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SPECTRAL-LINE FORMATION BY NONCOHERENT SCATTERING

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ABSTRACT

The (singular) eigenfunction-expansion technique is used to construct solutions to a general class of nongray-model problems in the theory of radiative transfer for plane-parallel media. The model considered allows for completely incoherent scattering and continuum absorption, both of which are assumed to be independent of depth. The radiation intensity is obtained for an arbitrary space-dependent Planck function in a semi-infinite atmosphere. In addition, simplified results are obtained for the special cases where the Planck function varies either linearly or exponentially with the optical depth in the continuum. Solutions are obtained in terms of H-functions which can be calculated once the frequency-dependent scattering and absorption coefficients are specified.

I. INTRODUCTION

Considerable attention has been given to the development of exact analytical techniques for solving nongray-model problems in the study of line formation (Busbridge 1953, 1955; Stibbs 1953; Busbridge and Stibbs 1954). These earlier papers were based on the condition of steady-state radiative equilibrium and included the effects of absorption in the continuum as well as the line, but were restricted in that the Planck function was assumed to vary linearly with the optical depth in the continuum. Further, since emphasis was placed on calculating the intensity of radiation escaping from the atmosphere, the formalism did not lead immediately to explicit results for the source function within the medium.

The more general resolvent method (Sobolev 1959), as utilized by Ivanov (1962, 1963) and Heaslet and Warming (1968), is an exact analytical technique which has been used to obtain source functions everywhere in the stellar interior, as well as the exit radiation. These solutions were obtained, however, by neglecting the absorption in the continuum.

The purpose of this paper is to illustrate that the (singular) eigenfunction-expansion method (Case 1960), used frequently in neutron-transport theory, yields an exact analytical solution to a problem of radiative transfer which is of interest in line-formation studies. The final results lead to a formal solution which gives the source function everywhere in the interior, as well as an expression for the emergent intensity. Though the procedure can be regarded as an alternative choice to the resolvent method, the problem treated here is more general than those previously analyzed with the resolvent method since continuum absorption is allowed. Hence the theoretical analysis can be considered to complement the numerical results of Hummer (1968), who investigated the problem by means of the discrete-ordinates method, and the solutions obtained by the discrete-ordinates and kernel-approximation methods (Hummer and Rybicki 1967).

Both the normal-mode expansion (eigenfunction) technique and the resolvent method can also be used to obtain formulae in thick atmospheres where numerical discrete-ordinate approaches may converge slowly. Furthermore, either of these analytical methods may be used to provide bench-mark solutions to model problems in order to evaluate the accuracy of detailed computer codes, such as those of Rybicki and Hummer (1967) and the references cited therein.

The eigenfunction-expansion procedure has some analytical advantage over the resolvent method in that the approach is more closely related to that classically used to solve boundary-value problems: After a particular solution appropriate to the Planck function is added to a sufficiently general solution of the homogeneous equation of transfer, the resulting complete solution is constrained to meet the boundary conditions of the problem. Furthermore, it will be shown that the eigenfunction procedure is able to account for the spatially dependent Planck function in either of two ways: For particularly simple approximations to the Planck function, such as those which increase linearly or exponentially with the optical depth in the continuum, it is possible to obtain the required particular solution to the equation of transfer by elementary methods; for other situations, it may be necessary to account for the Planck function by first constructing the infinite-medium Green's function and then using superposition of the localized sources to account for the general Planck function.

In § II the equation of transfer is considered, and a change of variables is made such that the space-angle frequency-dependent radiation intensity can be written in terms of a new function depending upon space and only one other variable. The exact solution to the homogeneous equation of transfer is formally developed in § III, and § IV is devoted to a determination of the required particular solution for a generalized emission spectrum. Finally, particular solutions are used in § V for the special cases in which the Planck function varies linearly or exponentially with optical depth.

