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AN EXACT SOLUTION OF THE MILNE PROBLEM IN THE PICKET-FENCE MODEL

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Abstract—The results of numerical calculations for several of the quantities of interest in a non-gray radiative transfer problem are presented. The model considered is the "uniform" or "random" picket-fence model, with the assumption of local thermodynamic equilibrium. The extrapolation distance, the integrated black-body radiation intensity and the exit angular intensities for the classical Milne problem are explicitly calculated for several parametrical representations of the model.

I. INTRODUCTION

THE CONCEPT of the picket-fence model as an approximation to non-gray radiative transfer for astrophysical applications was discussed by CHANDRASEKHAR[1] in a paper published in 1935. In this model the radiation absorption coefficient is represented by a set of discrete values over the entire frequency spectrum, and thus to some approximation the absorption by resonance lines is included in the basic formulation of the equations of transfer. The defining equations considered by Chandrasekhar are somewhat more general than those considered here in that the effect of radiation scattering was included. Our equations, in fact, correspond to Chandrasekhar's case $\varepsilon = 1$.

The Milne problem for the picket-fence model was also investigated by CHANDRASEKHAR,[1] and a solution was constructed within the limitations of the Eddington approximation. Various other approximate and numerical solutions have been developed and have been summarized by GINGERICH[2] who considered numerical solutions.

With the emphasis placed more explicitly on engineering applications, two papers by LICK[3] and GRIEF[4] have also discussed the picket-fence model in connection with studies of combined radiative and conductive heat transfer; however, because of the complexity of the equations involved, only approximate analytical methods were used.

More recently SIEWERT and ZWEIFEL[5] developed the normal modes of the equation of transfer in the picket-fence approximation with the assumption of local thermodynamic equilibrium. They also proved the necessary completeness and orthogonality theorems and constructed a rigorous solution to the Milne problem. This work has also been extended to the generalized picket-fence model[6] and applied to finite media problems by SIMMONS and FERZIGER.[7]

In this paper we investigate numerically the analytical solutions given by SIEWERT and ZWEIFEL[5] for the Milne problem and calculate explicitly several of the quantities of interest. Also, since these analytical solutions were written in terms of the so-called X-function notation more familiar in neutron transport theory, we summarize this earlier work[5] and utilize Chandrasekhar's H-function in order to be more consistent with classical methods in radiative transfer.[8]

In Section II we discuss very briefly the picket-fence model with local thermodynamic equilibrium, and in Section III we review the normal modes of the equation of transfer and discuss the necessary completeness and orthogonality theorems in the H-function notation. Section IV is devoted to the Milne problem, and there we give the results of our explicit calculations for the Milne-problem extrapolation distance, the incident radiation, the integrated Planck function, and the exit angular intensities (laws of darkening) for several sets of basic parameters.

II. THE PICKET-FENCE MODEL

The equations of radiative transfer considered initially are written as[8]

$$\mu\frac{\partial}{\partial z}I_\nu(z,\mu)+\rho(z)k_\nu I_\nu(z,\mu) = \rho(z)k_\nu B_\nu[T(z)] \tag{1}$$

and

$$\int_0^\infty \mathrm{d}\nu k_\nu B_\nu[T(z)] = \frac{1}{2}\int_0^\infty \mathrm{d}\nu k_\nu \int_{-1}^{1}\mathrm{d}\mu I_\nu(z,\mu). \tag{2}$$

Here $I_\nu(z,\mu)$ is the frequency-dependent angular intensity, k_ν is the absorption coefficient, $\rho(z)$ is the density of the medium as a function of position z, $T(z)$ is the *local* temperature and $B_\nu[T(z)]$ is the Planck black-body function:

$$B_\nu[T(z)] = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT(z)}\right)-1\right]^{-1}. \tag{3}$$

Equation (1) is simply the balance equation for the radiation of frequency ν at position z propagating with direction cosine μ relative to the *positive* z-axis, and equation (2) is a statement of energy conservation.

In the picket-fence model, the absorption coefficient is assumed to take either of two constant values, k_1 or k_2. As discussed by CHANDRASEKHAR,[1] we let $\Delta\nu_i$ represent the frequency region over which k_ν has the value k_i and integrate equations (1) and (2) to obtain

$$\mu\frac{\partial}{\partial z}I_i(z,\mu)+\rho(z)k_i I_i(z,\mu) = \frac{\rho(z)k_i w_i}{2\sum_{j=1}^{2}k_j w_j}\sum_{j=1}^{2}k_j\int_{-1}^{1}I_j(z,\mu)\,\mathrm{d}\mu, \qquad i = 1 \text{ or } 2. \tag{4}$$

Here $I_i(z,\mu)$ is defined by

$$I_i(z,\mu) \triangleq \int_{\Delta\nu_i} I_\nu(z,\mu)\,\mathrm{d}\nu, \tag{5}$$

and w_i is given by

$$w_i \triangleq \frac{\pi}{\bar{\sigma}T^4(z)} \int_{\Delta\nu_i} B_\nu[T(z)]\,\mathrm{d}\nu, \qquad i = 1 \text{ or } 2, \tag{6}$$

where $\bar{\sigma}$ is the Stefan–Boltzmann constant. To obtain equation (4) the Schwarzschild condition has been used; for the present model equation (2) takes the form

$$\sum_{j=1}^{2} k_j w_j \frac{\bar{\sigma}T^4(z)}{\pi} = \frac{1}{2}\sum_{j=1}^{2} k_j \int_{-1}^{1} I_j(z,\mu')\,\mathrm{d}\mu'. \tag{7}$$

We note that in general the w_i are functions of z; however, we proceed in the classical manner[1] and consider the case of constant w_1 and w_2.

