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Radiative Transfer in a Rayleigh-Scattering Atmosphere with True Absorption*

T. W. SCHNATZ AND C. E. SIEWERT
Department of Nuclear Engineering, North Carolina State University, Raleigh, North Carolina 27607

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The singular-eigenfunction-expansion technique is used to solve the equation of transfer for partially polarized light in a Rayleigh-scattering atmosphere with true absorption. The normal modes for the considered nonconservative vector equation of transfer are established; two discrete eigenvectors and two linearly independent continuum solutions are thus derived. Further, the necessary full-range completeness and orthogonality theorems are proved, so that all expansion coefficients can be determined explicitly, and, in order to illustrate the technique, an exact analytical solution for the infinite-medium Green's function is developed. Finally, a numerical tabulation of the required discrete eigenvalue, as a function of the single-scatter albedo, is given.

I. INTRODUCTION

In one of his classical papers on radiative transfer,[1] Chandrasekhar formulated explicitly the equations of transfer for the two components $I_l(\tau, \mu)$ and $I_r(\tau, \mu)$ of a polarized radiation field in a free-electron atmosphere; he also developed an approximate solution for the law of darkening appropriate to the considered Milne problem. This latter result was subsequently improved[2] as Chandrasekhar was able to observe the infinite limit of a discrete-ordinates procedure in order to establish a rigorous solution for the desired surface quantities.

More recently, as the study of neutron physics has developed, Wigner[3] has discussed a theory of neutron transport which takes into consideration the quantum mechanical effects of neutron polarization. The influence of neutron polarization on the scattering of fast neutrons by unpolarized nuclei has also been reported recently by Bell and Goad,[4] who used the P_l approximation to the transport solution.

Although most studies [for example, Refs. 5–8] of the scattering of polarized light have been based on Chandrasekhar's model,[9] the extension to include the effects of true absorption has been discussed briefly by Sobolev[10] and Simmons.[11] Further, Mullikin[12] recently extended his earlier work[13] on the conservative model and reported the results of a more general investigation, which accounted for true absorption by allowing the single-scatter albedo to be less than unity.

Since the principal interest relevant to many astrophysical studies of polarized light is in the evaluation of surface quantities, Chandrasekhar's invariance principles[9] have been widely used[5–8]; however, the singular-eigenfunction-expansion technique developed by Case[14] has been used to advantage by Siewert and Fraley[15] to construct rigorous analytical solutions, valid *anywhere* within the medium, to the Milne problem and other half-space problems. This latter method was also used by Mourad and Siewert[16] to establish full-range completeness and orthogonality theorems basic to the normal modes of a more general vector equation of transfer, also formulated by Chandrasekhar.[9] For this case, the mathematical model used to describe the scattering of polarized light by molecules was also shown to be appropriate for the theory of resonance line scattering.

Although Mourad and Siewert[16] did not find closed-form expressions for the more interesting half-range applications, the computational merits of their results have recently been confirmed for the half-space Milne problem.[17] The vector equation of transfer considered in Ref. 17 is inherently restricted to conservative media, but similar half-range methods should be available for the nonconservative model discussed here.

We consider, then, the equation of transfer

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \tfrac{1}{2} c \int_{-1}^{1} \mathbf{K}(\mu, \mu') \mathbf{I}(\tau, \mu') \, d\mu', \quad (1)$$

where the Rayleigh-scattering matrix is given by

$$\mathbf{K}(\mu, \mu') = \frac{3}{4} \begin{vmatrix} 2(1-\mu^2)(1-\mu'^2) + \mu'^2\mu^2 & \mu^2 \\ \mu'^2 & 1 \end{vmatrix}. \quad (2)$$

Here, τ is the optical variable, μ is the direction cosine (as measured from the *positive* τ axis) of the propagating radiation, and $c \in [0, 1]$ is the single-scatter albedo. Further, the desired intensities $I_l(\tau, \mu)$ and $I_r(\tau, \mu)$ are the two components of the vector $\mathbf{I}(\tau, \mu)$.

We prefer to make use of Sekera's[18] factorization

$$\mathbf{K}(\mu, \mu') = \tfrac{3}{4} \mathbf{Q}(\mu) \mathbf{Q}^{\mathrm{T}}(\mu'), \quad (3)$$

where the superscript "T" denotes the transpose operation, in order to write Eq. (1) in the form

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \tfrac{3}{8} c \mathbf{Q}(\mu) \int_{-1}^{1} \mathbf{Q}^{\mathrm{T}}(\mu') \mathbf{I}(\tau, \mu')\, d\mu', \tag{4a}$$

where

$$\mathbf{Q}(\mu) \stackrel{\text{DEF}}{=} \begin{vmatrix} \mu^2 & 2^{\frac{1}{2}}(1-\mu^2) \\ 1 & 0 \end{vmatrix}. \tag{4b}$$

II. EIGENVALUE SPECTRUM AND EIGENVECTORS

Since the development of the normal modes of Eq. (1) follows previously reported analysis of vector equations of transfer,[15,16,19,20] we should like only to summarize our results here.

