Technical Notes

Several Particular Solutions of the One-Speed Transport Equation

M. N. Özişik and C. E. Siewert

Departments of Mechanical and Nuclear Engineering
North Carolina State University
Raleigh, North Carolina 27607

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I. INTRODUCTION

Case's singular eigenfunction expansion technique[1,2] is one of the principal methods currently in vogue for analytically treating a reasonably broad class of problems in both neutron-transport theory and radiative transfer. One of the particular merits of the method is the systematic approach by which solutions to many model problems can be constructed. The desired solution is first written as a linear sum of sufficiently general normal modes (solutions of the homogeneous transport equation) and a particular solution appropriate to the inhomogeneous source term of interest. This sum of complementary and particular solutions is then constrained by the boundary conditions of the considered problem, and the unknown expansion coefficients scaling the various normal modes are determined by utilizing existing completeness and/or orthogonality theorems.

Workers in the fields of radiative transfer and astrophysics, in addition to neutron physicists with their established interest in the inhomogeneous transport equation, have made use of forcing functions of this same equation to characterize the black-body radiation function. Particular solutions similar to those developed by Lundquist and Horak[3] and Mendelson and Congdon[4] have been used recently[5] to construct solutions for the intensity of radiation and the net radiative heat flux in a finite plane-parallel medium with reflecting boundaries, and the expressions given here can be used to treat problems with more general inhomogeneous source terms. It follows then that there exists a genuine use for particular solutions corresponding to various inhomogeneous source terms. Of course, the infinite-medium Green's function[2] can be used to construct these solutions in general; however, solutions so established have obvious computational and/or

[1]K. M. CASE, *Ann. Phys.*, **9**, 1 (1960).

[2]K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley, Reading, Mass. (1967).

[3]C. A. LUNDQUIST and H. G. HORAK, *Ap. J.*, **121**, 175 (1955).

[4]M. R. MENDELSON and S. P. CONGDON, "Particular Solutions of the One-Speed Transport Equation," KAPL-P-3335, Knolls Atomic Power Laboratory (1967).

[5]M. N. ÖZIŞIK and C. E. SIEWERT, *Intern. J. Heat Mass Transfer*, **12**, 611 (1969).

analytical disadvantages, as is easily observed by considering the simplest source term—a constant.

II. ANALYSIS

We should like to illustrate this work with several derivations, with the remainder of our results to be displayed in the accompanying table. Consider then the one-speed transport equation written in an established notation[2]

$$\mu \frac{\partial}{\partial x}\psi(x,\mu) + \psi(x,\mu) = \frac{c}{2}\int_{-1}^{1}(1 + b\mu\mu')\,\psi(x,\mu')d\mu' + S(x,\mu) \quad . \qquad (1)$$

We consider temporarily only the case of isotropic scattering, and seek a particular solution to Eq. (1) for a source given by

$$S(x,\mu) = \exp(-x/\eta) \quad , \quad \eta \,\varepsilon\, (-1,1) \quad . \qquad (2)$$

If a solution of the form

$$\psi_p(\eta;x,\mu) = F(\eta;\mu)\exp(-x/\eta) \qquad (3)$$

is proposed and substituted into Eq. (1), there results

$$(\eta - \mu)F(\eta;\mu) = \frac{c}{2}\,\eta\int_{-1}^{1}F(\eta;\mu')d\mu' + \eta \quad , \qquad (4)$$

after like terms are factored out.

