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The linearized Boltzmann equation with Cercignani–Lampis boundary conditions: Heat transfer in a gas confined by two plane-parallel surfaces ☆

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ABSTRACT

The analytical discrete-ordinates method is used to solve the problem of heat transfer for a single-species gas confined by two plane-parallel surfaces. The formulation of the problem is based on the linearized Boltzmann equation for rigid-sphere interactions between gas particles and the Cercignani–Lampis kernel for gas–surface interactions. Accurate numerical results are reported for the density, temperature, and heat-flow perturbations from a reference (equilibrium) state and are compared with similar results from five kinetic models. An interesting finding of this work is that there are combinations of the four numerical values of the accommodation coefficients used to define the Cercignani–Lampis boundary conditions that give rise to heat flows that are larger for the transition regime than for the free-molecular regime, an effect not observed when the standard (Maxwell) boundary conditions are used.
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1. Introduction

Heat transfer in a gas confined by two plane-parallel surfaces is a classical problem in the field of rarefied gas dynamics (Cercignani, 1969; Williams, 1971a, 2001) which has been the subject of many investigations, most of them based on kinetic models [see, for example, Thomas et al. (1973) and the references quoted therein]. It was not until more recent years that a few works reporting approaches based on the linearized Boltzmann equation (LBE) for a single-species gas (Ohwada et al., 1989; Siewert, 2003a) or a binary mixture (Garcia and Siewert, 2007a) and even on the full (nonlinear) Boltzmann equation for a single-species gas (Ohwada, 1996) or a binary mixture (Kosuge et al., 2001) appeared in the literature. Nevertheless, since all of the existing works on this topic have been formulated in terms of Maxwell (specular-diffuse) boundary conditions, we thought that a study of the problem introducing an improved description of gas–surface interactions based on the Cercignani–Lampis (CL) scattering kernel (Cercignani and Lampis, 1971) would be of interest.

The functional form of the CL kernel (Cercignani and Lampis, 1971) is the result of what can be considered an inspired mathematical insight, and is corroborated by alternative derivations that add physical insight to the original idea. We can mention, for example, the work by Kuščer et al. (1971), which is based on a procedure that involves the solution of a Fokker–Plank equation for the motion of Brownian particles and relaxing to some extent Knudsen's assumption that the scattering of a Maxwellian distribution results also in a Maxwellian, and the work by Williams (1971b), which relies on an analogy between gas–surface interactions and electromagnetic-wave scattering from a rough surface. There is also a work by Cowling (1974) that provides an independent derivation of the CL kernel based on a minimum number of assumptions.

Basically, the CL kernel depends on two parameters, the surface accommodation coefficient for tangential momentum and the surface accommodation coefficient for the kinetic energy related to the normal component of the velocity. The CL kernel thus opens up the possibility of an improved description of gas–surface interactions when compared with the Maxwell kernel, which depends on one parameter only. Indeed, two recent works (Sazhin and Kulev, 2007; Pantazis et al., 2011) have reported studies where calculations based on the CL kernel agree better with experimental results than calculations based on the Maxwell kernel.

In this work, we use a modern version of the discrete-ordinates method known as the analytical discrete-ordinates (ADO) method (Barichello and Siewert, 1999) to solve the problem of heat transfer for a single-species gas described by the LBE for rigid-sphere inter-

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actions and CL boundary conditions. Accurate numerical results are reported for the main quantities of interest: density, temperature, and heat-flow perturbations. In addition, as our formulation is sufficiently general to allow us to include several kinetic models in our study by just a proper choice of input parameters, we report in this work a comparison between LBE results and the results of five different kinetic models.

2. Formulation of the problem

We consider a gas of rigid spheres confined by two parallel plates which are kept at different temperatures. Provided the difference between the plate temperatures is small, the velocity distribution function $f(z,\boldsymbol{v})$, where z is the spatial coordinate measured in a direction perpendicular to the bounding surfaces and $\boldsymbol{v}$ is the velocity of a gas particle, can be linearized about an absolute Maxwellian distribution as

$$f(z,\boldsymbol{v}) = f_0(v)[1+h(z,\boldsymbol{c})], \tag{1}$$

where v is the magnitude of the velocity vector $\boldsymbol{v}$,

$$f_0(v) = n(\lambda/\pi)^{3/2}\mathrm{e}^{-\lambda v^2}, \quad \lambda = m/(2kT_0), \tag{2}$$

is the Maxwellian distribution for n particles of mass m (per unit volume) in equilibrium at temperature T_0, $\boldsymbol{c} = \lambda^{1/2}\boldsymbol{v}$ is a dimensionless velocity variable, and k is the Boltzmann constant.

The perturbation $h(z,\boldsymbol{c})$ in Eq. (1) can then be described by the linearized Boltzmann equation (LBE) for rigid-sphere interactions (Garcia and Siewert, 2009, 2010),

$$c\mu\frac{\partial}{\partial z}h(z,\boldsymbol{c}) + \varepsilon_0 \nu(c)h(z,\boldsymbol{c}) = \varepsilon_0\int \mathrm{e}^{-c'^2}\mathcal{P}(\boldsymbol{c}':\boldsymbol{c})h(z,\boldsymbol{c}')d^3c', \tag{3}$$

where

$$\nu(c) = \frac{2c^2+1}{c}\int_0^c \mathrm{e}^{-x^2}dx + \mathrm{e}^{-c^2}, \tag{4}$$

with c denoting the magnitude of $\boldsymbol{c}$, is the collision frequency,

$$\mathcal{P}(\boldsymbol{c}':\boldsymbol{c}) = \frac{1}{\pi}\left(\frac{2}{|\boldsymbol{c}'-\boldsymbol{c}|}\exp\left\{\frac{|\boldsymbol{c}'\times\boldsymbol{c}|^2}{|\boldsymbol{c}'-\boldsymbol{c}|^2}\right\} - |\boldsymbol{c}'-\boldsymbol{c}|\right) \tag{5}$$

is the rigid-sphere scattering kernel (Pekeris, 1955), and

$$\varepsilon_0 = n\pi^{1/2}d^2, \tag{6}$$

with d denoting the diameter of the particles. We note that spherical coordinates $\{c,\theta,\phi\}$, with $\mu=\cos\theta$, are used to describe the dimensionless velocity vector $\boldsymbol{c}$, so that

$$h(z,\boldsymbol{c}) \Longleftrightarrow h(z,c,\mu,\phi).$$

To derive the boundary conditions that go with Eq. (3), we first take the reference temperature T_0 in Eq. (2) to be the average between the wall temperatures, so that we can write the temperature of the wall located at $z=-z_0$ (assumed to be at a higher temperature than the other wall) as

$$T_{w1} = T_0(1+\delta) \tag{7}$$

and the temperature of the wall at $z=z_0$ as

$$T_{w2} = T_0(1-\delta), \tag{8}$$

where $\delta = 2(T_{w1}-T_{w2})/(T_{w1}+T_{w2})$. Next, using the CL kernel (Cercignani and Lampis, 1971) to describe the interaction of the gas with the walls, we follow Cercignani (1975) and find that we can write the CL boundary conditions for the problem as

$$h(-z_0,c,\mu,\phi) - \mathcal{J}_1\{h\}(-z_0,c,\mu,\phi) = -[\alpha_{n,1}f(c,\mu)+\widehat{\alpha}_1 g(c,\mu)]\delta \tag{9a}$$

and

$$h(z_0,c,-\mu,\phi) - \mathcal{J}_2\{h\}(z_0,c,-\mu,\phi) = [\alpha_{n,2}f(c,\mu)+\widehat{\alpha}_2 g(c,\mu)]\delta, \tag{9b}$$

for $c\in[0,\infty)$, $\mu\in(0,1]$, and $\phi\in[0,2\pi]$. Here,

$$f(c,\mu) = 1-c^2\mu^2, \tag{10a}$$

$$g(c,\mu) = 1-c^2(1-\mu^2), \tag{10b}$$

$$\mathcal{J}_1\{h\}(-z_0,c,\mu,\phi) = \int_0^\infty\int_0^1\int_0^{2\pi} h(-z_0,c',-\mu',\phi') \times R_1(c',-\mu',\phi':c,\mu,\phi)c'^2 d\phi' d\mu' dc', \tag{11a}$$

$$\mathcal{J}_2\{h\}(z_0,c,-\mu,\phi) = \int_0^\infty\int_0^1\int_0^{2\pi} h(z_0,c',\mu',\phi') \times R_2(c',\mu',\phi':c,-\mu,\phi)c'^2 d\phi' d\mu' dc', \tag{11b}$$

and the CL kernel is given by (Siewert, 2003b)

$$R(c',\mp\mu',\phi':c,\pm\mu,\phi) = \frac{2c'\mu'}{\widehat{\alpha}\alpha_n\pi}S(c',\mp\mu':c,\pm\mu)T(c',\mp\mu',\phi':c,\pm\mu,\phi), \tag{12}$$

where

$$\widehat{\alpha} = \alpha_t(2-\alpha_t), \tag{13}$$

$$S(c',\mp\mu':c,\pm\mu) = \exp\{-[(c'\mu')^2+(1-\alpha_n)(c\mu)^2]/\alpha_n\} \times I_0[2(1-\alpha_n)^{1/2}c'\mu'c\mu/\alpha_n], \tag{14a}$$

and

$$T(c',\mp\mu',\phi':c,\pm\mu,\phi) = E(c',\mu':c,\mu)\exp\{-2c'r(\mu')cr(\mu) \times [|1-\alpha_t|-(1-\alpha_t)\cos(\phi'-\phi)]/\widehat{\alpha}\}, \tag{14b}$$

with

$$E(c',\mu':c,\mu) = \exp\{-[|1-\alpha_t|cr(\mu)-c'r(\mu')]^2/\widehat{\alpha}\}. \tag{15}$$

In these expressions, $I_n(x)$ denotes the nth-order modified Bessel function of the first kind and $r(x)=(1-x^2)^{1/2}$. In addition, $\alpha_t\in[0,2]$ denotes the accommodation coefficient of tangential momentum and $\alpha_n\in[0,1]$ that of the kinetic energy due to the normal component of the velocity. The subscript 1 attached to the CL kernel that appears in Eq. (11a) has the meaning that accommodation coefficients $\{\alpha_{t,1},\alpha_{n,1}\}$ should be used in Eqs. (12)–(15) when defining the boundary condition at $z=-z_0$. Similarly, the subscript 2 attached to the CL kernel in Eq. (11b) means that $\{\alpha_{t,2},\alpha_{n,2}\}$ should be used in Eqs. (12)–(15) when defining the boundary condition at $z=z_0$.