We prefer to write equation (4) in the more convenient matrix notation and also to introduce an optical variable,

$$\tau \triangleq k_2 \int_0^z \rho(z')\,\mathrm{d}z', \tag{8}$$

defined in terms of k_2, which is taken to be the smaller of the two absorption coefficients. Further, we denote the ratio k_1/k_2 by σ and thus write equation (4) as

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}(\tau,\mu) + \mathbf{\Sigma}\mathbf{I}(\tau,\mu) = \mathbf{C}\int_{-1}^{1}\mathbf{I}(\tau,\mu')\,\mathrm{d}\mu'. \tag{9}$$

Here $\mathbf{I}(\tau,\mu)$ is a two-component vector with elements $I_i(\tau,\mu)$, while the $\mathbf{\Sigma}$-matrix and the transfer matrix $\mathbf{C}$ are given by

$$\mathbf{\Sigma} = \begin{vmatrix} \sigma & 0 \\ 0 & 1 \end{vmatrix} \tag{10}$$

and

$$\mathbf{C} = \frac{1}{2(\sigma w_1 + w_2)} \begin{vmatrix} \sigma^2 w_1 & \sigma w_1 \\ \sigma w_2 & w_2 \end{vmatrix}. \tag{11}$$

We note that equation (9) is analogous to the two-group neutron transport problem discussed by Davison and Sykes;[9] however, here we have the simplifying fact that $\det \mathbf{C} = 0$.

III. BASIC FORMALISM

Since the normal modes or elementary solutions of equation (9) and the necessary completeness and orthogonality theorems were introduced by Siewert and Zweifel,[5] we should simply like to state here that part of the previously reported analysis that is germane to the Milne-problem solution given in Section IV. In addition to establishing these principal features, we believe we have improved the notation and that the use of a Chandrasekhar-type H-function instead of the X-function usually encountered in the solution of a class of singular integral equations facilitates the use of this method for radiative transfer applications.

A general solution to equation (9) may be written as

$$\mathbf{I}(\tau, \mu) = A_+\mathbf{I}_+ + A_-\mathbf{I}_-(\tau, \mu) + \int_{①} [A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]\, e^{-\tau/\eta}\, d\eta$$

$$+ \int_{②} A_3(\eta)\mathbf{\Phi}_3(\eta, \mu)\, e^{-\tau/\eta}\, d\eta, \tag{12}$$

where the regions of integration 1 and 2 imply respectively $\eta \in (-1/\sigma, 1/\sigma)$ and $\eta \in (-1, -1/\sigma)$ and $(1/\sigma, 1)$, A_+, A_-, $A_\alpha(\eta)$, $\alpha = 1, 2$ and 3, are arbitrary expansion coefficients, and

$$\mathbf{I}_+ = \begin{vmatrix} \dfrac{c_{12}}{\sigma} \\ c_{22} \end{vmatrix}, \qquad \mathbf{I}_-(\tau, \mu) = \begin{vmatrix} \dfrac{c_{12}}{\sigma}(\tau - \mu/\sigma) \\ c_{22}(\tau - \mu) \end{vmatrix}, \tag{13a, b}$$

$$\mathbf{\Phi}_1(\eta, \mu) = \begin{vmatrix} c_{12}\delta(\sigma\eta - \mu) \\ -c_{11}\delta(\eta - \mu) \end{vmatrix}, \quad \eta \in (-1/\sigma, 1/\sigma), \tag{13c}$$

$$\mathbf{\Phi}_2(\eta, \mu) = \begin{vmatrix} c_{12}\eta \dfrac{P}{\sigma\eta - \mu} + \delta(\sigma\eta - \mu)[-2\eta c_{12}T(\sigma\eta)] \\ c_{22}\eta \dfrac{P}{\eta - \mu} + \delta(\eta - \mu)[1 - 2\eta c_{22}T(\eta)] \end{vmatrix}, \quad \eta \in (-1/\sigma, 1/\sigma), \tag{13d}$$

and

$$\mathbf{\Phi}_3(\eta, \mu) = \begin{vmatrix} \dfrac{c_{12}\eta}{\sigma\eta - \mu} \\ c_{22}\eta \dfrac{P}{\eta - \mu} + \delta(\eta - \mu)[1 - 2\eta c_{11}T(1/\sigma\eta) - 2\eta c_{22}T(\eta)] \end{vmatrix}, \tag{13e}$$

$$\eta \in (-1, -1/\sigma) \quad \text{and} \quad (1/\sigma, 1).$$

In addition, the symbol P is used in the above equations to indicate that all ensuing integrals over η or μ are to be evaluated in the Cauchy principal-value sense, and $\delta(x)$ denotes the Dirac delta function; the notation here that $T(x) \triangleq \tanh^{-1}(x)$ should be noticed since a similar symbol is used for the temperature.