Noting that $\eta \,\varepsilon\, (-1,1)$ as is μ, we solve Eq. (4) to find

$$F(\eta;\mu) = \eta\left[\frac{P}{\eta - \mu} + \omega(\eta)\,\delta(\eta - \mu)\right]\left[\frac{c}{2}\int_{-1}^{1}F(\eta;\mu')d\mu' + 1\right] \quad , \qquad (5)$$

where P is a symbol used to ensure that all ensuing integrals over η or μ are to be evaluated in the Cauchy principal-value sense, $\delta(x)$ denotes the Dirac delta function, and $\omega(\eta)$ is, at this point, arbitrary. Equation (5) can now be integrated over μ from -1 to 1 to yield $\omega(\eta)$, and when this result is entered above, we find

Table of Particular Solutions

$S(x)$	$\psi_p(x,\mu)$	Remarks
1	$(1 - c)^{-1}$	$c \neq 1$
	$(b - 3)\,\frac{1}{2}x^2 + 3x\mu - 3\mu^2$	$c = 1$
x	$(1 - c)^{-1}\left[x - \frac{3\mu}{(3 - cb)}\right]$	$c \neq 1,\ cb \neq 3$
	$-\frac{1}{2}x^3 + \frac{3}{2}x^2\mu - 3x\mu^2 + 3\mu^3 + 9\frac{b}{5}\frac{1}{3 - b}\mu + \frac{b}{6}x^3$	$c = 1,\ b \neq 3$
x^2	$(1 - c)^{-1}\left[\frac{3}{3 - cb}\right]\left[x^2\left(1 - \frac{cb}{3}\right) - 2x\mu + 2\mu^2 + \frac{2c}{3(1 - c)}\right]$	$c \neq 1,\ cb \neq 3$
	$-\frac{1}{4}x^4 + x^3\mu - 3x^2\mu^2 + 6x\mu^3 - 6\mu^4\ \ \frac{3}{5}\left(-\frac{3}{2}x^2 + 3x\mu - 3\mu^2\right) + \frac{b}{12}x^4$	$c = 1$
$\exp(-x/\eta)$	$-\frac{2}{cR(\eta,\eta)}\left[\delta(\eta - \mu) - \frac{cb\eta^2}{2}\right]\exp(-x/\eta)$	$\eta \in (-1,1)$
	$\frac{1}{\Lambda(\eta)}\left[\frac{\eta}{\eta - \mu} + cb\eta^2\left(1 - \eta\tanh^{-1}\frac{1}{\eta}\right)\right]\exp(-x/\eta)$	$\eta \notin (-1,1),\ \Lambda(\eta) \neq 0$
$\cos x/\eta$	$\frac{1}{\Lambda(i\eta)}\left[\frac{\eta^2\cos x/\eta + \eta\mu\sin x/\eta}{\eta^2 + \mu^2} - cb\eta^2\left(1 - \eta\tan^{-1}\frac{1}{\eta}\right)\cos x/\eta\right]$	$\Lambda(i\eta) \neq 0$
$\sin x/\eta$	$\frac{-1}{\Lambda(i\eta)}\left[\frac{\eta\mu\cos x/\eta - \eta^2\sin x/\eta}{\eta^2 + \mu^2} + cb\eta^2\left(1 - \eta\tan^{-1}\frac{1}{\eta}\right)\sin x/\eta\right]$	$\Lambda(i\eta) \neq 0$
$\cosh x/\eta$	$\frac{1}{\Lambda(\eta)}\left[\frac{\eta^2\cosh x/\eta - \eta\mu\sinh x/\eta}{\eta^2 - \mu^2} + cb\eta^2\left(1 - \eta\tanh^{-1}\frac{1}{\eta}\right)\cosh x/\eta\right]$	$\eta \notin (-1,1)$ $\Lambda(\eta) \neq 0$
$\sinh x/\eta$	$\frac{-1}{\Lambda(\eta)}\left[\frac{\eta\mu\cosh x/\eta - \eta^2\sinh x/\eta}{\eta^2 - \mu^2} - cb\eta^2\left(1 - \eta\tanh^{-1}\frac{1}{\eta}\right)\sinh x/\eta\right]$	$\eta \notin (-1,1)$ $\Lambda(\eta) \neq 0$
$\int_{-1}^{1}A(\eta)\exp(-x/\eta)d\eta$	$-\frac{2A(\mu)}{cR(\mu,\mu)}\exp(-x/\mu) + b\int_{-1}^{1}A(\eta)\frac{1}{R(\eta,\eta)}\eta^2\exp(-x/\eta)\,d\eta$	
$E_N(x)$	$-\frac{2}{c}\mu^{N-2}\exp(-x/\mu), \quad \mu \in (0,1)$ $0 \quad , \quad \mu \in (-1,0)$	$b = 0$
$x\exp(-x/\eta)$	$\frac{1}{\Lambda(\eta)}\left[\frac{\eta}{\eta - \mu} + cb\eta^2(1 - \delta)\right]x\exp(-x/\eta) + \frac{c\eta^2}{(\eta - \mu)\Lambda^2(\eta)}\Big\{2b\eta(1 - \delta)$ $\times[c\eta - \mu - c(\eta - \mu)\delta] + R(\eta,\mu)\left(\delta - \frac{\eta^2}{\eta^2 - 1}\right)\Big\}\exp(-x/\eta)$ $-\frac{\eta^2}{\Lambda(\eta)}\frac{\mu}{(\eta - \mu)^2}\exp(-x/\eta)$	$\eta \notin (-1,1)$ $\delta = \eta\tanh^{-1}\frac{1}{\eta}$ $\Lambda(\eta) \neq 0$