Proposing solutions of the form

$$\mathbf{I}(\tau, \mu) = e^{-\tau/\eta} \mathbf{\Phi}(\eta, \mu), \tag{5}$$

we note that the eigenvalues η and associated eigenvectors $\mathbf{\Phi}(\eta, \mu)$ are to be determined from the reduced equation

$$(\eta - \mu)\mathbf{\Phi}(\eta, \mu) = \tfrac{3}{8} c\eta \mathbf{Q}(\mu) \int_{-1}^{1} \mathbf{Q}^{\mathrm{T}}(\mu') \mathbf{\Phi}(\eta, \mu')\, d\mu'. \tag{6}$$

If we now introduce the normalization

$$\mathbf{M}(\eta) \stackrel{\text{DEF}}{=} \int_{-1}^{1} \mathbf{Q}^{\mathrm{T}}(\mu) \mathbf{\Phi}(\eta, \mu)\, d\mu, \tag{7}$$

then clearly the discrete eigenvectors are given by

$$\mathbf{\Phi}(\eta, \mu) = [3c\eta/8(\eta - \mu)]\mathbf{Q}(\mu)\mathbf{M}(\eta), \tag{8}$$

where

$$\left(\mathbf{I} - \tfrac{3}{8} c\eta \int_{-1}^{1} \mathbf{Q}^{\mathrm{T}}(\mu)\mathbf{Q}(\mu) \frac{d\mu}{\eta - \mu}\right)\mathbf{M}(\eta) = \mathbf{0}; \tag{9}$$

here, $\mathbf{I}$ denotes the identity matrix.

Thus, we obtain the discrete eigenvalue spectrum from the zeros of the dispersion function $\Lambda(z)$ defined as

$$\Lambda(z) \stackrel{\text{DEF}}{=} 8 \det \left(\mathbf{I} + \tfrac{3}{8} cz \int_{-1}^{1} \mathbf{Q}^{\mathrm{T}}(\mu)\mathbf{Q}(\mu) \frac{d\mu}{\mu - z}\right), \tag{10}$$

where the factor 8 has been included in order to obtain the more convenient form

$$\Lambda(z) = \Lambda_1(z)\Lambda_2(z) + 12z^2(1 - c)\Lambda_0(z). \tag{11}$$

Here,

$$\Lambda_\alpha(z) = (-1)^\alpha + 3(1 - z^2)\Lambda_0(z) - (-1)^\alpha 3z^2(1 - c), \quad \alpha = 1 \text{ or } 2, \tag{12a}$$

and

$$\Lambda_0(z) = 1 + \tfrac{1}{2} cz \int_{-1}^{1} \frac{d\mu}{\mu - z}. \tag{12b}$$

We note from Eq. (10) that $\Lambda(z)$ is a function analytic in the complex plane cut from -1 to 1 and, as we discuss in Sec. VI, the argument principle[21] can be used to show that $\Lambda(z)$ has only two zeros, which appear as a pair $z = \pm\eta_0$ of real eigenvalues. Thus, there are two discrete eigenvectors $\mathbf{\Phi}_\pm(\mu)$ which, after judiciously normalizing the vector $\mathbf{M}(\eta)$, we write in the tractable form

$$\mathbf{\Phi}_\pm(\mu) = \tfrac{3}{2} c\eta_0 \frac{1}{\eta_0 \mp \mu} \begin{vmatrix} \Lambda_2(\eta_0)(1 - \mu^2) + 2\eta_0^2(1 - c) \\ 2\eta_0^2(1 - c) \end{vmatrix}. \tag{13}$$

For the continuum $\eta \in (-1, 1)$, we express the solution to Eq. (6) in the form

$$\mathbf{\Phi}(\eta, \mu) = \tfrac{3}{8} c\eta \left(\frac{P}{\eta - \mu} + \lambda(\eta)\delta(\eta - \mu)\right)\mathbf{Q}(\mu)\mathbf{M}(\eta), \tag{14}$$

where $\lambda(\eta)$ is as yet unspecified; further, the symbol P is used to denote that all ensuing integrals over η or μ are to be evaluated in the Cauchy principal-value sense, and $\delta(x)$ is the Dirac δ function. Multiplying Eq. (14) by $\mathbf{Q}^{\mathrm{T}}(\mu)$ and integrating over μ from -1 to 1 yields a homogeneous equation for $\mathbf{M}(\eta)$. The compatibility condition thus yields a quadratic equation in $\lambda(\eta)$. We find two independent solutions for $\lambda(\eta)$ and thus establish two linearly independent continuum eigenvectors. Choosing a convenient normalization for $\mathbf{M}(\eta)$, we write