The half-range completeness and orthogonality theorems necessary for half-space or finite media problems state[5] that a "well-behaved", but arbitrary, function $\mathbf{F}(\mu)$ may be expanded in the manner

$$\mathbf{F}(\mu) = A_+\mathbf{I}_+ + \int_0^{1/\sigma} [A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]\, d\eta + \int_{1/\sigma}^{1} A_3(\eta)\mathbf{\Phi}_3(\eta, \mu)\, d\eta, \quad \mu \in (0, 1), \tag{14}$$

where the expansion coefficients may be obtained readily by utilizing the following summary of orthogonality relations and normalization integrals:

$$\int_0^1 \tilde{\mathbf{\Phi}}_\alpha^\dagger(\xi', \mu)\mathbf{H}(\mu)\mathbf{\Phi}_\beta(\xi, \mu)\,d\mu = \xi H(\xi)N_\alpha(\xi)\delta(\xi'-\xi)\delta_{\alpha\beta},\ \xi,\xi \in (0, 1), \qquad \alpha, \beta = 1, 2 \text{ or } 3, \tag{15a}$$

$$\int_0^1 \tilde{\mathbf{I}}_+^\dagger \mathbf{H}(\mu)\mathbf{\Phi}_\beta(\xi, \mu)\,d\mu = 0, \qquad \xi \in (0, 1), \qquad \beta = 1, 2, \text{ or } 3, \tag{15b}$$

$$\int_0^1 \tilde{\mathbf{\Phi}}_\alpha^\dagger(\xi, \mu)\mathbf{H}(\mu)\mathbf{I}_+\,d\mu = 0, \qquad \xi \in (0, 1), \qquad \alpha = 1, 2, \text{ or } 3, \tag{15c}$$

and

$$\int_0^1 \tilde{\mathbf{I}}_+^\dagger \mathbf{H}(\mu)\mathbf{I}_+\,d\mu = N_+. \tag{15d}$$

Here the superscript tilde denotes the transpose operation, and the matrix $\mathbf{H}(\mu)$ is given by

$$\mathbf{H}(\mu) = \begin{vmatrix} \mu H(\mu/\sigma) & 0 \\ 0 & \mu H(\mu) \end{vmatrix}, \tag{16}$$

where $H(\mu)$ is Chandrasekhar's H-function[8] for characteristic function $\Psi(\mu) = c_{22} + c_{11}\Theta(\mu)$, with

$$\begin{aligned} \Theta(\mu) &= 1, \mu \in (-1/\sigma, 1/\sigma) \\ &= 0 \text{ otherwise.} \end{aligned} \tag{17}$$

Further,

$$N_\alpha(\xi) = [1-2\xi c_{11}T(\sigma\xi)-2\xi c_{22}T(\xi)]^2 + \pi^2\xi^2(c_{11}+c_{22})^2, \qquad \alpha = 1 \text{ or } 2, \tag{18a}$$

$$N_3(\xi) = \left[1-2\xi c_{11}T\left(\frac{1}{\sigma\xi}\right)-2\xi c_{22}T(\xi)\right]^2 + \pi^2\xi^2 c_{22}^2, \tag{18b}$$

and

$$N_+ = c_{22}\left[\frac{2}{3}\left(\frac{c_{11}}{\sigma^3}+c_{22}\right)\right]^{1/2}. \tag{18c}$$

In addition, the adjoint vectors $\mathbf{I}_+^\dagger$ and $\mathbf{\Phi}_3^\dagger(\xi, \mu)$ are obtained respectively simply by interchanging c_{ij} with c_{ji} in equations (13a) and (13e). Because of the degeneracy associated with that part of the eigenvalue spectrum for which $\eta \in (-1/\sigma, 1/\sigma)$, the adjoint vectors $\mathbf{\Phi}_\alpha^\dagger(\xi, \mu)$, $\alpha = 1$ or 2, take slightly different forms:

$$\mathbf{\Phi}_1^\dagger(\xi, \mu) = M_{11}(\xi)\mathbf{G}_1(\xi, \mu) + M_{12}(\xi)\mathbf{G}_2(\xi, \mu) \tag{19a}$$

and

$$\mathbf{\Phi}_2^\dagger(\xi, \mu) = M_{21}(\xi)\mathbf{G}_1(\xi, \mu) + M_{22}(\xi)\mathbf{G}_2(\xi, \mu). \tag{19b}$$

Here

$$M_{11}(\eta) = \frac{1}{c_{11}c_{22}}\{c_{22}(c_{11}+c_{22})\pi^2\eta^2+[1-2\eta c_{22}T(\eta)]^2+4\eta^2c_{11}c_{22}T^2(\sigma\eta)\}, \quad (20a)$$

$$M_{12}(\eta) = M_{21}(\eta) = \frac{1}{c_{22}}[1-2\eta c_{22}T(\eta)+2\eta c_{22}T(\sigma\eta)], \quad (20b)$$

and

$$M_{22}(\eta) = \frac{1}{c_{22}}(c_{11}+c_{22}). \quad (20c)$$

We note that $\mathbf{G}_1(\xi, \mu)$ and $\mathbf{G}_2(\xi, \mu)$ are obtained respectively by replacing c_{12} with c_{21} in equations (13c) and (13d).