Our aim in this work is to compute, for $z\in[-z_0,z_0]$, the density, temperature, and heat-flow perturbations, defined, respectively, as

$$N(z) = \frac{1}{\pi^{3/2}}\int \mathrm{e}^{-c^2}h(z,\boldsymbol{c})d^3c, \tag{16a}$$

$$T(z) = \frac{2}{3\pi^{3/2}}\int \mathrm{e}^{-c^2}h(z,\boldsymbol{c})(c^2-3/2)d^3c, \tag{16b}$$

and

$$Q(z) = \frac{1}{\pi^{3/2}}\int \mathrm{e}^{-c^2}h(z,\boldsymbol{c})(c^2-5/2)c\mu d^3c. \tag{16c}$$

As before (Siewert, 2003a; Garcia and Siewert, 2007a), we have found that only the first (azimuthally symmetric) component in a Fourier series expansion of $h(z,\boldsymbol{c})$ in terms of the azimuthal angle

ϕ is relevant to this problem, and so, introducing the dimensionless spatial variable

$$\tau = z\varepsilon_0, \tag{17}$$

where ε_0 is given by Eq. (6), and using

$$\psi(\tau, c, \mu) = h(\tau/\varepsilon_0, \boldsymbol{c}), \tag{18}$$

we find that Eq. (16) reduce to

$$N(\tau) = \frac{2}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 e^{-c^2} \psi(\tau, c, \mu) c^2 d\mu dc, \tag{19a}$$

$$T(\tau) = \frac{4}{3\pi^{1/2}} \int_0^\infty \int_{-1}^1 e^{-c^2} \psi(\tau, c, \mu)(c^2 - 3/2) c^2 d\mu dc, \tag{19b}$$

and

$$Q = \frac{2}{\pi^{1/2}} \int_0^\infty \int_{-1}^1 e^{-c^2} \psi(\tau, c, \mu)(c^2 - 5/2) c^3 \mu d\mu dc. \tag{19c}$$

It should be noted that, in order to avoid excessive notation, we have adopted here the frequently used (but dubious) procedure of not always introducing new labels for the dependent variables (in this case N, T, and Q) when an independent variable is changed. In addition, we note that we have omitted the τ-dependence on the left side of Eq. (19c), since, as in the case based on Maxwell boundary conditions (Siewert, 2003a), it can be shown that $Q(\tau)$ is a constant in this work.

We can now use Eq. (18) in Eqs. (3) and (9) to find that $\psi(\tau, c, \mu)$ is to be determined from the balance equation

$$c\mu \frac{\partial}{\partial \tau} \psi(\tau, c, \mu) + \nu(c) \psi(\tau, c, \mu) = \int_0^\infty \int_{-1}^1 e^{-c'^2} \mathcal{P}_0(c', \mu' : c, \mu) \times \psi(\tau, c', \mu') c'^2 d\mu' dc', \tag{20}$$

for $\tau \in (-a, a)$, $c \in [0, \infty)$, and $\mu \in [-1, 1]$, and the boundary conditions

$$\psi(-a, c, \mu) - \int_0^\infty \int_0^1 \psi(-a, c', -\mu') R_{0,1}(c', -\mu' : c, \mu) c'^2 d\mu' dc' = -[\alpha_{n,1} f(c, \mu) + \widehat{\alpha}_1 g(c, \mu)]\delta \tag{21a}$$

and

$$\psi(a, c, -\mu) - \int_0^\infty \int_0^1 \psi(a, c', \mu') R_{0,2}(c', \mu' : c, -\mu) c'^2 d\mu' dc' = [\alpha_{n,2} f(c, \mu) + \widehat{\alpha}_2 g(c, \mu)]\delta, \tag{21b}$$

for $c \in [0, \infty)$ and $\mu \in (0, 1]$. In these expressions, $a = z_0\varepsilon_0$. The scattering kernel

$$\mathcal{P}_0(c', \mu' : c, \mu) = \int_0^{2\pi} \mathcal{P}(\boldsymbol{c}' : \boldsymbol{c}) d\phi \tag{22}$$

that appears in Eq. (20) can be expressed as

$$\mathcal{P}_0(c', \mu' : c, \mu) = (1/2) \sum_{n=0}^\infty (2n+1) P_n(\mu') P_n(\mu) \mathcal{P}^{(n)}(c', c), \tag{23}$$

where $P_n(x)$ denotes a Legendre polynomial and $\{\mathcal{P}^{(n)}(c', c)\}$ are expansion coefficients (Garcia and Siewert, 2007b). In addition, the reflection functions $R_{0,1}(c', -\mu' : c, \mu)$ and $R_{0,2}(c', \mu' : c, -\mu)$ that appear in Eq. (21) are expressed by using, respectively, the pair $\{\widehat{\alpha}_1, \alpha_{n,1}\}$ and the pair $\{\widehat{\alpha}_2, \alpha_{n,2}\}$ either in the general definition of $R_0(c', \mp\mu' : c, \pm\mu)$ listed next or in one of the special forms listed after it. The general definition of the reflection function is

$$R_0(c', \mp\mu' : c, \pm\mu) = \frac{4c'\mu'}{\widehat{\alpha}\alpha_n} S(c', \mp\mu' : c, \pm\mu) U_0(c', \mp\mu' : c, \pm\mu), \tag{24}$$

where $S(c', \mp\mu' : c, \pm\mu)$ is given by Eq. (14a) and

$$U_0(c', \mp\mu' : c, \pm\mu) = \exp\{-[c'^2 r^2(\mu') + (1 - \widehat{\alpha}) c^2 r^2(\mu)]/\widehat{\alpha}\} \times I_0[2(1 - \widehat{\alpha})^{1/2} c' r(\mu') c r(\mu)/\widehat{\alpha}]. \tag{25}$$

Equation (24) is valid for all choices of $\widehat{\alpha}$ and α_n, except the following special cases (Siewert, 2003b):

$$\lim_{\hat{\alpha}\to 0} R_0(c', \mp\mu' : c, \pm\mu) = \frac{2\mu'}{\alpha_n r(\mu')} S(c', \mp\mu' : c, \pm\mu) \delta[c' r(\mu') - c r(\mu)], \tag{26a}$$

$$\lim_{\alpha_n \to 0} R_0(c', \mp\mu' : c, \pm\mu) = \frac{2}{\widehat{\alpha}} U_0(c', \mp\mu' : c, \pm\mu) \delta(c'\mu' - c\mu), \tag{26b}$$

and

$$\lim_{\hat{\alpha}\to 0} \lim_{\alpha_n \to 0} R_0(c', \mp\mu' : c, \pm\mu) = \frac{1}{c^2} \delta(c' - c) \delta(\mu' - \mu), \tag{26c}$$

where $\delta(x)$ denotes the Dirac delta distribution.

At this point, we believe it instructive to look briefly at how the CL boundary conditions derived in this work compare with the usual Maxwell boundary conditions for the considered heat-transfer problem. In the approach based on the Maxwell (specular/diffuse) boundary conditions (Siewert, 2003a), we have, instead of Eq. (21),

$$\psi(-a, c, \mu) - (1 - \alpha_1) \psi(-a, c, -\mu) - 4\alpha_1 \int_0^\infty \int_0^1 e^{-c'^2} \psi(-a, c', -\mu') c'^3 \mu' d\mu' dc' = \alpha_1 (c^2 - 2)\delta \tag{27a}$$

and

$$\psi(a, c, -\mu) - (1 - \alpha_2) \psi(a, c, \mu) - 4\alpha_2 \int_0^\infty \int_0^1 e^{-c'^2} \psi(a, c', \mu') c'^3 \mu' d\mu' dc' = -\alpha_2 (c^2 - 2)\delta, \tag{27b}$$

for $c \in [0, \infty)$ and $\mu \in (0, 1]$. Here, α_1 and α_2 are the accommodation coefficients at the boundaries $\tau = -a$ and $\tau = a$, respectively. Since

$$\lim_{\hat{\alpha}\to 1} \lim_{\alpha_n \to 1} R_0(c', \mp\mu' : c, \pm\mu) = 4c'\mu' e^{-c'^2}, \tag{28}$$

it is easy to see that the CL boundary condition with $\widehat{\alpha} = 1$ and $\alpha_n = 1$ reduces to the Maxwell boundary condition with $\alpha = 1$.