$$\mathbf{\Phi}_1(\eta, \mu) = \begin{vmatrix} \tfrac{3}{2} c\eta(1 - \eta^2)(1 - \mu^2)P/(\eta - \mu) \\ \quad + [(1 - \eta^2)\lambda_1(\eta) + 2\eta^2(1 - c)]\delta(\eta - \mu) \\ - 2\eta^2(1 - c)\delta(\eta - \mu) \end{vmatrix}, \tag{15a}$$

$$\mathbf{\Phi}_2(\eta, \mu) = \begin{vmatrix} \tfrac{3}{2} c\eta(1 - \eta^2)P/(\eta - \mu) + \lambda_1(\eta)\delta(\eta - \mu) \\ \tfrac{3}{2} c\eta(1 - \eta^2)P/(\eta - \mu) + \lambda_2(\eta)\delta(\eta - \mu) \end{vmatrix}, \tag{15b}$$

$$\lambda_\alpha(\eta) = (-1)^\alpha + 3(1 - \eta^2)\lambda_0(\eta) - (-1)^\alpha 3\eta^2(1 - c), \tag{16a}$$

$$\lambda_0(\eta) = 1 - c\eta \tanh^{-1}(\eta). \tag{16b}$$

Having determined the eigenvectors, we can express the general solution of Eq. (1) as a linear sum of the independent solutions:

$$\mathbf{I}(\tau, \mu) = A_+ e^{-\tau/\eta_0}\mathbf{\Phi}_+(\mu) + A_- e^{\tau/\eta_0}\mathbf{\Phi}_-(\mu) + \int_{-1}^{1} [A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)] e^{-\tau/\eta}\, d\eta, \tag{17}$$

where $A_\pm$, $A_1(\eta)$, and $A_2(\eta)$ are the arbitrary coefficients to be determined from the boundary conditions of a suitably defined physical problem.

We note that for the special case $c = 1$, the dispersion relation given by Eq. (11) and the established eigenvectors reduce to forms equivalent to those obtained by Siewert and Fraley.[15]

III. FULL-RANGE COMPLETENESS OF THE EIGENVECTORS

Theorem 1: The eigenvectors $\mathbf{\Phi}_\pm(\mu)$, $\mathbf{\Phi}_1(\eta, \mu)$, and $\mathbf{\Phi}_2(\eta, \mu)$ are complete on the full range, $\mu \in (-1, 1)$, in the sense that an arbitrary 2-component vector $\mathbf{\Psi}(\mu)$ satisfying the Hölder condition for $\mu \in (-1, 1)$ can be expanded in the form

$$\mathbf{\Psi}(\mu) = A_+\mathbf{\Phi}_+(\mu) + A_-\mathbf{\Phi}_-(\mu) + \int_{-1}^{1} A_1(\eta)\mathbf{\Phi}_1(\eta, \mu)\, d\eta + \int_{-1}^{1} A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)\, d\eta, \quad \mu \in (-1, 1). \tag{18}$$

In order to prove the theorem, we use the methods of Muskhelishvili[22] to convert Eq. (18) to the equivalent Riemann–Hilbert problem

$$\mu(1 - \mu^2)\mathbf{\Psi}'(\mu) = \mathbf{\Lambda}^+(\mu)\mathbf{N}^+(\mu) - \mathbf{\Lambda}^-(\mu)\mathbf{N}^-(\mu), \quad \mu \in (-1, 1), \tag{19}$$

where

$$\mathbf{\Psi}'(\mu) = \mathbf{\Psi}(\mu) - A_+\mathbf{\Phi}_+(\mu) - A_-\mathbf{\Phi}_-(\mu) \tag{20}$$

and

$$\mathbf{N}(z) = \frac{1}{2\pi i}\int_{-1}^{1} \eta(1 - \eta^2)\mathbf{A}(\eta)\frac{d\eta}{\eta - z}\,; \tag{21}$$

here, $\mathbf{A}(\eta)$ is a vector with components $A_1(\eta)$ and $A_2(\eta)$. We note that $\mathbf{N}(z)$ is analytic in the complex plane cut from -1 to 1. Also, it vanishes at least as fast as $1/z$ when z increases without bound. Further, the boundary values of $\mathbf{N}(z)$ as z approaches the cut from above (+) and below (−) can be shown to satisfy the following relations deducible from the Plemelj formulas[22]:

$$\pi i[\mathbf{N}^+(\mu) + \mathbf{N}^-(\mu)] = \int_{-1}^{1} \eta(1 - \eta^2)\mathbf{A}(\eta)\frac{P}{\eta - \mu}\, d\eta \tag{22a}$$

and

$$\mathbf{N}^+(\mu) - \mathbf{N}^-(\mu) = \mu(1 - \mu^2)\mathbf{A}(\mu). \tag{22b}$$