Thus all of the expansion coefficients in equation (14) can be obtained simply by taking scalar products of that equation with the appropriate adjoint vectors; thus

$$A_\alpha(\eta) = \frac{1}{\eta H(\eta)N_\alpha(\eta)}\int_0^1 \tilde{\mathbf{\Phi}}_\alpha^\dagger(\eta, \mu)\mathbf{H}(\mu)\mathbf{F}(\mu)\,d\mu, \alpha = 1, 2 \text{ or } 3, \quad (21a)$$

and

$$A_+ = \frac{1}{N_+}\int_0^1 \tilde{\mathbf{I}}_+^\dagger \mathbf{H}(\mu)\mathbf{F}(\mu)\,d\mu. \quad (21b)$$

IV. NUMERICAL RESULTS FOR THE MILNE PROBLEM

We should now like to establish the Milne problem solution given by SIEWERT and ZWEIFEL,[5] and will proceed to discuss the numerical evaluation of the several quantities of interest. We seek a diverging (for large τ) solution of equation (9) which satisfies the Milne-problem boundary conditions:[8]

$$\text{(i)} \quad \mathbf{I}_M(0, \mu) = \mathbf{0}, \ \mu \in (0, 1) \quad (22a)$$

$$\text{(ii)} \quad \lim_{\tau\to\infty} e^{-\tau}\mathbf{I}_M(\tau, \mu) = \mathbf{0}. \quad (22b)$$

A solution which meets this second boundary condition can be obtained from equation (12) simply by taking $A_\alpha(\eta) = 0, \eta < 0, \alpha = 1, 2$ and 3. Thus

$$\mathbf{I}_M(\tau, \mu) = A_+\mathbf{I}_+ + A_-\mathbf{I}_-(\tau, \mu) + \int_0^{1/\sigma} [A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]\,e^{-\tau/\eta}\,d\eta$$
$$+ \int_{1/\sigma}^1 A_3(\eta)\mathbf{\Phi}_3(\eta, \mu)\,e^{-\tau/\eta}\,d\eta, \quad (23)$$

where A_- is an arbitrary normalization constant, and A_+ and $A_\alpha(\eta)$, $\alpha = 1$, 2 or 3, are to be determined from the zero re-entrant condition; these expansion coefficients are thus determined from

$$A_-\begin{vmatrix}\dfrac{c_{12}}{\sigma^2}\\ c_{22}\end{vmatrix}\mu = A_+\mathbf{I}_+ + \int_0^{1/\sigma}[A_1(\eta)\boldsymbol{\Phi}_1(\eta,\mu)+A_2(\eta)\boldsymbol{\Phi}_2(\eta,\mu)]\,\mathrm{d}\eta$$
$$+\int_{1/\sigma}^{1}A_3(\eta)\boldsymbol{\Phi}_3(\eta,\mu)\,\mathrm{d}\eta,\ \mu\in(0,1). \tag{24}$$

In the manner discussed in the previous section, we now take scalar products of equation (24) and evaluate ensuing integrals to find

$$\frac{A_+}{A_-} \triangleq \tau_0 = \frac{1}{\alpha}\int_0^1 \Psi(\mu)H(\mu)\mu^2\,\mathrm{d}\mu, \tag{25a}$$

$$\frac{A_1(\eta)}{A_-} = \frac{-\alpha}{H(\eta)N_1(\eta)}[1-2\eta c_{22}T(\eta)+2\eta c_{22}T(\sigma\eta)], \tag{25b}$$

$$\frac{A_2(\eta)}{A_-} = \frac{-\alpha}{H(\eta)N_2(\eta)}[c_{11}+c_{22}], \tag{25c}$$

and

$$\frac{A_3(\eta)}{A_-} = \frac{-\alpha}{H(\eta)N_3(\eta)}[c_{22}], \tag{25d}$$

where

$$\alpha = \int_0^1 \Psi(\mu)H(\mu)\mu\,\mathrm{d}\mu = \left[\frac{2}{3}\left(\frac{c_{11}}{\sigma^3}+c_{22}\right)\right]^{1/2}. \tag{26}$$