To close this section, we note that at least one of the accommodation coefficients $\{\widehat{\alpha}_1, \widehat{\alpha}_2, \alpha_{n,1}, \alpha_{n,2}\}$ must be non-zero when defining the problem with CL boundary conditions, otherwise there is no driving term for the problem. Similarly, in the case of Maxwell boundary conditions, at least one of the accommodation coefficients $\{\alpha_1, \alpha_2\}$ must be non-zero.

3. The ADO solution

Following Siewert (2003a), we can express the general solution of a discrete-ordinates version of Eq. (20) defined at the ordinates $\pm\mu_i$, $i = 1, 2, \ldots, N$, where $\{\mu_i\}$ denote the nodes of the Gauss–Legendre quadrature of order N shifted to the interval [0, 1], as

$$\psi(\tau, c, \pm\mu_i) = \psi_*(\tau, c, \pm\mu_i) + \boldsymbol{P}(c) \sum_{j=3}^J [A_j \boldsymbol{\Phi}(\nu_j, \pm\mu_i) e^{-(a+\tau)/\nu_j} + B_j \boldsymbol{\Phi}(\nu_j, \mp\mu_i) e^{-(a-\tau)/\nu_j}], \tag{29}$$

for $\tau \in [-a, a]$ and $c \in [0, \infty)$. Here,

$$\psi_*(\tau, c, \mu) = A_1 c\mu + A_2(c^2 - 5/2) + B_1 + B_2[\tau(c^2 - 5/2) - \mu A(c)], \tag{30}$$

where $A(c)$ is the Chapman–Enskog function associated with thermal conductivity (Loyalka and Hickey, 1989; Siewert, 2002),

$$\boldsymbol{P}(c) = [P_0(2e^{-c} - 1) \quad P_1(2e^{-c} - 1) \quad \cdots \quad P_K(2e^{-c} - 1)], \tag{31}$$

is a row-vector of dimension $K+1$ with components that are the basis functions used to approximate the speed dependence of the solution, and, as discussed in detail by Siewert (2003c), the elementary solutions $\{\mathbf{\Phi}(\nu_j, \pm\mu_i)\}$ and the separation constants $\{\nu_j\}$ come from the solution of an eigensystem of order $J = N(K+1)$. We note that the summation in Eq. (29) starts at $j = 3$ because the solutions that correspond to separation constants that approach unbounded values as the quadrature order N is increased (ν_1 and ν_2) have been replaced by the exact solutions of Eq. (30). Once the boundary conditions are used to determine the coefficients A_j and B_j, $j = 1, 2, \ldots, J$, the solution of the problem becomes completely known.

As in other works (Siewert, 2003a,c), a projection scheme, based on multiplying the boundary conditions evaluated at $\mu_i, i = 1, 2, \ldots, N$, by

$$\mathrm{e}^{-c^2}\boldsymbol{P}^T(c)c^2,$$

where T denotes the transpose operation, and integrating the resulting equations over all c, is used to generate a system of $2J$ linear algebraic equations for the unknown coefficients $\{A_j\}$ and $\{B_j\}$. However, this problem has a particularity: a constant is a solution of Eq. (20) and satisfies homogeneous versions of Eq. (21). It turns out that the coefficient B_1 does not appear in the linear system that is obtained from the projection of the boundary conditions, and so an additional condition is needed to determine this coefficient. Our procedure is to solve the linear system for the other $2J-1$ coefficients (either by least squares or by combining two equations into one to obtain a square system) and then to determine B_1 from the particle conservation condition (Siewert, 1999)

$$\int_{-a}^{a} N(\tau)d\tau = 0. \tag{32}$$

Using Eqs. (19a), (29), and (30), we find from Eq. (32) that

$$B_1 = A_2 - \frac{1}{2a}\sum_{j=3}^{J}\nu_j(A_j + B_j)(1 - \mathrm{e}^{-2a/\nu_j})N_j, \tag{33}$$

where

$$N_j = 2\pi^{-1/2}\boldsymbol{P}_0\boldsymbol{N}(\nu_j). \tag{34}$$

In this equation, we define

$$\boldsymbol{N}(\nu_j) = \sum_{i=1}^{N} w_i[\mathbf{\Phi}(\nu_j, \mu_i) + \mathbf{\Phi}(\nu_j, -\mu_i)] \tag{35}$$

with $\{w_i\}$ denoting the weights of the Gauss–Legendre quadrature of order N shifted to [0, 1], and, in general,

$$\boldsymbol{P}_n = \int_0^\infty \mathrm{e}^{-c^2}\boldsymbol{P}(c)c^{n+2}dc. \tag{36}$$

Finally, upon substitution of Eq. (29) into Eq. (16), we find our final expressions for the desired density, temperature, and heat-flow perturbations. For the density and temperature perturbations, $N(\tau)$ and $T(\tau)$, we obtain (Siewert, 2003a)

$$N(\tau) = -A_2 - B_2\tau + B_1 + \sum_{j=3}^{J}\left[A_j\mathrm{e}^{-(a+\tau)/\nu_j} + B_j\mathrm{e}^{-(a-\tau)/\nu_j}\right]N_j \tag{37a}$$

and

$$T(\tau) = A_2 + B_2\tau + \sum_{j=3}^{J}\left[A_j\mathrm{e}^{-(a+\tau)/\nu_j} + B_j\mathrm{e}^{-(a-\tau)/\nu_j}\right]T_j, \tag{37b}$$

where N_j is given by Eq. (34) and T_j is given by

$$T_j = (4/3)\pi^{-1/2}[\boldsymbol{P}_2 - (3/2)\boldsymbol{P}_0]\boldsymbol{N}(\nu_j). \tag{38}$$

For the heat-flow perturbation Q, after neglecting the ADO contribution (since Q is a constant, as mentioned in Section 2), we find (Siewert, 2003a)

$$Q = -(5/4)\varepsilon_t B_2, \tag{39}$$

where

$$\varepsilon_t = \frac{16}{15\pi^{1/2}}\int_0^\infty \mathrm{e}^{-c^2}A(c)c^5dc. \tag{40}$$

4. Numerical results

We report in this section numerical results for some selected cases, which are defined in Table 1. We note that we have considered only $\alpha_t \in [0, 1]$ when defining these cases, since the kernel $R_0(c', \mp\mu' : c, \pm\mu)$ can be seen to be symmetrical about $\alpha_t = 1.0$, and so numerical results for $\alpha_t \in [1, 2]$ can be obtained from the numerical results for $\alpha_t \in [0, 1]$ simply by changing α_t to $2 - \alpha_t$. In addition, we note that two of the considered cases (Cases 3 and 4) make use of the special forms of the CL kernel defined by Eqs. (26a) and (26b), so that our solution could also be tested in these limiting (and usually more difficult) situations. The reported numerical results are thought to be correct to within ± 1 in the last reported digit, except when an entry is much smaller than its neighbor entries, in which case the error may be larger.

In Tables 2–5, we report our converged numerical results for the temperature and density profiles for several values of a, the dimensionless half-distance between plates. Since a is inversely proportional to the Knudsen number, it can be readily seen that the values of a used in these tables are representative of the free molecular flow, transition flow, and slip flow regimes. The converged results were obtained by varying the values of the approximating parameters K and L in our ADO solution as follows: $20 \leqslant K \leqslant 35$ and $60 \leqslant N \leqslant 220$. The expansion of the rigid-sphere kernel expressed by Eq. (23) was truncated at a finite L, and L was varied between 50 and 155 to analyze the effect of the choice of this parameter on the convergence of our results. The expansion coefficients $\mathcal{P}^{(n)}(c', c)$ in Eq. (23) were computed as described in Appendix A of Garcia and Siewert (2007b). Integrals over the speed were evaluated by using the transformation $u = \exp(-c)$ to map the interval $[0, \infty)$ onto [0, 1] and then applying a shifted Gauss–Legendre quadrature of order M, with M being varied between 100 and 400 to analyze its effect on convergence. The Chapman–Enskog function $A(c)$ was evaluated as described by Siewert (2002), with K_s, the number of splines used in the approximation, being varied between 322 and 1282.

While the high-quality results reported in Tables 2–5 required several hours of CPU time to be generated on a single core of an Intel Core i7–920 processor running at 2.67 GHz, results accurate enough for practical applications can be obtained in much less time. For example, without any special effort to speed up the developed program, we were able to obtain numerical results with at least three significant figures of accuracy for Tables 2–5 in about 40 min of CPU time on the mentioned Intel processor.

Table 1
Defining data for various cases.

Case	$\alpha_{t,1}$	$\alpha_{n,1}$	$\alpha_{t,2}$	$\alpha_{n,2}$	δ
1	0.25	0.50	0.75	0.25	1.0
2	0.50	0.75	0.25	0.25	1.0
3	1.00	0.00	1.00	0.25	1.0
4	0.00	1.00	0.25	1.00	1.0

Table 2
Temperature and density profiles (case 1) based on the LBE: $\tau = -a + 2a\eta$.