In establishing Eq. (19), we introduced the matrix

$$\mathbf{\Lambda}(z) \overset{\mathrm{DEF}}{=} \begin{vmatrix} (1 - z^2)\Lambda_1(z) + 2z^2(1 - c) & \Lambda_1(z) \\ -2z^2(1 - c) & \Lambda_2(z) \end{vmatrix}. \tag{23}$$

It can be seen from Eqs. (12) and Eq. (16) that the functions $\Lambda_\alpha(z)$, $\alpha = 1$ and 2, are also analytic in the complex plane cut from -1 to 1 and that the boundary values $\Lambda_\alpha^\pm(\mu)$ obey

$$\Lambda_\alpha^+(\mu) + \Lambda_\alpha^-(\mu) = 2\lambda_\alpha(\mu), \quad \alpha = 1 \text{ or } 2, \tag{24a}$$

and

$$\Lambda_\alpha^+(\mu) - \Lambda_\alpha^-(\mu) = 3\pi i c\mu(1 - \mu^2), \quad \alpha = 1 \text{ or } 2. \tag{24b}$$

Defining a vector $\mathbf{P}(z)$ with components $P_1(z)$ and $P_2(z)$ as

$$\mathbf{P}(z) = \mathbf{\Lambda}(z)\mathbf{N}(z) - \frac{1}{2\pi i}\int_{-1}^{1} \mu(1 - \mu^2)\mathbf{\Psi}'(\mu)\frac{d\mu}{\mu - z}, \tag{25}$$

we note that $\mathbf{P}(z)$ is analytic in the complex plane cut from -1 to 1. It follows from Eq. (19) that $\mathbf{P}(z)$ is continuous across the cut and thus is an entire function. If we consider the behavior of the functions $\Lambda_\alpha(z)$, $\alpha = 0$, 1, and 2, as z tends to infinity, we observe that

$$\Lambda_0(z) \sim (1 - c) - c/3z^2 - c/5z^4, \quad z \to \infty, \tag{26a}$$

$$\Lambda_1(z) \sim 2(1 - c) - 2c/5z^2, \quad z \to \infty, \tag{26b}$$

and

$$\Lambda_2(z) \sim 2(2 - c) - 6z^2(1 - c), \quad z \to \infty. \tag{26c}$$

The above results may be employed to deduce the behavior of $\mathbf{\Lambda}(z)$ for large z:

$$\mathbf{\Lambda}(z) \sim \begin{vmatrix} \frac{2}{5}(5 - 4c) & 2(1 - c) \\ -2z^2(1 - c) & -6z^2(1 - c) \end{vmatrix}, \quad z \to \infty, \tag{27}$$

and thus, from Eqs. (25) and (21), we conclude that, as z tends to infinity, $P_1(z)$ vanishes while $P_2(z)$ has a first-order pole. Liouville's theorem[23] then requires that $P_2(z)$ be a first-order polynomial, whereas $P_1(z)$ must be identically zero. Thus, we find

$$\mathbf{P}(z) = \begin{vmatrix} 0 \\ a + bz \end{vmatrix}, \tag{28}$$

where a and b are arbitrary constants.

Equation (25) can now be solved for $\mathbf{N}(z)$ to yield

$$\mathbf{N}(z) = \mathbf{\Lambda}^{-1}(z)\left(\frac{1}{2\pi i}\int_{-1}^{1} \mu(1 - \mu^2)\mathbf{\Psi}'(\mu)\frac{d\mu}{\mu - z} + \mathbf{P}(z)\right), \tag{29}$$

where the inverse of $\mathbf{\Lambda}(z)$ is given by

$$\mathbf{\Lambda}^{-1}(z) = \frac{1}{(1-z^2)\Lambda(z)} \times \begin{vmatrix} \Lambda_2(z) & -\Lambda_1(z) \\ 2z^2(1-c) & (1-z^2)\Lambda_1(z) + 2z^2(1-c) \end{vmatrix}. \tag{30}$$

Since $\Lambda(z)$ has zeros at $z = \pm\eta_0$, we note that $\mathbf{N}(z)$ is not a holomorphic function in the complex plane cut from -1 to 1 unless we impose on $\mathbf{\Psi}'(\mu)$ the two constraints

$$\begin{vmatrix} \Lambda_2(\eta_0) & -\Lambda_1(\eta_0) \\ 2\eta_0^2(1-c) & (1-\eta_0^2)\Lambda_1(\eta_0) + 2\eta_0^2(1-c) \end{vmatrix} \times \left(\frac{1}{2\pi i}\int_{-1}^{1}\mu(1-\mu^2)\mathbf{\Psi}'(\mu)\frac{d\mu}{\mu \mp \eta_0} + \mathbf{P}(\pm\eta_0)\right) = \mathbf{0}. \tag{31}$$