II. BASIC ANALYSIS

The transfer equation considered here is written as (Hummer 1968)

$$\mu \frac{\partial}{\partial \tau} I_x(\tau, \mu) = [\phi(x) + \beta][S_x(\tau) - I_x(\tau, \mu)] , \tag{1}$$

where $S_x(\tau)$ is the source function of the medium,

$$[\phi(x) + \beta]S_x(\tau) = \tfrac{1}{2}(1 - \epsilon)\phi(x) \int_{-\infty}^{\infty} \phi(x') \int_{-1}^{1} I_{x'}(\tau, \mu) d\mu dx' + [\rho\beta + \epsilon\phi(x)]B(\tau) , \tag{2}$$

and $B(\tau)$ is the Planck blackbody function at the line center at the local electron temperature. Here μ is the direction cosine of the photon-propagation vector, as measured from the *inward* normal to the free surface, and τ is the mean optical depth in the line:

$$d\tau = k_L(z)dz , \tag{3}$$

where $k_L(z)$ denotes the frequency-averaged line-scattering coefficient per unit volume at depth z. The variable x measures frequencies from the line center in a convenient unit, such as the Doppler half-width at some temperature, and the profile of the line-scattering coefficient $\phi(x)$ is normalized to unity on the interval $-\infty \leq x \leq \infty$. For example, the value of $\phi(x)$ could be that for the Voigt profile (Hummer and Rybicki 1967),

$$\phi(x) = \frac{a}{\pi^{3/2}} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{(x - y)^2 + a^2} , \qquad a \neq 0 , \tag{4a}$$

where a is the ratio of collision to Doppler widths. In the limit as $a \rightarrow 0$, equation (4a) becomes the Doppler profile

$$\phi(x) = \pi^{-1/2} \exp(-x^2) , \qquad a = 0 . \tag{4b}$$

Finally, β is the ratio of the continuum-absorption coefficient k_c to the average line coefficient k_L; ϵ is the probability that upon scattering a photon will be lost from the line by collisional de-excitation of the excited state; and ρ is the ratio of the continuum source function to the Planck function, and is frequently set equal to unity.

Equation (1) has been analyzed in line-formation studies using various numerical procedures by Avrett (1965), Avrett and Loeser (1966), Hummer and Rybicki (1967), Rybicki and Hummer (1967), and Hummer (1968). In the special case that $\beta = 0$, equation (1) has been solved by Ivanov (1962, 1963), Nagirner (1967), and Heaslet and Warming (1968), all of whom used the resolvent method.

For a given value of $B(\tau)$, the objective of the analysis here is to be able to solve equation (1) for $I_x(\tau, \mu)$, subject to the boundary condition that there is no incident radiation,

$$I_x(0, \mu) = 0, \qquad 0 \leq \mu \leq 1 . \tag{5}$$

The solution of equation (1) begins with a change of variables in the manner of Busbridge (1953) and Bednarz and Mika (1963), i.e., we introduce the variable

$$\xi = \mu\gamma_x, \qquad -\gamma \leq \xi \leq \gamma , \tag{6a}$$

where

$$\gamma_x = [\phi(x) + \beta]^{-1} \tag{6b}$$

and $\gamma = \max \gamma_x$. If we change variables such that $I_x(\tau, \mu)d\mu = I_x(\tau, \xi)d\xi$, we find that $I_x(\tau, \xi)$ satisfies the equation

$$\begin{aligned} \xi \frac{\partial}{\partial\tau} I_x(\tau, \xi) + I_x(\tau, \xi) = \tfrac{1}{2}(1 - \epsilon)\phi(x) \int_{-\infty}^{\infty} \phi(x') \int_{-\gamma_{x'}}^{\gamma_{x'}} I_{x'}(\tau, \xi')d\xi'dx' \\ + [\rho\beta + \epsilon\phi(x)]B(\tau) . \end{aligned} \tag{7}$$

Interchanging the orders of integration on the right-hand side of the above equation, we find

$$\begin{aligned} \xi \frac{\partial}{\partial\tau} I_x(\tau, \xi) + I_x(\tau, \xi) = \tfrac{1}{2}(1 - \epsilon)\phi(x) \int_{-\gamma}^{\gamma} \int_{M_{\xi'}(x')} \phi(x')I_{x'}(\tau, \xi')dx'd\xi' \\ + [\rho\beta + \epsilon\phi(x)]B(\tau) , \end{aligned} \tag{8}$$

where $x \in M_\xi(x)$ if and only if $[\phi(x) + \beta]|\xi| \leq 1$. The function $M_\xi(x)$ is an even function of ξ and can be determined directly from the frequency-dependent absorption and scattering coefficients.