$$F(\eta;\mu) = \left[\frac{2}{c} + \int_{-1}^{1} F(\eta;\mu')d\mu'\right] \phi_\eta(\mu) - \frac{2}{c}\delta(\eta - \mu) \quad , \qquad (6)$$

where

$$\phi_\eta(\mu) = \frac{c}{2}\eta\frac{P}{\eta - \mu} + [1 - c\eta \tanh^{-1}\eta]\delta(\eta - \mu) \quad . \qquad (7)$$

Since Case[1] has already shown that $\phi_\eta(\mu)\exp(-x/\eta)$ for $\eta \varepsilon (-1,1)$ is a solution of the homogeneous equation, the second term in Eq. (6) is sufficient for the determination of a particular solution. It follows, then, that

$$\psi_p(\eta;x,\mu) = -\frac{2}{c}\delta(\eta - \mu)\exp(-x/\eta) \quad , \quad \eta \varepsilon (-1,1) \quad , \qquad (8)$$

is a solution of Eq. (1) for $S(x,\mu)$ given by Eq. (2).

Clearly, if instead of Eq. (2) for $S(x,\mu)$, we have a source of the form

$$S(x,\mu) = \int_{-1}^{1} B(\eta)\exp(-x/\eta)d\eta \quad , \qquad (9)$$

where $B(\eta)$ is arbitrary (except that the integral above should exist), then the corresponding particular solution is

$$\psi_p(x,\mu) = \int_{-1}^{1} B(\eta)\,\psi_p(\eta;x,\mu)d\eta \quad , \qquad (10a)$$

or

$$\psi_p(x,\mu) = -\frac{2}{c}B(\mu)\exp(-x/\mu) \quad . \qquad (10b)$$

One special case of the above result is

$$S(x,\mu) = \int_0^1 \eta^{N-2}\exp(-x/\eta)d\eta \equiv E_N(x) \quad , \qquad (11)$$

where $E_N(x)$ is the exponential integral function.[2] The particular solution required here follows immediately from Eq. (10b):

$$\psi_p(x,\mu) = -\frac{2}{c}\mu^{N-2}\exp(-x/\mu) \quad , \quad \mu \varepsilon (0,1) \quad ,$$

$$= 0, \text{ otherwise.} \qquad (12)$$

We should now like to consider source terms of the form

$$S(x,\mu) = x^\alpha \exp(-x/\eta) \quad , \quad \eta \not\varepsilon (-1,1) \quad , \quad \Lambda(\eta) \neq 0 \quad ,$$
$$\text{and } \alpha = 0, 1, 2, 3, \ldots \quad , \qquad (13)$$