In addition, we observe that $\mathbf{\Lambda}(z)$ is singular at the branch points $z = \pm 1$, so that we must carefully investigate the end-point[22] behavior of $\mathbf{N}(z)$. Observing the limits as z tends to ± 1 in Eq. (29), we obtain

$$\lim_{z\to\pm1}\mathbf{N}(z) \sim \{(1-z^2)[-(2-3c)^2 + 12(1-c)\Lambda_0(z)]\}^{-1} \times \begin{vmatrix} -(2-3c) & -(2-3c) \\ 2(1-c) & 2(1-c) \end{vmatrix} \times \left(\frac{1}{2\pi i}\int_{-1}^{1}\mu(1-\mu^2)\mathbf{\Psi}'(\mu)\frac{d\mu}{\mu \mp 1} + \mathbf{P}(\pm 1)\right). \tag{32}$$

The singularities at $z = \pm 1$ introduced by the factor $(1-z^2)^{-1}$ in the above equation are termed special end-points by Muskhelishvili,[22] and, in order for $\mathbf{N}(z)$ to have the proper end-point behavior, we impose the additional constraints

$$\begin{vmatrix} 1 \\ 1 \end{vmatrix}^{\mathrm{T}}\left[\frac{1}{2\pi i}\int_{-1}^{1}\mu(1\pm\mu)\mathbf{\Psi}'(\mu)\,d\mu - \begin{vmatrix} 0 \\ \pm a + b \end{vmatrix}\right] = 0. \tag{33}$$

Equation (33) can be solved immediately for a and b to yield

$$a = \frac{1}{2\pi i}\int_{-1}^{1}\mu^2 \begin{vmatrix} 1 \\ 1 \end{vmatrix}^{\mathrm{T}}\mathbf{\Psi}'(\mu)\,d\mu \tag{34a}$$

and

$$b = \frac{1}{2\pi i}\int_{-1}^{1}\mu \begin{vmatrix} 1 \\ 1 \end{vmatrix}^{\mathrm{T}}\mathbf{\Psi}'(\mu)\,d\mu. \tag{34b}$$

Having determined $\mathbf{P}(z)$, we investigate more thoroughly the original constraints on $\mathbf{\Psi}'(\mu)$ as given by Eq. (31). Rewriting Eq. (31) as two separate equations, we observe that

$$\begin{vmatrix} \Lambda_2(\eta_0) \\ -\Lambda_1(\eta_0) \end{vmatrix}^{\mathrm{T}} \times \left(\int_{-1}^{1}\mu(1-\mu^2)\mathbf{\Psi}'(\mu)\frac{d\mu}{\mu\mp\eta_0} + 2\pi i\mathbf{P}(\pm\eta_0)\right) = 0 \tag{35a}$$

and

$$\begin{vmatrix} 2\eta_0^2(1-c) \\ (1-\eta_0^2)\Lambda_1(\eta_0) + 2\eta_0^2(1-c) \end{vmatrix}^{\mathrm{T}} \times \left(\int_{-1}^{1}\mu(1-\mu^2)\mathbf{\Psi}'(\mu)\frac{d\mu}{\mu\mp\eta_0} + 2\pi i\,\mathbf{P}(\pm\eta_0)\right) = 0. \tag{35b}$$

The above expressions can be rearranged to yield

$$\int_{-1}^{1}\mu\mathbf{\Phi}_{\pm}^{\mathrm{T}}(\mu)\mathbf{\Psi}'(\mu)\,d\mu + \Delta\int_{-1}^{1}\mu(\mu\pm\eta_0)\begin{vmatrix} 1 \\ 1 \end{vmatrix}^{\mathrm{T}}\mathbf{\Psi}'(\mu)\,d\mu - 2\pi i\Delta\begin{vmatrix} 0 \\ 1 \end{vmatrix}^{\mathrm{T}}\mathbf{P}(\pm\eta_0) = 0, \tag{36}$$

where

$$\Delta = 3c\eta_0^3(1-c)/(1-\eta_0^2). \tag{37}$$

The results given by Eqs. (34) for a and b can now be introduced into Eq. (36), thus reducing the constraints on $\mathbf{\Psi}'(\mu)$ to the explicit form

$$\int_{-1}^{1}\mu\mathbf{\Phi}_{\pm}^{\mathrm{T}}(\mu)\mathbf{\Psi}'(\mu)\,d\mu = 0. \tag{38}$$

In general, this condition is not met; however, noting $\mathbf{\Psi}'(\mu)$ as given by Eq. (20), we can determine A_+ and A_- such that Eq. (38) is satisfied. With A_+ and A_- so established and a and b given by Eqs. (34), the result for $\mathbf{N}(z)$ expressed by Eq. (29) exhibits the proper analytic properties; the completeness theorem is thus proved.