η	$a = 0.005$		$a = 0.05$		$a = 0.5$		$a = 5.0$	
	$T(\tau)$	$N(\tau)$	$T(\tau)$	$N(\tau)$	$T(\tau)$	$N(\tau)$	$T(\tau)$	$N(\tau)$
0.0	−1.8875(−1)	−7.0931(−3)	−1.0022(−1)	−4.2851(−2)	1.9353(−1)	−2.2431(−1)	6.9417(−1)	−7.0893(−1)
0.1	−1.9272(−1)	−4.6986(−3)	−1.2061(−1)	−2.6742(−2)	1.1310(−1)	−1.5461(−1)	5.2085(−1)	−5.4404(−1)
0.2	−1.9624(−1)	−3.0153(−3)	−1.3702(−1)	−1.6280(−2)	5.7045(−2)	−1.0628(−1)	3.8232(−1)	−4.0600(−1)
0.3	−1.9964(−1)	−1.6122(−3)	−1.5255(−1)	−7.9019(−3)	5.8089(−3)	−6.3466(−2)	2.4615(−1)	−2.6989(−1)
0.4	−2.0302(−1)	−4.0167(−4)	−1.6787(−1)	−9.2812(−4)	−4.3985(−2)	−2.3858(−2)	1.1044(−1)	−1.3419(−1)
0.5	−2.0642(−1)	6.5443(−4)	−1.8336(−1)	4.9202(−3)	−9.4088(−2)	1.3295(−2)	−2.5166(−2)	1.4196(−3)
0.6	−2.0988(−1)	1.5745(−3)	−1.9934(−1)	9.7629(−3)	−1.4595(−1)	4.8107(−2)	−1.6079(−1)	1.3703(−1)
0.7	−2.1344(−1)	2.3649(−3)	−2.1618(−1)	1.3621(−2)	−2.0129(−1)	8.0223(−2)	−2.9659(−1)	2.7275(−1)
0.8	−2.1719(−1)	3.0202(−3)	−2.3443(−1)	1.6412(−2)	−2.6283(−1)	1.0867(−1)	−4.3326(−1)	4.0876(−1)
0.9	−2.2124(−1)	3.5138(−3)	−2.5520(−1)	1.7859(−2)	−3.3655(−1)	1.3110(−1)	−5.7514(−1)	5.4481(−1)
1.0	−2.2622(−1)	3.6882(−3)	−2.8387(−1)	1.6498(−2)	−4.5622(−1)	1.3723(−1)	−7.9117(−1)	6.5477(−1)

Table 3
Temperature and density profiles (case 2) based on the LBE: $\tau = -a + 2a\eta$.

η	$a = 0.005$		$a = 0.05$		$a = 0.5$		$a = 5.0$	
	$T(\tau)$	$N(\tau)$	$T(\tau)$	$N(\tau)$	$T(\tau)$	$N(\tau)$	$T(\tau)$	$N(\tau)$
0.0	4.6337(−1)	−7.6105(−3)	5.1561(−1)	−4.0723(−2)	6.4664(−1)	−2.0516(−1)	8.5907(−1)	−6.7816(−1)
0.1	4.5942(−1)	−5.5959(−3)	4.9589(−1)	−2.8654(−2)	5.7461(−1)	−1.4878(−1)	6.9387(−1)	−5.2658(−1)
0.2	4.5604(−1)	−3.9882(−3)	4.8050(−1)	−1.9881(−2)	5.2424(−1)	−1.0705(−1)	5.6019(−1)	−3.9367(−1)
0.3	4.5282(−1)	−2.5220(−3)	4.6624(−1)	−1.2178(−2)	4.7837(−1)	−6.8583(−2)	4.2857(−1)	−2.6215(−1)
0.4	4.4967(−1)	−1.1406(−3)	4.5243(−1)	−5.1079(−3)	4.3408(−1)	−3.1738(−2)	2.9735(−1)	−1.3093(−1)
0.5	4.4654(−1)	1.8408(−4)	4.3870(−1)	1.5298(−3)	3.8988(−1)	4.1160(−3)	1.6620(−1)	2.1310(−4)
0.6	4.4339(−1)	1.4705(−3)	4.2475(−1)	7.8556(−3)	3.4460(−1)	3.9269(−2)	3.5035(−2)	1.3137(−1)
0.7	4.4019(−1)	2.7337(−3)	4.1027(−1)	1.3959(−2)	2.9686(−1)	7.3853(−2)	−9.6284(−2)	2.6264(−1)
0.8	4.3686(−1)	3.9896(−3)	3.9481(−1)	1.9926(−2)	2.4448(−1)	1.0792(−1)	−2.2833(−1)	3.9433(−1)
0.9	4.3331(−1)	5.2627(−3)	3.7748(−1)	2.5893(−2)	1.8263(−1)	1.4154(−1)	−3.6450(−1)	5.2727(−1)
1.0	4.2906(−1)	6.6527(−3)	3.5401(−1)	3.2500(−2)	8.3091(−2)	1.7715(−1)	−5.6097(−1)	6.6251(−1)

Table 4
Temperature and density profiles (case 3) based on the LBE: $\tau = -a + 2a\eta$.

η	$a = 0.005$		$a = 0.05$		$a = 0.5$		$a = 5.0$	
	$T(\tau)$	$N(\tau)$	$T(\tau)$	$N(\tau)$	$T(\tau)$	$N(\tau)$	$T(\tau)$	$N(\tau)$
0.0	−2.7138(−1)	9.0504(−3)	−4.5321(−2)	3.4894(−2)	3.1319(−1)	−4.1635(−2)	7.3134(−1)	−6.0713(−1)
0.1	−2.8138(−1)	6.5181(−3)	−9.4164(−2)	2.2310(−2)	1.5815(−1)	−6.8518(−2)	4.8109(−1)	−5.3172(−1)
0.2	−2.8964(−1)	4.4890(−3)	−1.2998(−1)	1.3915(−2)	6.6520(−2)	−6.5096(−2)	3.3921(−1)	−3.9971(−1)
0.3	−2.9732(−1)	2.6790(−3)	−1.6179(−1)	7.2525(−3)	−8.2440(−3)	−5.0865(−2)	2.0492(−1)	−2.6654(−1)
0.4	−3.0470(−1)	1.0251(−3)	−1.9148(−1)	1.7334(−3)	−7.4705(−2)	−3.0852(−2)	7.1861(−2)	−1.3362(−1)
0.5	−3.1191(−1)	−5.0326(−4)	−2.1999(−1)	−2.9510(−3)	−1.3695(−1)	−7.6602(−3)	−6.0963(−2)	−8.1852(−4)
0.6	−3.1904(−1)	−1.9236(−3)	−2.4800(−1)	−7.0039(−3)	−1.9771(−1)	1.6963(−2)	−1.9377(−1)	1.3198(−1)
0.7	−3.2619(−1)	−3.2458(−3)	−2.7613(−1)	−1.0583(−2)	−2.5947(−1)	4.1523(−2)	−3.2675(−1)	2.6487(−1)
0.8	−3.3347(−1)	−4.4739(−3)	−3.0513(−1)	−1.3834(−2)	−3.2541(−1)	6.4301(−2)	−4.6059(−1)	3.9804(−1)
0.9	−3.4107(−1)	−5.6024(−3)	−3.3642(−1)	−1.6934(−2)	−4.0166(−1)	8.2449(−2)	−5.9957(−1)	5.3114(−1)
1.0	−3.4985(−1)	−6.5745(−3)	−3.7648(−1)	−2.0385(−2)	−5.2114(−1)	8.5112(−2)	−8.1162(−1)	6.3675(−1)

Table 5
Temperature and density profiles (case 4) based on the LBE: $\tau = -a + 2a\eta$.

η	$a = 0.005$		$a = 0.05$		$a = 0.5$		$a = 5.0$	
	$T(\tau)$	$N(\tau)$	$T(\tau)$	$N(\tau)$	$T(\tau)$	$N(\tau)$	$T(\tau)$	$N(\tau)$
0.0	−6.3079(−1)	−4.7272(−2)	−4.5810(−1)	−2.0936(−1)	−5.7684(−2)	−5.5322(−1)	6.4729(−1)	−8.6714(−1)
0.1	−6.3231(−1)	−3.5164(−2)	−4.6375(−1)	−1.4641(−1)	−7.7477(−2)	−3.3999(−1)	5.2777(−1)	−5.9696(−1)
0.2	−6.3375(−1)	−2.5331(−2)	−4.6942(−1)	−1.0180(−1)	−1.0739(−1)	−2.2416(−1)	3.8158(−1)	−4.4340(−1)
0.3	−6.3520(−1)	−1.6266(−2)	−4.7542(−1)	−6.3063(−2)	−1.4218(−1)	−1.3353(−1)	2.3364(−1)	−2.9465(−1)
0.4	−6.3667(−1)	−7.6376(−3)	−4.8184(−1)	−2.7621(−2)	−1.8015(−1)	−5.5179(−2)	8.5578(−2)	−1.4648(−1)
0.5	−6.3820(−1)	7.2321(−4)	−4.8872(−1)	5.7920(−3)	−2.2046(−1)	1.6706(−2)	−6.2486(−2)	1.5980(−3)
0.6	−6.3980(−1)	8.9337(−3)	−4.9617(−1)	3.8024(−2)	−2.6266(−1)	8.5709(−2)	−2.1056(−1)	1.4968(−1)
0.7	−6.4149(−1)	1.7098(−2)	−5.0431(−1)	6.9804(−2)	−3.0661(−1)	1.5480(−1)	−3.5869(−1)	2.9784(−1)
0.8	−6.4331(−1)	2.5337(−2)	−5.1340(−1)	1.0197(−1)	−3.5263(−1)	2.2753(−1)	−5.0703(−1)	4.4643(−1)
0.9	−6.4534(−1)	3.3854(−2)	−5.2403(−1)	1.3596(−1)	−4.0209(−1)	3.1060(−1)	−6.5606(−1)	5.9773(−1)
1.0	−6.4796(−1)	4.3509(−2)	−5.3918(−1)	1.7824(−1)	−4.6709(−1)	4.4056(−1)	−8.1617(−1)	8.0584(−1)

Table 6
Temperature profiles $T(-a+2a\eta)$ for case 1 with $a=0.01$.