Since all of the unknown expansion coefficients are now determined, the solution is complete:

$$\mathbf{I}_M(\tau,\mu) = \frac{3\sigma^2F}{4(c_{12}+\sigma^2c_{22})}\{\tau_0\mathbf{I}_+ + \mathbf{I}_-(\tau,\mu) - \alpha\int_0^{1/\sigma}\{[1-2\eta c_{22}T(\eta)+2\eta c_{22}T(\sigma\eta)]\boldsymbol{\Phi}_1(\eta,\mu)$$
$$+[c_{11}+c_{22}]\boldsymbol{\Phi}_2(\eta,\mu)\}\,e^{-\tau/\eta}\frac{\mathrm{d}\eta}{H(\eta)N_1(\eta)} - \alpha\int_{1/\sigma}^{1} c_{22}\boldsymbol{\Phi}_3(\eta,\mu)\,e^{-\tau/\eta}\frac{\mathrm{d}\eta}{H(\eta)N_3(\eta)}, \tag{27}$$

where the normalization

$$2\widetilde{\begin{vmatrix}1\\1\end{vmatrix}}\int_{-1}^{1}\mathbf{I}_M(\tau,\mu)\mu\,\mathrm{d}\mu \triangleq -F \tag{28}$$

has been imposed. The surface quantity $\mathbf{I}_M(0, -\mu, \mu \in (0, 1)$ may be readily obtained from equation (27) by utilizing equation (24) and the half-space S-matrix[10,11] for this problem:

$$\mathbf{I}_M(0, -\mu) = \frac{3\sigma^2 F\alpha}{4(c_{12}+\sigma^2 c_{22})} \begin{vmatrix} \dfrac{c_{12}}{\sigma} H\left(\dfrac{\mu}{\sigma}\right) \\ c_{22} H(\mu) \end{vmatrix}, \; \mu \in (0, 1). \tag{29}$$

The intensity of the integrated black-body radiation is found from the Schwarzschild condition, equation (7), to be

$$B(\tau) \triangleq \frac{\tilde{\sigma} T^4(\tau)}{\pi} = \frac{1}{2(\sigma w_1 + w_2)} \begin{vmatrix} \tilde{\sigma} \\ 1 \end{vmatrix} \int_{-1}^{1} \mathbf{I}_M(\tau, \mu)\, \mathrm{d}\mu, \tag{30a}$$

or more explicitly,

$$B(\tau) = \frac{3}{4} \frac{F\sigma}{(w_1 + \sigma w_2)} \left\{ \tau_0 + \tau + \int_0^{1/\sigma} A_2(\eta)\, e^{-\tau/\eta}\, \mathrm{d}\eta + \int_{1/\sigma}^{1} A_3(\eta)\, e^{-\tau/\eta}\, \mathrm{d}\eta \right\}, \tag{30b}$$

where the solution given by equation (27) has been used.

Since solutions for the several quantities of interest here, viz. the Milne-problem extrapolation distance, the integrated Planck function and the distribution of exit radiation, have been established analytically, we proceed to evaluate numerically these solutions and thus to establish quantitatively the effects of the various parameters. Further these numerical calculations should serve to indicate the transport corrections to the Eddington (or diffusion) theory approximation.

Once the required H-functions have been constructed, there remains only to evaluate numerically the various explicitly given functions. However, since computational experience with the normal mode technique has been rather limited, several precautionary measures were taken to establish confidence that our results were valid to the degree of accuracy reported here. All calculations were performed in double-precision arithmetic on an IBM 360/75 computer, and the 81-point improved Gaussian quadrature scheme discussed by Kronrod[12] was the basic method used for the numerical evaluation of required integrals.

In the usual manner,[8] the necessary H-functions were obtained by solving numerically the nonlinear integral equation

$$\frac{1}{H(\mu)} = \int_0^1 \nu \Psi(\nu) H(\nu) \frac{\mathrm{d}\nu}{\nu + \mu}, \quad \mu \notin (-1, 0). \tag{31}$$

Replacement of the integration process by numerical quadrature results in a set of nonlinear algebraic equations which subsequently may be solved by iteration. The iterative process was initiated by setting $H(\mu)$ to unity and was considered converged when successive iterates differed by less than 10^{-14}; these converged results were then used to verify

Chandrasekhar's identities:[8]

$$\int_0^1 H(\mu)\Psi(\mu)\,\mathrm{d}\mu = 1, \tag{32a}$$

$$\int_0^1 H(\mu)\Psi(\mu)\mu\,\mathrm{d}\mu = \left[2\int_0^1 \Psi(\mu)\mu^2\,\mathrm{d}\mu\right]^{1/2} = \alpha. \tag{32b}$$

The errors thus encountered were less than 10^{-13} for all cases considered. As an additional measure, the converged H-functions were found to satisfy the alternative integral equation,[8]

$$\frac{1}{H(\mu)} = 1-\mu\int_0^1 \Psi(\nu)H(\nu)\frac{\mathrm{d}\nu}{\nu+\mu}, \qquad \mu\notin(-1,0), \tag{33}$$

to within a difference of 10^{-13}.

With the H-functions so established the expansion coefficients A_+ and $A_\alpha(\eta)$, $\alpha = 1, 2$ and 3, were readily available, and thus the remaining quantities of interest were obtained by numerical integration. Initially, we used the 81-point quadrature scheme in each of the intervals $(0, 1/\sigma)$ and $(1/\sigma, 1)$; however, in a number of cases (generally for large σ and small w_1) a rather sharp peak occurs in the coefficient $A_3(\eta)$. This peak is due to the pronounced variation, for these cases, of $N_3(\eta)$ which appears in the denominator of $A_3(\eta)$. In order to minimize the errors in the numerical integration, it was deemed advisable to increase the quadrature nodal density by subdividing the interval $(1/\sigma, 1)$ and to apply 81-point scheme in each of the subintervals. This subdivision was continued until it failed to alter the reported values of the integrated black-body radiation intensity. The H-function and thus both the extrapolation distance and the law of darkening, being independent of $A_3(\eta)$, were found to be rather insensitive to the quadrature order.