We proceed to decompose $I_x(\tau, \xi)$ by defining

$$I_x(\tau, \xi) = I^*_x(\tau, \xi) + I_x{}^p(\tau, \xi) , \tag{9}$$

where the intensity $I^*_x(\tau, \xi)$ is the solution of the homogeneous transfer equation,

$$\xi \frac{\partial}{\partial\tau} I^*_x(\tau, \xi) + I^*_x(\tau, \xi) = \tfrac{1}{2}(1 - \epsilon)\phi(x) \int_{-\gamma}^{\gamma} \int_{M_{\xi'}(x')} \phi(x')I^*_{x'}(\tau, \xi')dx'd\xi' , \tag{10}$$

and $I_x{}^p(\tau, \xi)$ is a particular solution associated with the inhomogeneous-source term in equation (8).

The free-surface boundary condition on $I^*_x(\tau, \xi)$ follows from equation (5):

$$I^*_x(0, \xi) = -I_x{}^p(0, \xi) , \qquad 0 \leq \xi \leq \gamma . \tag{11}$$

Here we assume that $I_x{}^p(\tau, \xi)$ is known, and in a subsequent section we discuss how to construct this particular solution.

In order to separate the variables in equation (10), we follow the approach used by Stewart, Kuščer, and McCormick (1966), and propose the substitution

$$I^*_x(\tau, \xi) = (1-\epsilon)\phi(x)G(\tau, \xi) + [1-\Theta(\xi)]e^{-\tau/\xi}[I^*_x(0, \xi) - (1-\epsilon)\phi(x)G(0, \xi)]\,, \quad (12)$$

where

$$\Theta(\xi) = 0 \quad \text{for} \quad 0 \leq \xi \leq \gamma\,, \qquad \Theta(\xi) = 1 \quad \text{otherwise}\,. \quad (13)$$

The form of equation (12) is specifically chosen so that an arbitrary condition on $I^*_x(0, \xi)$, $0 \leq \xi \leq \gamma$, may be satisfied. The second term of the right-hand side contains a Heaviside step-function because only the incident intensity is constrained, as is seen from equation (11).

The equations that define $G(\tau, \xi)$ and the corresponding boundary condition, $G(0, \xi)$ for $0 \leq \xi \leq \gamma$, are obtained by entering equation (12) into equation (10). We find

$$\begin{aligned}\xi \frac{\partial}{\partial \tau} G(\tau, \xi) + G(\tau, \xi) &= \tfrac{1}{2}(1-\epsilon) \int_{-\gamma}^{\gamma} G(\tau, \xi') \int_{M_{\xi'}(x')} \phi^2(x') dx' d\xi' \\ &+ \tfrac{1}{2} \int_0^{\gamma} e^{-\tau/\xi'} \int_{M_{\xi'}(x')} [\phi(x') I^*_{x'}(0, \xi') - (1-\epsilon)\phi^2(x') G(0, \xi')] dx' d\xi'\,,\end{aligned} \quad (14)$$

and the variables separate by selecting

$$\tfrac{1}{2} \int_{M_\xi(x)} [\phi(x) I^*_x(0, \xi) - (1-\epsilon)\phi^2(x) G(0, \xi)] dx = 0\,, \qquad 0 \leq \xi \leq \gamma\,. \quad (15)$$

Using equation (11), we rewrite equation (15) as

$$\Psi(\xi) G(0, \xi) = \Gamma(\xi)\,, \qquad 0 \leq \xi \leq \gamma\,, \quad (16)$$

where we have introduced the definitions

$$\Gamma(\xi) = -\tfrac{1}{2} \int_{M_\xi(x)} \phi(x) I_x^p(0, \xi) dx\,, \qquad 0 \leq \xi \leq \gamma\,, \quad (17)$$

and

$$\Psi(\xi) = \tfrac{1}{2}(1-\epsilon) \int_{M_\xi(x)} \phi^2(x) dx\,. \quad (18)$$

Equation (16) thus specifies the boundary value for $\xi \in (0, \gamma)$ of the function $G(\tau, \xi)$ which, from equations (14), (15), and (18), satisfies

$$\xi \frac{\partial}{\partial \tau} G(\tau, \xi) + G(\tau, \xi) = \int_{-\gamma}^{\gamma} \Psi(\xi') G(\tau, \xi') d\xi'\,. \quad (19)$$