η	BGK	S	GJ	MRS	CES	LBE
0.0	−1.9241(−1)	−1.9353(−1)	−1.9472(−1)	−1.8915(−1)	−1.7064(−1)	−1.7376(−1)
0.1	−1.9580(−1)	−1.9674(−1)	−1.9791(−1)	−1.9247(−1)	−1.7760(−1)	−1.8039(−1)
0.2	−1.9887(−1)	−1.9962(−1)	−2.0072(−1)	−1.9565(−1)	−1.8367(−1)	−1.8613(−1)
0.3	−2.0186(−1)	−2.0243(−1)	−2.0344(−1)	−1.9882(−1)	−1.8953(−1)	−1.9165(−1)
0.4	−2.0484(−1)	−2.0522(−1)	−2.0611(−1)	−2.0203(−1)	−1.9534(−1)	−1.9713(−1)
0.5	−2.0782(−1)	−2.0802(−1)	−2.0879(−1)	−2.0531(−1)	−2.0120(−1)	−2.0265(−1)
0.6	−2.1086(−1)	−2.1087(−1)	−2.1150(−1)	−2.0869(−1)	−2.0719(−1)	−2.0828(−1)
0.7	−2.1397(−1)	−2.1379(−1)	−2.1427(−1)	−2.1219(−1)	−2.1340(−1)	−2.1413(−1)
0.8	−2.1721(−1)	−2.1685(−1)	−2.1715(−1)	−2.1588(−1)	−2.1997(−1)	−2.2032(−1)
0.9	−2.2067(−1)	−2.2012(−1)	−2.2023(−1)	−2.1988(−1)	−2.2717(−1)	−2.2711(−1)
1.0	−2.2479(−1)	−2.2406(−1)	−2.2391(−1)	−2.2472(−1)	−2.3624(−1)	−2.3569(−1)

Table 7
Density profiles $N(-a+2a\eta)$ for case 1 with $a=0.01$.

η	BGK	S	GJ	MRS	CES	LBE
0.0	−1.0245(−2)	−9.3118(−3)	−8.0905(−3)	−1.3747(−2)	−1.2787(−2)	−1.2343(−2)
0.1	−7.1535(−3)	−6.4059(−3)	−5.6705(−3)	−9.1227(−3)	−8.3784(−3)	−7.9855(−3)
0.2	−4.8391(−3)	−4.2779(−3)	−3.8665(−3)	−5.8350(−3)	−5.3325(−3)	−5.0090(−3)
0.3	−2.8247(−3)	−2.4502(−3)	−2.2936(−3)	−3.0857(−3)	−2.8171(−3)	−2.5716(−3)
0.4	−1.0120(−3)	−8.2435(−4)	−8.7128(−4)	−7.1386(−4)	−6.6292(−4)	−5.0411(−4)
0.5	6.4347(−4)	6.4408(−4)	4.3824(−4)	1.3484(−3)	1.2044(−3)	1.2664(−3)
0.6	2.1651(−3)	1.9785(−3)	1.6566(−3)	3.1307(−3)	2.8217(−3)	2.7743(−3)
0.7	3.5645(−3)	3.1907(−3)	2.7973(−3)	4.6379(−3)	4.2035(−3)	4.0303(−3)
0.8	4.8435(−3)	4.2823(−3)	3.8692(−3)	5.8472(−3)	5.3434(−3)	5.0220(−3)
0.9	5.9884(−3)	5.2397(−3)	4.8769(−3)	6.6817(−3)	6.1985(−3)	5.6937(−3)
1.0	6.9006(−3)	5.9642(−3)	5.8144(−3)	6.7193(−3)	6.5050(−3)	5.7262(−3)

Table 8
Temperature profiles $T(-a+2a\eta)$ for case 2 with $a=0.1$.

η	BGK	S	GJ	MRS	CES	LBE
0.0	5.1308(−1)	5.0873(−1)	5.0364(−1)	5.2468(−1)	5.5188(−1)	5.4608(−1)
0.1	4.9396(−1)	4.9123(−1)	4.8699(−1)	5.0368(−1)	5.1999(−1)	5.1596(−1)
0.2	4.7823(−1)	4.7714(−1)	4.7387(−1)	4.8584(−1)	4.9624(−1)	4.9328(−1)
0.3	4.6342(−1)	4.6399(−1)	4.6172(−1)	4.6883(−1)	4.7454(−1)	4.7243(−1)
0.4	4.4900(−1)	4.5122(−1)	4.4999(−1)	4.5213(−1)	4.5361(−1)	4.5227(−1)
0.5	4.3466(−1)	4.3854(−1)	4.3836(−1)	4.3547(−1)	4.3275(−1)	4.3218(−1)
0.6	4.2017(−1)	4.2570(−1)	4.2662(−1)	4.1859(−1)	4.1139(−1)	4.1168(−1)
0.7	4.0527(−1)	4.1246(−1)	4.1449(−1)	4.0126(−1)	3.8892(−1)	3.9023(−1)
0.8	3.8963(−1)	3.9847(−1)	4.0164(−1)	3.8311(−1)	3.6446(−1)	3.6705(−1)
0.9	3.7257(−1)	3.8303(−1)	3.8740(−1)	3.6341(−1)	3.3620(−1)	3.4056(−1)
1.0	3.5091(−1)	3.6297(−1)	3.6863(−1)	3.3883(−1)	2.9534(−1)	3.0277(−1)

Table 9
Density profiles $N(-a+2a\eta)$ for case 2 with $a=0.1$.

η	BGK	S	GJ	MRS	CES	LBE
0.0	−5.7052(−2)	−4.8636(−2)	−4.1138(−2)	−7.4084(−2)	−6.7476(−2)	−6.6730(−2)
0.1	−4.1375(−2)	−3.4604(−2)	−2.9001(−2)	−5.4119(−2)	−4.6711(−2)	−4.6772(−2)
0.2	−2.9377(−2)	−2.4284(−2)	−2.0310(−2)	−3.8448(−2)	−3.2277(−2)	−3.2485(−2)
0.3	−1.8559(−2)	−1.5155(−2)	−1.2703(−2)	−2.4187(−2)	−1.9790(−2)	−1.9932(−2)
0.4	−8.4136(−3)	−6.7036(−3)	−5.6937(−3)	−1.0762(−2)	−8.4012(−3)	−8.3828(−3)
0.5	1.3060(−3)	1.3176(−3)	9.5650(−4)	2.0917(−3)	2.2815(−3)	2.4842(−3)
0.6	1.0758(−2)	9.0693(−3)	7.4097(−3)	1.4527(−2)	1.2502(−2)	1.2854(−2)
0.7	2.0076(−2)	1.6685(−2)	1.3809(−2)	2.6649(−2)	2.2457(−2)	2.2854(−2)
0.8	2.9410(−2)	2.4317(−2)	2.0327(−2)	3.8550(−2)	3.2363(−2)	3.2606(−2)
0.9	3.9015(−2)	3.2223(−2)	2.7272(−2)	5.0348(−2)	4.2599(−2)	4.2301(−2)
1.0	5.0084(−2)	4.1605(−2)	3.6082(−2)	6.2578(−2)	5.5058(−2)	5.3001(−2)

Since our general approach based on the LBE for rigid-sphere interactions can be easily adapted to include various kinetic models [see details in Garcia and Siewert (2006)], we report in Tables 6–13 a comparison between temperature and density profiles obtained from the LBE and five kinetic models: the BGK (Bhatnagar et al., 1954), S (Shakhov, 1968), GJ (Gross and Jackson, 1959), MRS (Garcia and Siewert, 2006), and CES (Barichello and Siewert, 2003) models.

In Table 14, we report converged numerical results obtained from the LBE and the considered kinetic models for the normalized heat-flow perturbation

$$q = Q/Q_{\rm fm}, \tag{41}$$

where Q is given by Eq. (39) and $Q_{\rm fm}$ is the heat-flow perturbation for free molecular flow. As shown in Appendix A, $Q_{\rm fm}$ is given by

Table 10
Temperature profiles $T(-a+2a\eta)$ for case 3 with $a = 1.0$.

