953) to compute the limiting values of $\Lambda(z)$ as the branch cut is approached from above and below to find

$$\Lambda^{\pm}(x) = R^{\pm}(x) + iI^{\pm}(x), \quad x \in (-1, 1), \tag{3}$$

where

$$R^{\pm}(x) = 1 + \frac{\alpha}{2}J(x) \mp \frac{\beta x \pi}{2} \tag{4}$$

and

$$I^{\pm}(x) = \frac{\beta}{2}J(x) \pm \frac{\alpha x \pi}{2}, \tag{5}$$

with

$$J(x) = x\ln\left(\frac{1-x}{1+x}\right). \tag{6}$$

In order to compute the number of zeros of $\Lambda(z)$, we use the argument principle (Ahlfors, 1953) and therefore evaluate the change in the argument of $\Lambda^{+}(x)$ for x going from -1 to 1 and of $\Lambda^{-}(x)$ for x going from 1 to -1. However, since $\Lambda^{-}(-x) = \Lambda^{+}(x)$, it is sufficient to study just $\Lambda^{+}(x)$ as x varies from -1 to 1. Continuing, we let $\Theta^{\pm}(x)$ denote the arguments of $\Lambda^{\pm}(x)$ and write

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http://dx.doi.org/10.1016/j.anucene.2014.02.015
0306-4549/

$$\Theta^{\pm}(x) = \arctan[I^{\pm}(x)/R^{\pm}(x)]. \tag{7}$$

It is important to note that care must be taken in the use of the arctan function in Eq. (7) so that the resulting argument functions are continuous.

To summarize our findings regarding the zeros of $\Lambda(z)$, we let

$$\gamma = \frac{\pi}{2}\left(\frac{\alpha^2+\beta^2}{\beta}\right), \tag{8a}$$

$$\varpi = \exp(-\alpha\pi/\beta), \tag{8b}$$

and

$$\xi = \left(\frac{1-\varpi}{1+\varpi}\right), \tag{8c}$$

and conclude that

$$\gamma\xi > 1 \Rightarrow \text{two zeros} \ni [-1,1] \tag{9a}$$

and

$$\gamma\xi \leqslant 1 \Rightarrow \text{no zeros}. \tag{9b}$$

We note that in the event that $\gamma\xi = 1$ in Eq. (9b), we have $\Lambda^{+}(\xi) = \Lambda^{-}(-\xi) = 0$, $\xi \in (0,1)$, for any non-zero ratio of α/β. In addition, we find that the dispersion function $\Lambda(z)$ has no zeros for $\alpha \leqslant 0$ and that the endpoints ± 1 cannot be zeros of $\Lambda(z)$.

3. An analytical expression for ν_0

We now assume that the condition given by Eq. (9a) is satisfied and look for an analytical expression for ν_0, keeping in mind that $-\nu_0$ is the companion eigenvalue. We write

$$\log[\Lambda^{+}(x)/\Lambda^{-}(x)] = \ln|\Lambda^{+}(x)/\Lambda^{-}(x)| + i\Theta(x), \tag{10}$$

where

$$\Theta(x) = \Theta^{+}(x) - \Theta^{-}(x). \tag{11}$$

Clearly, $\Theta(x) \in [0, 2\pi]$ for $x \in (0,1)$ since $\Theta^{\pm}(0) = 0$ and $\Theta^{\pm}(1) = \pm\pi + \arctan(\beta/\alpha)$.

Once a Wiener–Hopf factorization of $\Lambda(z)$ is established along the lines of previous works (Case, 1960; Siewert, 1980), we obtain

$$(\nu_0^2 - z^2)\Lambda(\infty)X(z)X(-z) = \Lambda(z), \tag{12}$$

where

$$X(z) = \frac{1}{1-z}\exp\left\{\frac{1}{2\pi i}\int_0^1 [\ln|\Lambda^{+}(x)/\Lambda^{-}(x)| + i\Theta(x)]\frac{dx}{x-z}\right\} \tag{13}$$

is the X-function used by Muskhelishvili (1953) and Case (1960). We can now solve Eq. (12) to get

$$\nu_0 = \left\{z^2 + \frac{\Lambda(z)}{(1-c)X(z)X(-z)}\right\}^{1/2}. \tag{14}$$

We note that Eq. (14) is an identity in the complex z plane and so, since the right-hand side of Eq. (14) depends only on c and z, that equation gives an explicit expression for ν_0 for any value of z. A particularly simple expression for ν_0 can be found by taking the limit of $z \to \infty$ in Eq. (14). We find

$$\nu_0 = \left\{\frac{3-2c}{3(1-c)} + \frac{i}{\pi}\int_0^1 x[\ln|\Lambda^{+}(x)/\Lambda^{-}(x)| + i\Theta(x)]dx\right\}^{1/2}. \tag{15}$$

Table 1
The discrete eigenvalue ν_0 for selected values of $c = \alpha + i\beta$.

α	β	$\Re\{\nu_0\}$	$\Im\{\nu_0\}$
0.5	0.1	1.03101573694256	3.97452079194778(−2)
0.9	0.1	1.50104911758437	5.47242004863977(−1)
1.0	0.1	1.33951917593349	1.23658702230985
1.0	1.0	4.61488844429075(−1)	2.18244298742223(−1)
1.0	3.0	2.01131500532114(−1)	3.85807890710501(−2)
3.0	0.1	1.01992873143149(−2)	2.51300678037113(−1)
3.0	1.0	8.69421709883194(−2)	2.15778216073148(−1)
3.0	3.0	1.22266126305228(−1)	1.03545315562501(−1)

We have evaluated Eqs. (14) and (15) to obtain easily confirmed correct results, a sample of which is listed in the accompanying table. In order to generate the accurate results reported in Table 1, we have used Eqs. (14) and (15) with a low-order (say 80 quadrature points) Gaussian scheme to generate (Fortran) results good to five or six figures that provide sufficiently accurate results to start a numerical iteration computation. Two different iteration schemes have been successfully implemented: in addition to the usual Newton scheme based on $\Lambda(z)$, we have replaced z in Eq. (14) with ν_0 thereby defining an equation with only ν_0 to be deduced by iteration. Both schemes achieved convergence to 15-digit accuracy after two or three iterations. Finally, a simple check of our numerical results was performed by evaluating the integral of Eq. (1) analytically, taking $z = \nu_0$ and expressing c in terms of ν_0 in the resulting equation. Then, we confirmed that the values of c in Table 1 can be recovered from the corresponding values of ν_0 in that table.

4. Concluding remarks

Finally, a comment about Eq. (9b) when the equality holds. In contrast to the finding of Bowden and Williams (1964), we find that there are no zeros of $\Lambda(z)$ embedded in the continuum $[-1,1]$. While $\Lambda^{+}(x)$ has a zero at $x = \xi$ and $\Lambda^{-}(x)$ has a zero at $x = -\xi$, the two limiting values $\Lambda^{\pm}(x)$ do not have zeros at the same value of x, and so $\Lambda(z)$ does not exist on the cut $[-1,1]$; clearly then, $\Lambda(z)$ cannot have a zero there on the cut. This situation can be contrasted with work reported by Siewert (1977) and Arthur et al. (1977) where the boundary values of the dispersion function do have zeros at a common point on the relevant branch cut, and so in that case (Siewert, 1977; Arthur et al., 1977) the dispersion function does have a zero embedded in the continuum.

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