Examination of equation (19) shows that the even function $\Psi(\xi)$ is a characteristic function for the "pseudo-problems" discussed by Chandrasekhar (1950). Furthermore, $\Psi(\xi) > 0$ with the exception that $\Psi(\infty) = 0$. The boundary condition on $G(\tau, \xi)$ for large τ follows from the fact that $I^*_x(\tau, \mu)$ should be bounded:

$$\lim_{\tau \to \infty} G(\tau, \xi) \to \text{a finite value}\,. \quad (20)$$

In summary, it has been shown that the solution of the considered equation of transfer, subject to the applicable boundary conditions, may be reduced to the form

$$\begin{aligned} I_x(\tau, \xi) &= I_x^p(\tau, \xi) + (1-\epsilon)\phi(x) G(\tau, \xi) \\ &- [1-\Theta(\xi)] e^{-\tau/\xi} [I_x^p(0, \xi) + (1-\epsilon)\phi(x)\Gamma(\xi)/\Psi(\xi)]\,,\end{aligned} \quad (21)$$

where $G(\tau, \xi)$ is the solution of the homogeneous equation (19) subject to the boundary conditions given by equations (16) and (20). The solution for $G(\tau, \xi)$ is developed in § III to complete the formal description of $I^*_x(\tau, \xi)$. The determination of $I_x^p(\tau, \xi)$ is discussed in §§ IV and V.

III. NORMAL-MODE EXPANSION

The solution for $G(\tau, \xi)$ will be constructed by utilizing the normal-mode expansion technique developed by Case (1960) for purposes of calculating the intensity everywhere within a given medium. This method of solution was subsequently introduced into the study of line formation in stellar atmospheres by Siewert and McCormick (1967) for the coherent-scattering model. It has also been used to treat nongray problems in radiative transfer by Siewert and Zweifel (1966*a*, *b*) who studied semi-infinite atmospheres, and by Simmons and Ferziger (1968), who considered finite atmospheres.

The analysis begins with the substitution of

$$G(\tau, \xi) = \Phi(\eta, \xi)e^{-\tau/\eta} \tag{22}$$

into equation (19), and this shows that any eigenvalues η have corresponding eigenfunctions determined from the equation

$$(\eta - \xi)\Phi(\eta, \xi) = \eta \int_{-\gamma}^{\gamma} \Psi(\xi')\Phi(\eta, \xi')d\xi' \,. \tag{23}$$

This last result is a homogeneous equation for $\Phi(\eta, \xi)$, and hence the solutions may be arbitrarily normalized such that

$$\int_{-\gamma}^{\gamma} \Psi(\xi)\Phi(\eta, \xi)d\xi = 1 \,. \tag{24}$$

The continuum solutions of equation (23) are

$$\Phi(\eta, \xi) = \eta \frac{P}{\eta - \xi} + \frac{\lambda(\eta)}{\Psi(\eta)} \delta(\eta - \xi) \,, \qquad -\gamma < \eta < \gamma \,, \tag{25}$$

where P is a mnemonic symbol used to indicate that all ensuing integrals over η or ξ are to be evaluated as Cauchy principal-value integrals. The function $\lambda(\eta)$ follows from the normalization relation and is given by

$$\lambda(\eta) = 1 + \eta P \int_{-\gamma}^{\gamma} \frac{\Psi(\xi)d\xi}{\xi - \eta} \,, \qquad -\gamma < \eta < \gamma \,. \tag{26}$$

Any discrete eigenvalues are given by the roots of the equation

$$\Lambda(\eta) = 0 \,, \qquad \eta \notin (-\gamma, \gamma) \,, \tag{27}$$

where

$$\Lambda(z) = 1 + z \int_{-\gamma}^{\gamma} \frac{\Psi(\xi)d\xi}{\xi - z} = 1 + 2z^2 \int_{0}^{\gamma} \frac{\Psi(\xi)d\xi}{\xi^2 - z^2} \,. \tag{28}$$

The occurrence of at most one pair of real roots $\pm\eta_0$ (where η_0 is defined to be positive) depends upon the value of $\Psi(\gamma)$, as shown by Bednarz and Mika (1963). Situations are encountered in line-formation studies where the discrete eigenvalues do and do not exist. If $\lambda(\gamma) < 0$, a pair of real roots exists, whereas if $\lambda(\gamma) > 0$, no roots exist; if $\lambda(\gamma) = 0$, there is a pair of discrete roots embedded in the continuum. For example, when $\beta = 0$, the eigenvalues may or may not occur, depending upon the value of $(1 - \epsilon)$ and whether or not a frequency cutoff is imposed; if there is no frequency cutoff, discrete roots can occur only if $\epsilon = 0$ (Ferziger 1969). Since the existence of η_0 affects only the appearance of the following equations and not the calculational method, we assume henceforth that η_0 exists and is finite. If η_0 does not occur, the following equations are modified by removing the terms depending upon η_0. In any event, a function defined for either $0 < \xi < \gamma$ or $-\gamma < \xi < \gamma$ may be expanded in a complete set of eigenfunctions consisting of continuum eigenfunctions given by equation (25) plus, respectively, one or both of the discrete eigenfunctions,