where the usual dispersion function is given by[2]

$$\Lambda(\eta) = c\,R(\eta,\eta)\left[1 - \eta\tanh^{-1}\frac{1}{\eta}\right] + 1 - c \quad , \qquad (14)$$

with

$$R(\eta,\mu) = 1 + b(1-c)\eta\mu \quad . \qquad (15)$$

Rewriting Eq. (1) for this case,

$$\mu\frac{\partial}{\partial x}\psi_\alpha(\eta;x,\mu) + \psi_\alpha(\eta;x,\mu)$$
$$= \frac{c}{2}\int_{-1}^{1}(1 + b\mu\mu')\,\psi_\alpha(\eta;x,\mu')d\mu' + x^\alpha\exp(-x/\eta) \quad , \qquad (16)$$

we observe that $\psi_1(\eta;x,\mu)$ can be obtained by differentiating the lowest-order result:

$$\psi_1(\eta;x,\mu) = \eta^2\frac{\partial}{\partial\eta}\psi_0(\eta;x,\mu) \quad , \qquad (17)$$

and similarly

$$\psi_\alpha(\eta;x,\mu) = \left[\eta^2\frac{\partial}{\partial\eta}\right]^\alpha \psi_0(\eta;x,\mu) \quad . \qquad (18)$$

The required result for $\psi_0(\eta;x,\mu)$ can be readily obtained by proposing the form $F(\eta;\mu)\exp(-x/\eta)$; we find

$$\psi_0(\eta;x,\mu) = \frac{1}{\Lambda(\eta)}\left[\frac{\eta}{\eta-\mu} + cb\eta^2\left(1 - \eta\tanh^{-1}\frac{1}{\eta}\right)\right]\exp(-x/\eta) \quad ,$$
$$\eta \not\varepsilon (-1,1) \quad , \quad \Lambda(\eta) \neq 0 \quad . \qquad (19)$$

We note that for $b = 0$, Eq. (19) reduces to the expression reported by Freund,[6] who developed several particular solutions for the case of isotropic scattering.

In addition, we should like to illustrate the procedure used here to develop several particular solutions for the conservative anisotropic-scattering model. For this case we consider

$$\mu\frac{\partial}{\partial x}\psi(x,\mu) + \psi(x,\mu)$$
$$= \frac{1}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' + \frac{1}{2}b\mu j(x) + S(x) \quad , \qquad (20)$$

where the current $j(x)$ is given by

$$j(x) = \int_{-1}^{1}\psi(x,\mu')\,\mu' d\mu' \quad . \qquad (21)$$

Integrating Eq. (20) with respect to μ yields

$$\frac{d}{dx}j(x) = 2S(x) \quad , \qquad (22)$$

or, alternatively,

$$j(x) = 2\left[K + \int_0^x S(x')dx'\right] \quad , \qquad (23)$$

where K is a constant. Making use of this expression, we write Eq. (20) in the form

$$\mu\frac{\partial}{\partial x}\psi(x,\mu) + \psi(x,\mu) = \frac{1}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' + Q(x,\mu) \quad , \qquad (24)$$

with

$$Q(x,\mu) = S(x) + b\mu\left[K + \int_0^x S(x')dx'\right] \quad . \qquad (25)$$

If we now consider source functions of the form $S(x) = x^\alpha$, $\alpha = 0, 1, 2, 3 \ldots$, then $Q(x,\mu)$ becomes

$$Q(x,\mu) = x^\alpha + b\mu\left[K_\alpha + \frac{1}{(\alpha+1)}x^{\alpha+1}\right] \quad . \qquad (26)$$