Further confidence in the results was provided by a number of numerical checks, which we shall describe. The free-surface boundary condition, equation (22a), is difficult to verify pointwise since it involves numerical integration of generalized functions. However, a verification of this condition which avoids the distributional nature of the solution can be accomplished by computing numerous moments of equation (24). A satisfactory check for each component is then

$$\varepsilon_i(k) = \left|1-\frac{M_{iR}(k)}{M_{iL}(k)}\right|, \tag{34}$$

where $M_{iR}(k)$ and $M_{iL}(k)$ are, respectively, the result of operating on the right- and left-hand sides of the ith component of equation (24) with the operator

$$\mathrm{O}(k)[f(\mu)] \triangleq \int_0^1 \mu^k[f(\mu)]\,\mathrm{d}\mu,\ k = 0, 1, 2, \ldots. \tag{35}$$

This form of $\varepsilon_i(k)$ is preferred since it is independent of both the normalization of the problem and the absolute magnitude of the quantities involved.

A second check of the free-surface condition was devised in a similar manner using "weight function moments" obtained by operating on equation (24) with

$$\begin{vmatrix} \tilde{\sigma}^k \\ 1 \end{vmatrix} \int_0^1 \mu^k \mathbf{H}(\mu)\,\mathrm{d}\mu,\ k = 0, 1, 2, \ldots. \tag{36}$$

The resulting equations were compared by a rearrangement to the form of equation (34); in this case, of course, a scalar comparison has been made, whereas equation (34) was a vector check.

Confidence in the results may be additionally substantiated by a pointwise comparison of the law of darkening as given by equation (27) with that given by equation (29), and also by comparing the moments of these two equations after using the operator (35). Since equation (29) depends only upon an evaluation of the H-function (a procedure in which, from above indications, we are fairly confident) then these latter comparisons will be essentially a test of equation (27) and thus some indication of how accurately the numerical integrations have been performed. As was accomplished above, such comparisons can be made independent of the absolute magnitude and the particular normalization by dividing through by the term associated with $I(0, -\mu)$.

There is, of course, no rigorous basis for concluding that the checks which we have described provide any criterion to validate the accuracy of the results. However, we feel that because of their number and diversity the above tests provide some degree of confidence in the calculations. We are further convinced that such precautionary measures should be taken until more computational experience in the normal-mode technique is developed.

We have utilized the preceding equations to compute several quantities of interest in the present problem. The results shown in the accompanying tables have been arranged to demonstrate the effects of individual parametrical variations. Table 1 lists the extrapolation distance for each of the considered cases. The intensity of the integrated black-body radiation is shown in Table 2, while the emergent angular distributions (laws of darkening) are presented in Tables 3, 4 and 5. Finally, in Tables 6, 7 and 8, we compare

TABLE 1. THE EXTRAPOLATION DISTANCE

σ	w_1	τ_0
5	0.2	0.650810
5	0.4	0.584184
5	0.6	0.499781
5	0.8	0.376044
10	0.2	0.657946
10	0.4	0.602537
10	0.6	0.529955
10	0.8	0.409874
2	0.5	0.566812
5	0.5	0.545117
8	0.5	0.560306
10	0.5	0.569535