$$\Phi(\pm\eta_0, \xi) = \frac{\eta_0}{\eta_0 \mp \xi}, \tag{29}$$

if they occur.

As a generalization of the work reported by Siewert and McCormick (1967) and that of Stewart *et al.* (1966), it may be shown that the eigenfunctions given by equations (25) and (29) satisfy the following orthogonality relations:

$$\int_{-\gamma}^{\gamma} \Phi(\eta, \xi)\Phi(\eta', \xi)\xi\Psi(\xi)d\xi = 0\,, \qquad \eta \neq \eta'\,; \tag{30}$$

$$\int_0^{\gamma} \Phi(\eta, \xi)\Phi(\eta', \xi)\xi\Psi(\xi)H(\xi)d\xi = -\frac{\eta\Phi(\eta', \eta)}{H(-\eta)}\Theta(\eta) - \frac{\eta'\Phi(\eta, \eta')}{H(-\eta')}\Theta(\eta')\,, \quad \eta \neq \eta'\,, \tag{31}$$

where η and η' in equations (30) and (31) may be either the discrete or continuum eigenvalues. The function $\Theta(\eta)$ is defined by equation (13), and $H(\eta)$ is the H-function of Chandrasekhar (1950) for the characteristic function $\Psi(\xi)$:

$$H(z) = 1 + zH(z)\int_0^{\gamma} \frac{\Psi(\xi)H(\xi)d\xi}{\xi + z}. \tag{32}$$

This H-function has been tabulated for the Doppler line shape by Ivanov and Nagirner (1965) and for the Lorentz profile by Warming (1970) for the case where $\beta = 0$ and η_0 does not exist. In the event that η_0 does exist, note that $H^{-1}(-\eta_0) = 0$, so that the right-hand side of equation (31) vanishes for $\eta = \eta_0$ and $0 < \eta' < \gamma$. The discrete-normalization conditions corresponding to equations (30) and (31) are

$$\int_{-\gamma}^{\gamma} [\Phi(\pm\eta_0, \xi)]^2 \xi\Psi(\xi)d\xi = N(\pm\eta_0)\,, \tag{33}$$

$$\int_0^{\gamma} [\Phi(\eta_0, \xi)]^2 \xi\Psi(\xi)H(\xi)d\xi = H(\eta_0)N(\eta_0)\,, \tag{34}$$

where $N(\pm\eta_0)$ can be calculated either directly from equations (29) and (33) or from

$$N(\pm\eta_0) = \pm\eta_0^2 \frac{d}{dz}\Lambda(z)\,|_{z=\eta_0}\,. \tag{35}$$

Finally, the continuum-normalization conditions corresponding to equations (30) and (31) are

$$\int_{-\gamma}^{\gamma} \Phi(\eta, \xi)\Phi(\eta', \xi)\xi\Psi(\xi)d\xi = N(\eta)\delta(\eta - \eta')\,, \qquad -\gamma < \eta, \eta' < \gamma\,; \tag{36}$$

$$\int_0^{\gamma} \Phi(\eta, \xi)\Phi(\eta', \xi)\xi\Psi(\xi)H(\xi)d\xi = H(\eta)N(\eta)\delta(\eta - \eta')\,, \qquad 0 < \eta, \eta' < \gamma\,; \tag{37}$$

where $N(\eta)$ follows from

$$N(\eta) = \frac{\eta}{\Psi(\eta)} \{[\lambda(\eta)]^2 + [\pi\eta\Psi(\eta)]^2\} , \quad -\gamma < \eta < \gamma . \quad (38)$$