For the moment, we view $Q(x,\mu)$ in Eq. (24) as an inhomogeneous source term, and thus seek a particular solution corresponding to the $Q(x,\mu)$ given by Eq. (26). Solutions $\psi_\alpha(x,\mu)$ to Eq. (24) for source functions of the form x^α have been derived by Lundquist and Horak,[3] and can be generated from the recursive relation

$$\psi_\alpha(x,\mu)$$
$$= I_\alpha(x,\mu) - 3\alpha!\sum_{\beta=2,4,6}^{\alpha}\frac{1}{(\alpha-\beta)!}\frac{1}{\beta+3}\psi_{\alpha-\beta}(x,\mu) \quad , \qquad (27)$$

[6] HANS-DIETER FREUND, *Atomkernenergie*, **14**, 222 (1969).

where

$$I_\alpha(x,\mu) = -3\alpha! \sum_{r=0}^{\alpha+2} (-1)^r \mu^r \frac{x^{\alpha+2-r}}{(\alpha+2-r)!} \quad . \tag{28}$$

Further, solutions $\overline{\psi}_\alpha(x,\mu)$ to Eq. (24) corresponding to the remaining portion of $Q(x,\mu)$,

$$b\,\mu \left[K_\alpha + \frac{x^{\alpha+1}}{\alpha+1}\right] \quad ,$$

are found to be

$$\overline{\psi}_\alpha(x,\mu) = b\left[K_\alpha \mu + \frac{x^{\alpha+2}}{(\alpha+1)(\alpha+2)}\right] \quad , \tag{29}$$

and thus, the complete particular solution we seek takes the form

$$\psi_p(x,\mu) = \psi_\alpha(x,\mu) + \overline{\psi}_\alpha(x,\mu) \tag{30}$$

This result must now be entered into Eq. (23) to yield the value of K_α. We find

$$K_\alpha = \left(1 - \frac{b}{3}\right)^{-1} \left[\frac{1}{2}\int_{-1}^{1} \psi_\alpha(x,\mu)\,\mu\,d\mu - \frac{x^{\alpha+1}}{\alpha+1}\right] \quad , \tag{31}$$

where $\psi_\alpha(x,\mu)$ is given by Eqs. (27) and (28). Although K_α can be expressed explicitly in terms of nested series, we prefer to write the result recursively in a manner similar to Lundquist and Horak's solution[3] for $\psi_\alpha(x,\mu)$:

$$\left(1 - \frac{b}{3}\right)K_\alpha = \frac{-x^{\alpha+1}}{\alpha+1} + \frac{1}{2}\int_{-1}^{1} I_\alpha(x,\mu)\,\mu\,d\mu - 3\alpha! \times \sum_{\beta=2,4,6}^{\alpha} \frac{1}{(\alpha-\beta)!}\,\frac{1}{\beta+3}\left[\left(1 - \frac{b}{3}\right)K_{\alpha-\beta} + \frac{x^{\alpha-\beta+1}}{\alpha-\beta+1}\right] \quad . \tag{32}$$

The fact, that for α even, $I_\alpha(0,\mu)$ is an even function of μ can be used to show that $K_\alpha = 0$, α even; further, Eq. (32) can be evaluated at $x = 0$ to yield a more tractable recursive relation for α odd:

$$K_\alpha = \frac{3\alpha!}{(\alpha+4)}\left(1 - \frac{b}{3}\right)^{-1} - 3\alpha! \sum_{\beta=2,4,6}^{\alpha} \frac{1}{(\alpha-\beta)!}\,\frac{1}{\beta+3}\,K_{\alpha-\beta} \quad , \quad \alpha \text{ odd}. \tag{33}$$

Finally we should like to comment on the fact that some of the particular solutions established here are generalized functions.[7] Since for physically meaningful applications, the complete solution $\psi(x,\mu)$ must not be a generalized function, then for these problems the distributional nature of the particular solutions should be negated by contributions to $\psi(x,\mu)$ from the complementary solution; though this has not been proven in general, several explicit examples have shown this to be true.

ACKNOWLEDGMENT

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[7] I. M. GEL'FAND and G. E. SHILOV, *Generalized Functions*, Vol. 1, Academic Press, New York, N.Y. (1964).