TABLE 2. THE INTEGRATED BLACK-BODY FUNCTION

τ	$B(\tau)/F$											
	$\sigma = 5$				$\sigma = 10$				$w_1 = 0.5$			
	$w_1 = 0.2$	$w_1 = 0.4$	$w_1 = 0.6$	$w_1 = 0.8$	$w_1 = 0.2$	$w_1 = 0.4$	$w_1 = 0.6$	$w_1 = 0.8$	$\sigma = 2$	$\sigma = 5$	$\sigma = 8$	$\sigma = 10$
0.00	0.35215	0.32566	0.32566	0.35215	0.28577	0.25237	0.25237	0.28577	0.40825	0.32275	0.27217	0.24896
0.02	0.40400	0.38755	0.39690	0.43704	0.36339	0.34077	0.35295	0.41028	0.44846	0.38909	0.35639	0.34272
0.05	0.45984	0.45575	0.47752	0.53554	0.44024	0.43456	0.46522	0.55520	0.49547	0.46318	0.44878	0.44491
0.10	0.53707	0.55205	0.59430	0.68186	0.53875	0.56217	0.62578	0.77182	0.56452	0.56914	0.57887	0.58762
0.20	0.66695	0.71722	0.80092	0.94953	0.68905	0.76760	0.90048	1.16545	0.68890	0.75364	0.79882	0.82450
0.30	0.78118	0.86389	0.98927	1.20146	0.81130	0.93789	1.13936	1.52816	0.80489	0.91946	0.98945	1.02530
0.40	0.88713	0.99996	1.16664	1.44417	0.92072	1.08885	1.35576	1.86964	0.91647	1.07427	1.16251	1.20479
0.50	0.98809	1.12910	1.33643	1.68038	1.02354	1.22834	1.55706	2.19525	1.02528	1.22166	1.32408	1.37078
0.60	1.08581	1.25348	1.50068	1.91162	1.12271	1.36075	1.74797	2.50875	1.13221	1.36377	1.47784	1.52796
0.80	1.27538	1.49294	1.81758	2.36307	1.31527	1.61364	2.11002	3.10972	1.34227	1.63735	1.77078	1.82659
1.00	1.46047	1.72490	2.12431	2.80404	1.50389	1.85824	2.45676	3.68747	1.54907	1.90196	2.05250	2.11372
1.20	1.64314	1.95261	2.42476	3.23789	1.69048	2.09879	2.79529	4.25078	1.75381	2.16128	2.32826	2.39504
1.40	1.82442	2.17781	2.72123	3.66681	1.87588	2.33713	3.12923	4.80492	1.95720	2.41738	2.60069	2.67320
1.60	2.00483	2.40147	3.01514	4.09229	2.06053	2.57418	3.46045	5.35314	2.15964	2.67147	2.87115	2.94952
1.80	2.18469	2.62416	3.30736	4.51534	2.24468	2.81042	3.79000	5.89747	2.36144	2.92427	3.14039	3.22470
2.00	2.36417	2.84621	3.59846	4.93665	2.42848	3.04611	4.11848	6.43920	2.56276	3.17621	3.40885	3.49915
3.00	3.25901	3.95220	5.04616	7.03017	3.34504	4.22081	5.75385	9.13003	3.56577	4.43019	4.74591	4.86640
4.00	4.15233	5.05583	6.48965	9.11590	4.26014	5.39335	7.38539	11.81125	4.56653	5.68108	6.08010	6.23087

TABLE 3. THE LAWS OF DARKENING WITH $\sigma = 5$

μ	$\frac{I_i(0, -\mu)}{I_1(0, -1)+I_2(0, -1)}$							
	$w_1 = 0.2$		$w_1 = 0.4$		$w_1 = 0.6$		$w_1 = 0.8$	
	$i = 1$	$i = 2$	$i = 1$	$i = 2$	$i = 1$	$i = 2$	$i = 1$	$i = 2$
0.0	0.05544	0.22177	0.10179	0.15268	0.15191	0.10127	0.21912	0.05478
0.1	0.06300	0.32880	0.11978	0.25092	0.18302	0.17959	0.26888	0.10360
0.2	0.06854	0.40489	0.13331	0.32274	0.20711	0.23997	0.30852	0.14366
0.3	0.07344	0.47333	0.14541	0.38739	0.22903	0.29529	0.34529	0.18143
0.4	0.07795	0.53805	0.15665	0.44832	0.24966	C.34785	0.38041	0.21792
0.5	0.08220	0.60058	0.16728	0.50700	0.26939	0.39866	0.41442	0.25356
0.6	0.08626	0.66170	0.17746	0.56418	0.28843	0.44828	0.44758	0.28862
0.7	0.09016	0.72186	0.18728	0.62032	0.30692	0.49705	0.48009	0.32323
0.8	0.09394	0.78132	0.19680	0.67570	0.32496	0.54517	0.51205	0.35752
0.9	0.09762	0.84025	0.20609	0.73050	0.34262	0.59280	0.54355	0.39154
1.0	0.10122	0.89878	0.21516	0.78484	0.35995	0.64005	0.57465	0.42535

TABLE 4. THE LAWS OF DARKENING WITH $\sigma = 10$

μ	$\frac{I_i(0, -\mu)}{I_1(0, -1)+I_2(0, -1)}$							
	$w_1 = 0.2$		$w_1 = 0.4$		$w_1 = 0.6$		$w_1 = 0.8$	
	$i = 1$	$i = 2$	$i = 1$	$i = 2$	$i = 1$	$i = 2$	$i = 1$	$i = 2$
0.0	0.04484	0.17935	0.07836	0.11753	0.11647	0.07765	0.17510	0.04377
0.1	0.05134	0.32064	0.09291	0.24785	0.14099	0.18372	0.21525	0.11433
0.2	0.05592	0.40636	0.10369	0.33317	0.15983	0.26008	0.24712	0.17077
0.3	0.05983	0.47987	0.11319	0.40633	0.17687	0.32722	0.27657	0.22259
0.4	0.06335	0.54793	0.12190	0.47359	0.19278	0.38946	0.30460	0.27172
0.5	0.06657	0.61300	0.13002	0.53744	0.20788	0.44871	0.33162	0.31908
0.6	0.06958	0.67622	0.13770	0.59914	0.22234	0.50598	0.35786	0.36521
0.7	0.07241	0.73822	0.14500	0.65936	0.23627	0.56187	0.38347	0.41045
0.8	0.07511	0.79934	0.15200	0.71855	0.24975	0.61676	0.40854	0.45501
0.9	0.07768	0.85983	0.15873	0.77695	0.26284	0.67088	0.43314	0.49904
1.0	0.08016	0.91984	0.16523	0.83477	0.27558	0.72442	0.45733	0.54267