η	BGK	S	GJ	MRS	CES	LBE
0.0	3.5897(−1)	3.0354(−1)	2.4775(−1)	4.4730(−1)	4.1077(−1)	4.2458(−1)
0.1	2.1489(−1)	1.6430(−1)	1.1715(−1)	2.8939(−1)	1.9681(−1)	2.2315(−1)
0.2	1.1826(−1)	7.6611(−2)	3.6387(−2)	1.8147(−1)	8.9481(−2)	1.1381(−1)
0.3	3.4405(−2)	2.7957(−3)	−3.0751(−2)	8.6033(−2)	7.0200(−3)	2.6469(−2)
0.4	−4.3170(−2)	−6.4194(−2)	−9.1060(−2)	−3.5822(−3)	−6.4752(−2)	−5.0917(−2)
0.5	−1.1756(−1)	−1.2772(−1)	−1.4782(−1)	−9.0383(−2)	−1.3184(−1)	−1.2370(−1)
0.6	−1.9084(−1)	−1.9005(−1)	−2.0326(−1)	−1.7632(−1)	−1.9777(−1)	−1.9523(−1)
0.7	−2.6493(−1)	−2.5326(−1)	−2.5945(−1)	−2.6316(−1)	−2.6567(−1)	−2.6849(−1)
0.8	−3.4231(−1)	−3.2003(−1)	−3.1904(−1)	−3.5326(−1)	−3.3995(−1)	−3.4750(−1)
0.9	−4.2773(−1)	−3.9554(−1)	−3.8710(−1)	−4.5140(−1)	−4.3039(−1)	−4.4058(−1)
1.0	−5.4595(−1)	−5.0669(−1)	−4.9003(−1)	−5.8341(−1)	−5.9424(−1)	−5.9649(−1)

Table 11
Density profiles $N(-a+2a\eta)$ for case 3 with $a = 1.0$.

η	BGK	S	GJ	MRS	CES	LBE
0.0	−1.6948(−1)	−9.7674(−2)	−8.6629(−3)	−3.2479(−1)	−1.3420(−1)	−1.6833(−1)
0.1	−1.5220(−1)	−9.6192(−2)	−2.4139(−2)	−2.7482(−1)	−1.6333(−1)	−1.8676(−1)
0.2	−1.2187(−1)	−8.0530(−2)	−2.7182(−2)	−2.1057(−1)	−1.4109(−1)	−1.5609(−1)
0.3	−8.5731(−2)	−5.8560(−2)	−2.3923(−2)	−1.4226(−1)	−1.0283(−1)	−1.1134(−1)
0.4	−4.6311(−2)	−3.3023(−2)	−1.6769(−2)	−7.2289(−2)	−5.7212(−2)	−6.0395(−2)
0.5	−4.9880(−3)	−5.4192(−3)	−7.1529(−3)	−1.5772(−3)	−8.1546(−3)	−6.6283(−3)
0.6	3.7302(−2)	2.3235(−2)	3.9024(−3)	6.9388(−2)	4.2051(−2)	4.8066(−2)
0.7	7.9771(−2)	5.2084(−2)	1.5550(−2)	1.4020(−1)	9.1570(−2)	1.0217(−1)
0.8	1.2156(−1)	8.0190(−2)	2.6967(−2)	2.1024(−1)	1.3815(−1)	1.5368(−1)
0.9	1.6136(−1)	1.0614(−1)	3.7213(−2)	2.7781(−1)	1.7766(−1)	1.9844(−1)
1.0	1.9585(−1)	1.2633(−1)	4.5837(−2)	3.3305(−1)	1.9575(−1)	2.1776(−1)

Table 12
Temperature profiles $T(-a+2a\eta)$ for case 4 with $a = 10.0$.

η	BGK	S	GJ	MRS	CES	LBE
0.0	7.3797(−1)	6.4347(−1)	5.0219(−1)	8.3236(−1)	7.9063(−1)	7.9732(−1)
0.1	5.8790(−1)	5.1141(−1)	4.0103(−1)	6.6079(−1)	6.4134(−1)	6.4463(−1)
0.2	4.3091(−1)	3.6994(−1)	2.8300(−1)	4.8869(−1)	4.7168(−1)	4.7454(−1)
0.3	2.7242(−1)	2.2641(−1)	1.5992(−1)	3.1668(−1)	3.0188(−1)	3.0439(−1)
0.4	1.1354(−1)	8.2355(−2)	3.5376(−2)	1.4469(−1)	1.3207(−1)	1.3424(−1)
0.5	−4.5469(−2)	−6.1867(−2)	−8.9642(−2)	−2.7305(−2)	−3.7733(−2)	−3.5904(−2)
0.6	−2.0455(−1)	−2.0619(−1)	−2.1486(−1)	−1.9933(−1)	−2.0754(−1)	−2.0605(−1)
0.7	−3.6373(−1)	−3.5065(−1)	−3.4023(−1)	−3.7143(−1)	−3.7735(−1)	−3.7619(−1)
0.8	−5.2318(−1)	−4.9548(−1)	−4.6590(−1)	−5.4381(−1)	−5.4715(−1)	−5.4636(−1)
0.9	−6.8360(−1)	−6.4162(−1)	−5.9266(−1)	−7.1720(−1)	−7.1694(−1)	−7.1672(−1)
1.0	−8.5815(−1)	−8.0699(−1)	−7.4393(−1)	−9.0170(−1)	−8.9293(−1)	−8.9436(−1)

Table 13
Density profiles $N(-a+2a\eta)$ for case 4 with $a = 10.0$.

η	BGK	S	GJ	MRS	CES	LBE
0.0	−8.9091(−1)	−8.5157(−1)	−8.1727(−1)	−9.1448(−1)	−9.2543(−1)	−9.2411(−1)
0.1	−6.4201(−1)	−5.8504(−1)	−5.1708(−1)	−6.9011(−1)	−6.7893(−1)	−6.8061(−1)
0.2	−4.7728(−1)	−4.3302(−1)	−3.7637(−1)	−5.1618(−1)	−5.0891(−1)	−5.0999(−1)
0.3	−3.1723(−1)	−2.8738(−1)	−2.4797(−1)	−3.4387(−1)	−3.3911(−1)	−3.3983(−1)
0.4	−1.5796(−1)	−1.4279(−1)	−1.2195(−1)	−1.7181(−1)	−1.6930(−1)	−1.6969(−1)
0.5	1.1664(−3)	1.5871(−3)	3.5423(−3)	1.9911(−4)	5.0712(−4)	4.5855(−4)
0.6	1.6030(−1)	1.4599(−1)	1.2900(−1)	1.7223(−1)	1.7031(−1)	1.7060(−1)
0.7	3.1959(−1)	2.9059(−1)	2.5479(−1)	3.4435(−1)	3.4012(−1)	3.4075(−1)
0.8	4.7947(−1)	4.3600(−1)	3.8188(−1)	5.1681(−1)	5.0993(−1)	5.1091(−1)
0.9	6.4222(−1)	5.8532(−1)	5.1520(−1)	6.9079(−1)	6.7982(−1)	6.8144(−1)
1.0	8.5031(−1)	7.9633(−1)	7.3043(−1)	8.9484(−1)	8.8592(−1)	8.8797(−1)

$$Q_{\text{fm}} = \pi^{-1/2}\delta\left(\frac{\widehat{\alpha}_1\widehat{\alpha}_2}{\widehat{\alpha}_1+\widehat{\alpha}_2-\widehat{\alpha}_1\widehat{\alpha}_2}+\frac{\alpha_{n,1}\alpha_{n,2}}{\alpha_{n,1}+\alpha_{n,2}-\alpha_{n,1}\alpha_{n,2}}\right), \tag{42a}$$

when $\widehat{\alpha}_1+\widehat{\alpha}_2\neq 0$ and $\alpha_{n,1}+\alpha_{n,2}\neq 0$,

$$Q_{\text{fm}} = \pi^{-1/2}\delta\,\frac{\widehat{\alpha}_1\widehat{\alpha}_2}{\widehat{\alpha}_1+\widehat{\alpha}_2-\widehat{\alpha}_1\widehat{\alpha}_2}, \tag{42b}$$

when $\widehat{\alpha}_1+\widehat{\alpha}_2\neq 0$ and $\alpha_{n,1}+\alpha_{n,2}=0$, and

Table 14
The normalized heat-flow perturbation $q = Q/Q_{fm}$.