In addition to the above orthogonality relations, it is useful to know the moment relations

$$\int_{-\gamma}^{\gamma} \xi^j \Phi(\eta, \xi)\Psi(\xi)d\xi = [\eta\Lambda(\infty)]^j , \qquad j = 0, 1 ; \quad (39)$$

$$\int_{0}^{\gamma} \xi^j \Phi(\eta, \xi)\Psi(\xi)H(\xi)d\xi = \eta^j[\Lambda(\infty)]^{j/2} - \eta^j \frac{\Theta(\eta)}{H(-\eta)} , \qquad j = 0, 1 ; \quad (40)$$

where

$$\Lambda(\infty) = 1 - \int_{-\gamma}^{\gamma} \Psi(\xi)d\xi = \left[1 - \int_{0}^{\gamma} \Psi(\xi)H(\xi)d\xi\right]^2 . \quad (41)$$

With these mathematical preliminaries established, we proceed to develop a solution for $G(\tau, \xi)$ that is constrained to meet the appropriate boundary conditions, equations (16) and (20). If we write the solution to equation (19) as a linear sum of the nondiverging normal modes,

$$G(\tau, \xi) = A(\eta_0)\Phi(\eta_0, \xi) \exp(-\tau/\eta_0) + \int_{0}^{\gamma} A(\eta)\Phi(\eta, \xi)e^{-\tau/\eta}d\eta , \quad (42)$$

then the unknown expansion coefficients $A(\eta_0)$ and $A(\eta)$ may be determined by utilizing equation (16):

$$\Gamma(\xi) = \Psi(\xi)[A(\eta_0)\Phi(\eta_0, \xi) + \int_{0}^{\gamma} A(\eta)\Phi(\eta, \xi)d\eta] , \quad 0 \leq \xi \leq \gamma . \quad (43)$$

That $A(\eta_0)$ and $A(\eta)$ may be obtained from equation (43) is a consequence of Mika's half-range completeness proof (Mika 1965). Thus we multiply equation (43) by $\Phi(\omega, \xi)$ $\xi H(\xi)$ for $\omega = \eta_0$ or $0 < \omega < \gamma$, integrate, and utilize equations (31), (34), and (37) to find the final analytical form for the expansion coefficients:

$$A(\omega)H(\omega)N(\omega) = \int_{0}^{\gamma} \Gamma(\xi)\Phi(\omega, \xi)\xi H(\xi)d\xi , \quad \omega = \eta_0 \text{ or } \in (0, \gamma) . \quad (44)$$

The integrals on the right-hand side do not appear to simplify in general, but may be evaluated after $I_x^p(0, \xi)$ is used in equation (17) to construct $\Gamma(\xi)$. Thus equations (42) and (44) completely specify the function $G(\tau, \xi)$ in terms of the only remaining unknown, the particular solution $I_x^p(\tau, \xi)$; consequently, the total intensity finally appears as

$$\begin{aligned} I_x(\tau, \xi) = {} & I_x^p(\tau, \xi) \\ & + (1 - \epsilon)\phi(x)\left[A(\eta_0)\Phi(\eta_0, \xi) \exp(-\tau/\eta_0) + \int_{0}^{\gamma} A(\eta)\Phi(\eta, \xi)e^{-\tau/\eta}d\eta\right] \quad (45) \\ & - [1 - \Theta(\xi)]e^{-\tau/\xi}[I_x^p(0, \xi) + (1 - \epsilon)\phi(x)\Gamma(\xi)/\Psi(\xi)] . \end{aligned}$$

To supplement this general solution for the depth-dependent intensity, we now proceed to the determination of the two physical quantities of principal interest: the intensity emerging from the surface and the source function. The special result for the value of $G(0, -\xi)$, $0 \leq \xi \leq \gamma$, follows from equations (31) and (42) and is especially simple:

$$G(0, -\xi) = H(\xi)\xi^{-1}\int_0^\gamma G(0, \xi')\Phi(-\xi, \xi')\xi'\Psi(\xi')H(\xi')d\xi' , \quad 0 \leq \xi \leq \gamma . \quad (46)$$

Therefore, using equations (16), (21), and (46), we find that

$$I_x(0, -\xi) = I_x{}^p(0, -\xi) + (1 - \epsilon)\phi(x)H(\xi)\xi^{-1}\int_0^\gamma \Gamma(\xi')\Phi(-\xi , \xi')\xi' H(\xi')d\xi' , \quad 0 \leq \xi \leq \gamma . \quad (47)$$

Note that the expansion coefficients given by equation (44) need not be calculated if only the surface quantity is desired.