Although the proof of Theorem 1 can be pursued to yield explicit expressions for all expansion coefficients $A_\pm$, $A_1(\eta)$, and $A_2(\eta)$, we prefer to use the alternative full-range orthogonality theorem developed in the next section to establish these results.

IV. ORTHOGONALITY, NORMALIZATION INTEGRALS, AND ADJOINT FUNCTIONS

Theorem 2: The eigenvectors $\mathbf{\Phi}_\pm(\mu)$, $\mathbf{\Phi}_1(\eta, \mu)$, and $\mathbf{\Phi}_2(\eta, \mu)$ are orthogonal on the full range, with respect to weight function μ, i.e.,

$$\int_{-1}^{1}\mu\mathbf{\Phi}_i^{\mathrm{T}}(\eta', \mu)\mathbf{\Phi}_j(\eta, \mu)\,d\mu = 0, \quad \eta \neq \eta', \quad i, j = +, -, 1, \text{ or } 2. \tag{39}$$

We begin the proof by premultiplying Eq. (6) by $\mathbf{\Phi}^{\mathrm{T}}(\eta', \mu)/\eta$. Equation (6) with η replaced by η' is then transposed and postmultiplied by $\mathbf{\Phi}(\eta, \mu)/\eta'$. The resulting equations are integrated over μ from -1 to I and subtracted to yield

$$\left(\frac{1}{\eta'} - \frac{1}{\eta}\right)\int_{-1}^{1} \mu \mathbf{\Phi}^{\mathrm{T}}(\eta', \mu)\mathbf{\Phi}(\eta, \mu)\, d\mu = 0. \quad (40)$$

Thus, the proof is established. However, some difficulty still remains, since the continuum eigenvectors are degenerate. A vanishing scalar product between the continuum eigenvectors is not guaranteed by the theorem. In fact, if we define

$$\delta(\eta - \eta')\eta M_{ij}(\eta) \overset{\mathrm{DEF}}{=} \int_{-1}^{1} \mu \mathbf{\Phi}_i^{\mathrm{T}}(\eta', \mu)\mathbf{\Phi}_j(\eta, \mu)\, d\mu,$$
$$\eta \text{ and } \eta' \in (-1, 1), \quad i, j = 1 \text{ or } 2, \quad (41)$$

we find upon evaluating the above integrals that

$$M_{12}(\eta) = M_{21}(\eta) = (1 - \eta^2)\Lambda_1^+(\eta)\Lambda_1^-(\eta) - 4\eta^2(1 - c)[1 - 3\eta^2(1 - c)], \quad (42a)$$

$$M_{11}(\eta) = (1 - \eta^2)^2\Lambda_1^+(\eta)\Lambda_1^-(\eta) + 4\eta^2(1 - c) \times [(1 - \eta^2)\lambda_1(\eta) + 2\eta^2(1 - c)], \quad (42b)$$

and

$$M_{22}(\eta) = \Lambda_1^+(\eta)\Lambda_1^-(\eta) + \Lambda_2^+(\eta)\Lambda_2^-(\eta). \quad (42c)$$

A Schmidt-type procedure can now be used to develop a set of adjoint eigenvectors such that, if we define the scalar product as

$$\langle i \mid j\rangle \overset{\mathrm{DEF}}{=} \int_{-1}^{1} \mu \mathbf{\Phi}_i^{\mathrm{T}\dagger}(\eta, \mu)\mathbf{\Phi}_j(\eta, \mu)\, d\mu,$$
$$i, j = +, -, 1, \text{ and } 2, \quad (43)$$

where the adjoint vectors are defined as

$$\mathbf{\Phi}_{\pm}^{\dagger}(\mu) = \mathbf{\Phi}_{\pm}(\mu), \quad (44a)$$

$$\mathbf{\Phi}_1^{\dagger}(\eta, \mu) = M_{22}(\eta)\mathbf{\Phi}_1(\eta, \mu) - M_{12}(\eta)\mathbf{\Phi}_2(\eta, \mu), \quad (44b)$$

and

$$\mathbf{\Phi}_2^{\dagger}(\eta, \mu) = M_{11}(\eta)\mathbf{\Phi}_2(\eta, \mu) - M_{21}(\eta)\mathbf{\Phi}_1(\eta, \mu), \quad (44c)$$

then the desired orthogonality property is established, viz.,

$$\langle i \mid j\rangle = 0, \quad i \neq j. \quad (45)$$

With the definitions given by Eqs. (44), the necessary normalization integrals can be evaluated straightforwardly. We find