Case	$2a$	BGK	S	GJ	MRS	CES	LBE
1	2.0(−3)	1.00020	1.00029	1.00009	1.00103	1.00038	1.00049
	2.0(−2)	1.00167	1.00261	1.00087	1.00839	1.00299	1.00402
	2.0(−1)	9.98910(−1)	1.00838	1.00241	1.01249	9.86209(−1)	9.92887(−1)
	2.0	7.96819(−1)	8.61368(−1)	8.99611(−1)	7.25869(−1)	7.12211(−1)	7.14537(−1)
	2.0(1)	2.61923(−1)	3.47531(−1)	4.41597(−1)	1.91174(−1)	1.94519(−1)	1.94621(−1)
2	2.0(−3)	9.99527(−1)	9.99618(−1)	9.99660(−1)	9.99462(−1)	9.98963(−1)	9.99058(−1)
	2.0(−2)	9.95415(−1)	9.96328(−1)	9.96752(−1)	9.94674(−1)	9.90539(−1)	9.91412(−1)
	2.0(−1)	9.59862(−1)	9.68428(−1)	9.73038(−1)	9.49847(−1)	9.31759(−1)	9.36287(−1)
	2.0	7.46857(−1)	8.02065(−1)	8.41233(−1)	6.78491(−1)	6.78424(−1)	6.78504(−1)
	2.0(1)	2.59750(−1)	3.41505(−1)	4.31239(−1)	1.90548(−1)	1.94707(−1)	1.94655(−1)
3	2.0(−3)	9.98823(−1)	9.98972(−1)	9.98881(−1)	9.99303(−1)	9.95873(−1)	9.96414(−1)
	2.0(−2)	9.88400(−1)	9.89871(−1)	9.89247(−1)	9.91226(−1)	9.62809(−1)	9.67618(−1)
	2.0(−1)	8.96303(−1)	9.08554(−1)	9.10971(−1)	8.88830(−1)	7.62719(−1)	7.85222(−1)
	2.0	5.28936(−1)	5.74667(−1)	5.97221(−1)	4.86434(−1)	4.21802(−1)	4.29341(−1)
	2.0(1)	1.62227(−1)	2.14606(−1)	2.67137(−1)	1.19986(−1)	1.19668(−1)	1.20136(−1)
4	2.0(−3)	9.99269(−1)	9.99419(−1)	9.99325(−1)	9.99764(−1)	9.98222(−1)	9.98481(−1)
	2.0(−2)	9.92952(−1)	9.94437(−1)	9.93669(−1)	9.96643(−1)	9.83901(−1)	9.86270(−1)
	2.0(−1)	9.37125(−1)	9.50526(−1)	9.50437(−1)	9.36426(−1)	8.81435(−1)	8.93036(−1)
	2.0	6.34222(−1)	7.01123(−1)	7.40937(−1)	5.65949(−1)	5.42482(−1)	5.47518(−1)
	2.0(1)	1.76169(−1)	2.39701(−1)	3.12113(−1)	1.26279(−1)	1.27844(−1)	1.28098(−1)

Table 15
The heat-flow perturbation Q and the normalized heat-flow perturbation $q = Q/Q_{fm}$ based on the LBE and CL boundary conditions with $\alpha_{t,1} = \alpha_{n,2} = 1 - \epsilon$ and $\alpha_{t,2} = \alpha_{n,1} = \epsilon$.

a	$\epsilon = 0.05$		$\epsilon = 0.10$		$\epsilon = 0.15$	
	Q	q	Q	q	Q	q
1.0(−3)	8.56455(−2)	1.03025	1.64813(−1)	1.01243	2.39528(−1)	1.00636
1.0(−2)	1.06255(−1)	1.27817	1.81327(−1)	1.11387	2.51818(−1)	1.05799
1.0(−1)	2.19909(−1)	2.64534	2.68479(−1)	1.64924	3.12831(−1)	1.31433
1.0	2.48205(−1)	2.98573	2.59046(−1)	1.59129	2.68456(−1)	1.12790
2.0	1.92550(−1)	2.31624	1.98800(−1)	1.22121	2.04128(−1)	8.57627(−1)
4.0	1.32492(−1)	1.59378	1.35419(−1)	8.31866(−1)	1.37869(−1)	5.79245(−1)
8.0	8.15922(−2)	9.81496(−1)	8.26930(−2)	5.07975(−1)	8.36001(−2)	3.51238(−1)
1.6(+1)	4.61405(−2)	5.55037(−1)	4.64904(−2)	2.85586(−1)	4.67758(−2)	1.96524(−1)
3.2(+1)	2.46873(−2)	2.96970(−1)	2.47871(−2)	1.52265(−1)	2.48680(−2)	1.04481(−1)

Table 16
The normalized heat-flow perturbation $q = Q/Q_{fm}$: alternative spatial variable.

Case	$2a_t$	BGK	S	GJ	MRS	CES	LBE
1	2.0(−3)	1.00020	1.00043	1.00020	1.00069	1.00026	1.00033
	2.0(−2)	1.00167	1.00369	1.00173	1.00598	1.00224	1.00295
	2.0(−1)	9.98910(−1)	1.00456	9.93834(−1)	1.01855	9.96885(−1)	1.00216
	2.0	7.96819(−1)	7.91460(−1)	7.78622(−1)	8.15880(−1)	7.87806(−1)	7.92007(−1)
	2.0(1)	2.61923(−1)	2.61922(−1)	2.60962(−1)	2.63437(−1)	2.62405(−1)	2.62590(−1)
2	2.0(−3)	9.99527(−1)	9.99430(−1)	9.99241(−1)	9.99644(−1)	9.99291(−1)	9.99356(−1)
	2.0(−2)	9.95415(−1)	9.94575(−1)	9.92985(−1)	9.96460(−1)	9.93402(−1)	9.94012(−1)
	2.0(−1)	9.59862(−1)	9.54968(−1)	9.46996(−1)	9.66007(−1)	9.49683(−1)	9.53492(−1)
	2.0	7.46857(−1)	7.41674(−1)	7.35455(−1)	7.55742(−1)	7.45069(−1)	7.46222(−1)
	2.0(1)	2.59750(−1)	2.59749(−1)	2.59405(−1)	2.60285(−1)	2.60923(−1)	2.60828(−1)
3	2.0(−3)	9.98823(−1)	9.98462(−1)	9.97500(−1)	9.99546(−1)	9.97178(−1)	9.97547(−1)
	2.0(−2)	9.88401(−1)	9.84888(−1)	9.76693(−1)	9.94542(−1)	9.73958(−1)	9.77357(−1)
	2.0(−1)	8.96303(−1)	8.70875(−1)	8.31056(−1)	9.25816(−1)	8.17184(−1)	8.36390(−1)
	2.0	5.28936(−1)	5.05344(−1)	4.73001(−1)	5.70754(−1)	4.69449(−1)	4.81455(−1)
	2.0(1)	1.62227(−1)	1.62225(−1)	1.59974(−1)	1.65606(−1)	1.60544(−1)	1.61387(−1)
4	2.0(−3)	9.99269(−1)	9.99132(−1)	9.98499(−1)	9.99847(−1)	9.98784(−1)	9.98958(−1)
	2.0(−2)	9.92952(−1)	9.91777(−1)	9.86545(−1)	9.97993(−1)	9.88763(−1)	9.90425(−1)
	2.0(−1)	9.37125(−1)	9.28535(−1)	9.05063(−1)	9.60953(−1)	9.12870(−1)	9.22675(−1)
	2.0	6.34222(−1)	6.26705(−1)	6.10014(−1)	6.58047(−1)	6.15385(−1)	6.22641(−1)
	2.0(1)	1.76169(−1)	1.76169(−1)	1.75086(−1)	1.77814(−1)	1.75611(−1)	1.76091(−1)

$$Q_{fm} = \pi^{-1/2}\delta\,\frac{\alpha_{n,1}\alpha_{n,2}}{\alpha_{n,1}+\alpha_{n,2}-\alpha_{n,1}\alpha_{n,2}}, \tag{42c}$$

when $\widehat{\alpha}_1 + \widehat{\alpha}_2 = 0$ and $\alpha_{n,1} + \alpha_{n,2} \neq 0$.

We note from the results for Case 1 in Table 14 that here, differently than for the case based on the Maxwell boundary conditions (Siewert, 2003a), the normalized heat-flow perturbation q can be larger than one. This effect shows up when the gas interaction with one of the walls occurs preferentially in the tangential direction, while for the other wall the interaction is dominated by the normal component of the CL kernel. In fact, it can be noted from Eq. (42a) that if one of the α_n coefficients, say $\alpha_{n,1}$, approaches zero, and the value of $\widehat{\alpha}$ at the opposite wall ($\widehat{\alpha}_2$) also approaches zero, then $q \to \infty$, since $Q_{\rm fm} \to 0$ but Q does not $\to 0$. Three cases where q can be seen to reach a maximum value well above unity are shown in Table 15.

5. Concluding remarks

In this work, we have used the ADO method to solve the problem of heat transfer for a single-species gas described by the LBE for rigid-sphere interactions and CL boundary conditions. On comparing the results reported in Tables 6–14 for the LBE and the five kinetic models, we can see that the results of the CES model are the ones that agree better with the LBE results. We can also see that while all of the models behave reasonably well for small values of the parameter a, say $a \leqslant 0.1$, only the CES and MRS results are consistent with the LBE results for all of the considered values of a in these tables.

At this point, it is important to emphasize that the comparison results of Tables 6–14 were obtained from a formulation based on the dimensionless spatial variable defined by Eq. (17). Since this choice of spatial variable preserves the physical distance between the plates (when measured, for example, in cm) for the LBE and the models, we believe a comparison based on the results of Tables 6–14 is, among many possibilities, the one that makes more sense, even though the numerical results of the BGK, S, and GJ models are found not to agree always well with those of the LBE. As observed in a previous work (Garcia and Siewert, 2010), improved agreement between the results of these three models and the LBE results can be obtained when the problems being solved are formulated in terms of alternative dimensionless spatial variables. We have confirmed that this is also the case for the problem solved in this work, when a spatial variable based on thermal conductivity (Siewert, 2003a),

$$\tau_t = z\varepsilon_0/\varepsilon_t, \tag{43}$$

where ε_t is defined by Eq. (40), is used. As an illustration of this fact, we show some numerical results for the normalized heat-flow perturbation in Table 16, where the dimensionless half-distance between plates a_t is given by

$$a_t = z_0\varepsilon_0/\varepsilon_t. \tag{44}$$

It should be noted that the physical distance between plates is not preserved when this choice of spatial variable is made and the value of a_t is taken to be the same for the LBE and the models, as done in Table 16. The reason is that the numerical value of ε_t is different for the LBE and the models (Garcia and Siewert, 2006), except the CES model which shares with the LBE the same (exact) value of ε_t.