TABLE 5. THE LAWS OF DARKENING WITH $w_1 = 0.5$

μ	$\frac{I_i(0,-\mu)}{I_1(0,-1)+I_2(0,-1)}$							
	$\sigma = 2$		$\sigma = 5$		$\sigma = 8$		$\sigma = 10$	
	$i = 1$	$i = 2$	$i = 1$	$i = 2$	$i = 1$	$i = 2$	$i = 1$	$i = 2$
0.0	0.16133	0.16133	0.12573	0.12573	0.10536	0.10536	0.09616	0.09616
0.1	0.19382	0.21975	0.14985	0.21510	0.12604	0.21452	0.11532	0.21555
0.2	0.21975	0.26702	0.16824	0.28210	0.14167	0.29178	0.12979	0.29743
0.3	0.24388	0.31159	0.18484	0.34285	0.15566	0.35976	0.14272	0.36837
0.4	0.26702	0.35478	0.20035	0.40027	0.16863	0.42293	0.15468	0.43375
0.5	0.28953	0.39712	0.21510	0.45564	0.18086	0.48319	0.16594	0.49584
0.6	0.31159	0.43890	0.22928	0.50962	0.19252	0.54156	0.17664	0.55581
0.7	0.33331	0.48028	0.24299	0.56262	0.20371	0.59860	0.18689	0.61431
0.8	0.35478	0.52136	0.25633	0.61490	0.21452	0.65468	0.19676	0.67175
0.9	0.37603	0.56221	0.26935	0.66662	0.22499	0.71003	0.20630	0.72840
1.0	0.39712	0.60288	0.28210	0.71790	0.23517	0.76483	0.21555	0.78445

TABLE 6. DIFFUSION AND TRANSPORT THEORY RESULTS FOR THE EXTRAPOLATION DISTANCE

σ	w_1	τ_0	
		Diffusion theory*	Transport theory
5	0.2	0.6019	0.650810
10	0.2	0.6065	0.657946

* CHANDRASEKHAR.[1]

our results with those given by CHANDRASEKHAR[1] who used the Eddington or diffusion theory approximation.

For each case, the checks mentioned above were performed through the tenth moment. Of these; the maximum difference occurred consistently in the first component of the free-surface boundary check and was less 5×10^{-6}. The comparison of the law of darkening as given by equations (27) and (29) was made at 20 equally spaced points covering the interval $(-1, 0)$ and here the difference was never greater than 2×10^{-6}, with the maximum difference always occurring at $\mu = 0$. Because of the close agreement of these checks, as well as the other precautionary measures taken, we have confidence in the values to the degree of precision reported.

TABLE 7. DIFFUSION AND TRANSPORT THEORY RESULTS FOR $B(\tau)$

τ	$B(\tau)/F$			
	$\sigma = 5, w_1 = 0.2$		$\sigma = 10, w_1 = 0.2$	
	Diffusion theory*	Transport theory	Diffusion theory*	Transport theory
0.00	0.4127	0.35215	0.3398	0.28577
0.02	0.4445	0.40400	0.3949	0.36339
0.05	0.4893	0.45984	0.4660	0.44024
0.10	0.5577	0.53707	0.5620	0.53875
0.20	0.6778	0.66695	0.7040	0.68905
0.30	0.7841	0.78118	0.8162	0.81130
0.40	0.8829	0.88713	0.9155	0.92072
0.50	0.9774	0.98809	1.0100	1.02354
0.60	1.0696	1.08581	1.1027	1.12271
0.80	1.2506	1.27538	1.2863	1.31527
1.00	1.4299	1.46047	1.4693	1.50389
1.20	1.609	1.64314	1.652	1.69048
1.40	1.787	1.82442	1.835	1.87588
1.60	1.966	2.00483	2.018	2.06053
1.80	2.145	2.18469	2.201	2.24468
2.00	2.323	2.36417	2.384	2.42848
3.00	3.216	3.25901	3.299	3.34504
4.00	4.109	4.15233	4.213	4.26014

* CHANDRASEKHAR.[1]

TABLE 8. DIFFUSION AND TRANSPORT THEORY RESULTS FOR THE LAWS OF DARKENING

μ	$\dfrac{I_1(0, -\mu) + I_2(0, -\mu)}{I_1(0, -1) + I_2(0, -1)}$			
	$\sigma = 5, w_1 = 0.2$		$\sigma = 10, w_1 = 0.2$	
	Diffusion theory*	Transport theory	Diffusion theory*	Transport theory
0.0	0.3241	0.27722	0.2684	0.22419
0.1	0.4234	0.39180	0.3963	0.37198
0.2	0.4997	0.47343	0.4808	0.46228
0.4	0.6330	0.61600	0.6163	0.61127
0.6	0.7578	0.74796	0.7514	0.74580
0.8	0.8797	0.87526	0.8768	0.87445
0.9	0.9400	0.93787	0.9386	0.93752
1.0	1.0000	1.00000	1.0000	1.00000

* CHANDRASEKHAR.[1]

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