$$\langle \pm \mid \pm\rangle = M_{\pm} \quad (46a)$$

and

$$\langle 1 \mid 1\rangle = \langle 2 \mid 2\rangle = M(\eta)\delta(\eta - \eta'), \quad (46b)$$

where

$$M_{\pm} = \pm\tfrac{3}{8}c\eta_0^3\mathbf{M}^{\mathrm{T}}(\eta_0) \times \left(\tfrac{3}{8}c\int_{-1}^{1}\mathbf{Q}^{\mathrm{T}}(\mu)\mathbf{Q}(\mu)\frac{d\mu}{(\mu - \eta_0)^2} - \frac{1}{\eta_0^2}\mathbf{I}\right)\mathbf{M}(\eta_0) \quad (47a)$$

and

$$M(\eta) = \eta(1 - \eta^2)^2\Lambda^+(\eta)\Lambda^-(\eta). \quad (47b)$$

We now proceed to determine explicit expressions for $A_{\pm}$, $A_1(\eta)$, and $A_2(\eta)$ in the full-range expansion given by Eq. (18). If we multiply Eq. (18) by $\mu\mathbf{\Phi}_{\pm}^{\mathrm{T}\dagger}(\mu)$ and integrate over μ from -1 to 1, we find

$$A_{\pm}M_{\pm} = \int_{-1}^{1} \mu\mathbf{\Phi}_{\pm}^{\mathrm{T}\dagger}(\mu)\mathbf{\Psi}(\mu)\, d\mu, \quad (48a)$$

where we have utilized the results given by Eqs. (45) and (46). Similarly, we take scalar products of Eq. (18) with the adjoint vectors $\mathbf{\Phi}_1^{\dagger}(\eta, \mu)$ and $\mathbf{\Phi}_2^{\dagger}(\eta, \mu)$ to find

$$A_i(\eta)M(\eta) = \int_{-1}^{1} \mu\mathbf{\Phi}_i^{\mathrm{T}\dagger}(\eta, \mu)\mathbf{\Psi}(\mu)\, d\mu, \quad i = 1 \text{ or } 2. \quad (48b)$$

We note that the above results are identical with those which follow from the completeness proof given in Sec. III.

V. THE INFINITE-MEDIUM GREEN'S FUNCTION

We now illustrate the utility of the above theory by constructing a solution for the infinite-medium Green's function. We seek a bounded solution to the equation

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \tfrac{3}{8}c\mathbf{Q}(\mu)\int_{-1}^{1}\mathbf{Q}^{\mathrm{T}}(\mu')\mathbf{I}(\tau, \mu')\, d\mu' + \mathbf{S}(\tau, \mu), \quad (49)$$

where $\mathbf{S}(\tau, \mu)$ is defined as

$$\mathbf{S}(\tau, \mu) = \delta(\tau)\begin{vmatrix} s_l\delta(\mu - \mu_l) \\ s_r\delta(\mu - \mu_r) \end{vmatrix},$$
$$\mu, \quad \mu_l, \quad \text{and} \quad \mu_r \in (-1, 1). \quad (50)$$

Here, s_l and s_r represent the source strengths of each of the two polarization states. In the usual manner,[24] we neglect the source term in Eq. (49) and require the solutions of the resulting homogeneous equation to satisfy the "jump" boundary condition

$$\mu[\mathbf{I}(0, \mu_l, \mu_r; 0^+, \mu) - \mathbf{I}(0, \mu_l, \mu_r; 0^-, \mu)] = \begin{vmatrix} s_l\delta(\mu - \mu_l) \\ s_r\delta(\mu - \mu_r) \end{vmatrix}, \quad (51)$$

where the argument list has been extended to include the location of the source as well as the parameters μ_l and μ_r.

The solution is separated into two parts which are respectively bounded in the two half-spaces $\tau \gtrless 0$. Thus, we write

$$\mathbf{I}(0, \mu_l, \mu_r; \tau, \mu) = A_+\mathbf{\Phi}_+(\mu)e^{-\tau/\eta_0} + \int_0^1 [A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]e^{-\tau/\eta}\, d\eta, \quad \tau > 0, \tag{52a}$$

$$\mathbf{I}(0, \mu_l, \mu_r; \tau, \mu) = -A_-\mathbf{\Phi}_-(\mu)e^{\tau/\eta_0} - \int_{-1}^0 [A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]e^{-\tau/\eta}\, d\eta, \quad \tau < 0, \tag{52b}$$

where the negative signs appearing in Eq. (52b) were included for convenience. Applying the "jump" boundary condition given by Eq. (51), we obtain the full-range expansion

$$\frac{1}{\mu}\begin{vmatrix} s_l\delta(\mu - \mu_l) \\ s_r\delta(\mu - \mu_r) \end{vmatrix} = A_+\mathbf{\Phi}_+(\mu) + A_-\mathbf{\Phi}_-(\mu) + \int_{-1}^1 A_1(\eta)\mathbf{\Phi}_1(\eta, \mu)\, d\eta + \int_{-1}^1 A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)\, d\eta, \quad \mu,\ \mu_l,\ \text{and}\ \mu_r \in (-1, 1). \tag{53}$$