Finally, a comment regarding the way we obtained the numerical results reported in Tables 14 and 16 for Case 4 (the most difficult of all cases) is important. To develop confidence on the accuracy of these six-digit results, we applied Richardson extrapolation (Dahlquist and Björk, 1974) in the parameter N to four sequences of results generated with different values of N for two values of the parameter K. More specifically, we verified that the last terms of the accelerated sequences obtained with $N = 2^n, n = 1, 2, \ldots, 8,$ and $N = 3 \cdot 2^n, n = 1, 2, \ldots, 7,$ for both $K = 30$ and $K = 35$ were all in agreement with each other.

Appendix A. The heat-flow perturbation for the case of free molecular flow

In the regime of free molecular flow, our problem is formulated by Eq. (20) in the absence of collisions, i.e.

$$c\mu\frac{\partial}{\partial\tau}\psi(\tau, c, \mu) = 0, \tag{A.1}$$

subject to the CL boundary conditions expressed by Eq. (21). The solution of this problem can be formally written as

$$\psi(\tau, c, \mu) = F_+(c, \mu) \tag{A.2a}$$

and

$$\psi(\tau, c, -\mu) = F_-(c, \mu), \tag{A.2b}$$

for $\tau \in [-a, a], c \in [0, \infty)$, and $\mu \in (0, 1]$. Here, $F_+(c, \mu)$ and $F_-(c, \mu)$ are solutions to the following system of coupled integral equations:

$$F_+(c, \mu) - \int_0^\infty \int_0^1 F_-(c', \mu')R_{0,1}(c', -\mu' : c, \mu)c'^2 d\mu' dc' = -[\alpha_{n,1}f(c, \mu) + \widehat{\alpha}_1 g(c, \mu)]\delta \tag{A.3a}$$

and

$$F_-(c, \mu) - \int_0^\infty \int_0^1 F_+(c', \mu')R_{0,2}(c', \mu' : c, -\mu)c'^2 d\mu' dc' = [\alpha_{n,2}f(c, \mu) + \widehat{\alpha}_2 g(c, \mu)]\delta, \tag{A.3b}$$

for $c \in [0, \infty)$ and $\mu \in (0, 1]$.

Denoting as $Q_{\rm fm}$ the heat-flow perturbation for the case of free molecular flow, we can use Eqs. (19c) and (A.2) to write

$$Q_{\rm fm} = Q_+ - Q_-, \tag{A.4}$$

where

$$Q_\pm = \frac{2}{\pi^{1/2}}\int_0^\infty \int_0^1 e^{-c^2}F_\pm(c, \mu)(c^2 - 5/2)c^3\mu d\mu dc. \tag{A.5}$$

If we now multiply Eq. (A.3) by

$$2\pi^{-1/2}e^{-c^2}(c^2 - 5/2)c^3\mu$$

and integrate the resulting equations over $c \in [0, \infty)$ and $\mu \in (0, 1]$, we find that $Q_{\rm fm}$ can be expressed as

$$Q_{\rm fm} = \frac{2}{\pi^{1/2}}\int_0^\infty \int_0^1 e^{-c^2}\Phi_-(c, \mu)[\alpha_{n,1}f(c, \mu) + \widehat{\alpha}_1 g(c, \mu)]c^3\mu d\mu dc \tag{A.6a}$$

or

$$Q_{\rm fm} = \frac{2}{\pi^{1/2}}\int_0^\infty \int_0^1 e^{-c^2}\Phi_+(c, \mu)[\alpha_{n,2}f(c, \mu) + \widehat{\alpha}_2 g(c, \mu)]c^3\mu d\mu dc, \tag{A.6b}$$

where

$$\Phi_+(c, \mu) = -F_+(c, \mu) - (c^2 - 5/2)\delta \tag{A.7a}$$

and

$$\Phi_-(c, \mu) = F_-(c, \mu) - (c^2 - 5/2)\delta. \tag{A.7b}$$

Continuing with our derivation, we now use Eq. (A.7) in Eq. (A.3) to find

$$\Phi_+(c, \mu) + \int_0^\infty \int_0^1 \Phi_-(c', \mu')R_{0,1}(c', -\mu' : c, \mu)c'^2 d\mu' dc' = -2(c^2 - 5/2)\delta \tag{A.8a}$$

and

$$\Phi_-(c, \mu) + \int_0^\infty \int_0^1 \Phi_+(c', \mu')R_{0,2}(c', \mu' : c, -\mu)c'^2 d\mu' dc' = -2(c^2 - 5/2)\delta, \tag{A.8b}$$

for $c \in [0, \infty)$ and $\mu \in (0, 1]$. Multiplying Eq. (A.8) by

$$e^{-c^2}f(c, \mu)c^3\mu$$

and integrating the resulting equations over $c \in [0, \infty)$ and $\mu \in (0, 1]$, we find

$$f_+ + (1 - \alpha_{n,1})f_- = \delta/2 \tag{A.9a}$$

and

$$f_- + (1 - \alpha_{n,2})f_+ = \delta/2, \tag{A.9b}$$

where

$$f_\pm = \int_0^\infty \int_0^1 e^{-c^2}\Phi_\pm(c,\mu)f(c,\mu)c^3\mu d\mu dc. \tag{A.10}$$

Similarly, multiplying Eq. (A.8) by

$$e^{-c^2}g(c,\mu)c^3\mu$$

and integrating the resulting equations over $c \in [0, \infty)$ and $\mu \in (0, 1]$, we find

$$g_+ + (1 - \widehat{\alpha}_1)g_- = \delta/2 \tag{A.11a}$$

and

$$g_- + (1 - \widehat{\alpha}_2)g_+ = \delta/2, \tag{A.11b}$$

where

$$g_\pm = \int_0^\infty \int_0^1 e^{-c^2}\Phi_\pm(c,\mu)g(c,\mu)c^3\mu d\mu dc. \tag{A.12}$$

We can now use the definitions expressed by Eqs. (A.10) and (A.12) to rewrite Eq. (A.6) as

$$Q_{\rm fm} = 2\pi^{-1/2}(\alpha_{n,1}f_- + \widehat{\alpha}_1 g_-) \tag{A.13a}$$

and

$$Q_{\rm fm} = 2\pi^{-1/2}(\alpha_{n,2}f_+ + \widehat{\alpha}_2 g_+). \tag{A.13b}$$

At this point, we can solve Eq. (A.9) for f_+ and f_- and Eq. (A.11) for g_+ and g_-, and use the expressions obtained for these quantities in either of Eq. (A.13) to determine $Q_{\rm fm}$. This way, we conclude that the heat-flow perturbation for free molecular flow and the CL boundary conditions expressed by Eq. (21) is given by

$$Q_{\rm fm} = \pi^{-1/2}\delta\left(\frac{\widehat{\alpha}_1\widehat{\alpha}_2}{\widehat{\alpha}_1 + \widehat{\alpha}_2 - \widehat{\alpha}_1\widehat{\alpha}_2} + \frac{\alpha_{n,1}\alpha_{n,2}}{\alpha_{n,1} + \alpha_{n,2} - \alpha_{n,1}\alpha_{n,2}}\right), \tag{A.14}$$

provided $\widehat{\alpha}_1 + \widehat{\alpha}_2 \neq 0$ and $\alpha_{n,1} + \alpha_{n,2} \neq 0$. For $\widehat{\alpha}_1 + \widehat{\alpha}_2 \neq 0$ and $\alpha_{n,1} + \alpha_{n,2} = 0$, we find

$$Q_{\rm fm} = \pi^{-1/2}\delta\frac{\widehat{\alpha}_1\widehat{\alpha}_2}{\widehat{\alpha}_1 + \widehat{\alpha}_2 - \widehat{\alpha}_1\widehat{\alpha}_2}, \tag{A.15}$$

while for $\widehat{\alpha}_1 + \widehat{\alpha}_2 = 0$ and $\alpha_{n,1} + \alpha_{n,2} \neq 0$ we find

$$Q_{\rm fm} = \pi^{-1/2}\delta\frac{\alpha_{n,1}\alpha_{n,2}}{\alpha_{n,1} + \alpha_{n,2} - \alpha_{n,1}\alpha_{n,2}}. \tag{A.16}$$

Finally, we list some integrals that we have been able to perform analytically and that were instrumental for deriving the result expressed by Eqs. (A.14)–(A.16):

$$\int_0^\infty \int_0^1 e^{-c^2}R_0(c', \mp\mu' : c, \pm\mu)\mu d\mu c^3(c^2 - 5/2)dc$$
$$= c'\mu' e^{-c'^2}[(c'^2 - 5/2) + \alpha_n f(c', \mu') + \widehat{\alpha} g(c', \mu')], \tag{A.17a}$$

$$\int_0^\infty \int_0^1 e^{-c^2}R_0(c', \mp\mu' : c, \pm\mu)f(c,\mu)\mu d\mu c^3 dc$$
$$= (1 - \alpha_n)c'\mu' e^{-c'^2}f(c', \mu'), \tag{A.17b}$$

and

$$\int_0^\infty \int_0^1 e^{-c^2}R_0(c', \mp\mu' : c, \pm\mu)g(c,\mu)\mu d\mu c^3 dc$$
$$= (1 - \widehat{\alpha})c'\mu' e^{-c'^2}g(c', \mu'). \tag{A.17c}$$

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