It is clear that all required expansion coefficients can now be obtained simply by taking scalar products of Eq. (53) with the appropriate adjoint vectors. Thus,

$$A_\pm = \frac{1}{M_\pm}\int_{-1}^1 \mathbf{\Phi}_\pm^{\mathrm{T}\dagger}(\mu)\begin{vmatrix} s_l\delta(\mu - \mu_l) \\ s_r\delta(\mu - \mu_r) \end{vmatrix} d\mu \tag{54a}$$

and

$$A_i(\eta) = \frac{1}{M(\eta)}\int_{-1}^1 \mathbf{\Phi}_i^{\mathrm{T}\dagger}(\eta, \mu)\begin{vmatrix} s_l\delta(\mu - \mu_l) \\ s_r\delta(\mu - \mu_r) \end{vmatrix} d\mu, \quad i = 1 \text{ or } 2. \tag{54b}$$

Since the integrals above are elementary, we write the expanded form only for the discrete coefficient:

$$A_\pm = \frac{3c\eta_0}{2M_\pm}\left(s_l\frac{[\Lambda_2(\eta_0)(1 - \mu_l^2) + 2\eta_0^2(1 - c)]}{\eta_0 \mp \mu_l} + s_r\frac{2\eta_0^2(1 - c)}{\eta_0 \mp \mu_r}\right). \tag{55}$$

Having established all of the unknown expansion coefficients, we consider the construction of the Green's function to be completed.

VI. DISCRETE EIGENVALUES

One of the most important parameters in the above formalism is the discrete eigenvalue η_0. As pointed out in Sec. II, η_0 is a real number greater than unity and is the positive zero of the dispersion function $\Lambda(z)$ given by Eq. (11). Sobolev[10] has calculated η_0 for several values of c. In Table I below, we present values for the discrete eigenvalue as a function of the single-scatter albedo c; for convenience, we include those values reported by Sobolev.[10] The calculation was performed on the IBM 360 Model 75 digital computer, using Newton's iteration technique,[25] and the results are believed to be accurate to within $\pm 5 \times 10^{-7}$.

According to the argument principle,[21] the change in the argument as z traverses some closed contour of a function analytic inside that contour is 2π times the number of enclosed zeros. The dispersion function given by Eq. (11) is analytic in the entire plane cut from -1 to 1, and it can be shown that

$$\lim_{z\to\infty} \Lambda(z) = 8(1 - c)(1 - \tfrac{7}{10}c). \tag{56}$$

We choose as our contour one part designated γ which encompasses, but is arbitrarily close to, the cut, and a second part termed R at infinity. Noting Eq. (56), we see that the change in the argument of $\Lambda(z)$ as z traverses R is zero. On γ, we must consider the boundary values of $\Lambda(z)$, namely,

$$\Lambda^\pm(\mu) = -1 + 9(1 - \mu^2)^2[\lambda_0^2(\mu) - \tfrac{1}{4}\pi^2c^2\mu^2] + 3\mu^2(1 - c)[4\lambda_0(\mu) - 3\mu^2(1 - c) + 2] \pm i\pi c\mu[9(1 - \mu^2)^2\lambda_0(\mu) + 6\mu^2(1 - c)]. \tag{57}$$

Since $\Lambda(-z) = \Lambda(z)$ and the complex conjugate of $\Lambda^+(\mu)$ equals $\Lambda^-(\mu)$, it is sufficient to determine the

TABLE I. Table of η_0.

c	η_0 Sobolev[10]	η_0 Present work
0.1		1.000001
0.2		1.000709
0.3		1.007230
0.4		1.025904
0.5		1.062363
0.581	1.11	1.110624
0.6		1.125231
0.7		1.232743
0.712	1.25	1.250329
0.798	1.43	1.427842
0.8		1.433478
0.861	1.67	1.665949
0.9		1.924622

argument change of $\Lambda^+(\mu)$ along that part of γ for which $\mu \in (0, 1)$; the total change will then be four times the calculated value. An investigation of both the imaginary and real parts of $\Lambda^+(\mu)$, $\mu \in (0, 1)$, reveals a change in the argument of π. Thus, the total change will be 4π and the number of enclosed zeros two. Further, since $\Lambda^*(z) = \Lambda(z^*)$ and $\Lambda(z) = \Lambda(-z)$, it follows that the zeros occur as a $\pm$ pair, which upon closer inspection can be shown to be real for $c \in (0, 1)$ and to coalesce at infinity for $c = 1$. Here the symbol * denotes complex conjugate.

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