For the source function given by equation (2), equation (45) may be used to show that

$$[\phi(x) + \beta]S_x(\tau) = (1 - \epsilon)\phi(x)\Big[\tfrac{1}{2}\int_{-\gamma}^{\gamma}\int_{M_\xi(x')} \phi(x')I_{x'}{}^p(\tau, \xi)dx'd\xi + A(\eta_0)\exp(-\tau/\eta_0) + \int_0^\gamma A(\eta)e^{-\tau/\eta}d\eta\Big] + [\rho\beta + \epsilon\phi(x)]B(\tau) , \quad (48)$$

where equations (17), (18), and (24) have been utilized in the derivation. In the event that the source function only on the surface is required, equation (47) may be used to give

$$[\phi(x) + \beta]S_x(0) = (1 - \epsilon)\phi(x)\Big[\int_0^\gamma \Gamma(\xi)H(\xi)d\xi + \tfrac{1}{2}\int_{-\gamma}^{\gamma}\int_{M_\xi(x')} \phi(x')I_{x'}{}^p(0, \xi)dx'd\xi\Big] + [\rho\beta + \epsilon\phi(x)]B(0) \quad (49)$$

after equations (17), (18), and (40) are utilized. Equation (49) may also be written in the form

$$[\phi(x) + \beta]S_x(0) = \lim_{\xi\to 0^+} I_x{}^p(0, -\xi) + (1 - \epsilon)\phi(x)\int_0^\gamma \Gamma(\xi)H(\xi)d\xi = \lim_{\xi\to 0^+} I_x(0, -\xi) . \quad (50)$$

It should be noted that, after the change of variables, equation (50) may be written as

$$S_x(0) = \lim_{\mu\to 0^+} I_x\left[0, -\frac{\mu}{\phi(x) + \beta}\right] = \lim_{\mu\to 0^+} I_x(0, -\mu) ,$$

a result in agreement with that obtainable directly from equation (1).

IV. PARTICULAR SOLUTION FOR A GENERALIZED EMISSION SPECTRUM

We realize that equations (44) and (45) prescribe the distribution everywhere in the medium once a particular solution $I_x{}^p(\tau, \xi)$ is found from the equation

$$\xi\frac{\partial}{\partial\tau} I_x{}^p(\tau, \xi) + I_x{}^p(\tau, \xi) = \tfrac{1}{2}(1 - \epsilon)\phi(x)\int_{-\gamma}^{\gamma}\int_{M_{\xi'}(x')} \phi(x')I_{x'}{}^p(\tau, \xi')dx'd\xi' + [\rho\beta + \epsilon\phi(x)]B(\tau) . \quad (51)$$

The advantage of the eigenfunction method is that until now we have been able to avoid saying anything specific about the Planck function $B(\tau)$. In order to find $I_x^p(\tau, \xi)$ due to an arbitrary emission, we first determine the function $I_x(\tau, \xi; \tau_0)$ as the solution of equation (51) with $B(\tau)$ replaced by $\delta(\tau - \tau_0)$. Then multiplying $I_x(\tau, \xi; \tau_0)$ by an arbitrary $B(\tau_0)$ and integrating over τ_0 would yield $I_x^p(\tau, \xi)$ for the assumed emission spectrum of the source:

$$I_x^p(\tau, \xi) = \int_0^\infty B(\tau_0) I_x(\tau, \xi; \tau_0) d\tau_0 \,. \tag{52}$$

It should be pointed out, however, that the integration over space may be reduced in practical calculations when $\beta \neq 0$ since the distance over which the source from one depth has influence is limited; this is because the ionization of the elements responsible for continuous opacity by line radiation and the subsequent recombination provide an additional, very strong coupling between the line radiation and the electron gas.

The solution for the Green's function is equivalent to the solution of

$$\xi \frac{\partial}{\partial \tau} I_x(\tau, \xi; \tau_0) + I_x(\tau, \xi; \tau_0) = \tfrac{1}{2}(1 - \epsilon)\phi(x) \int_{-\gamma}^{\gamma} \int_{M_{\xi'}(x')} \phi(x') I_{x'}(\tau, \xi'; \tau_0) dx' d\xi' \tag{53}$$

with the boundary condition

$$\xi[I_x(\tau_0^+, \xi; \tau_0) - I_x(\tau_0^-, \xi; \tau_0)] = [\rho\beta + \epsilon\phi(x)] \,, \qquad -\gamma \le \xi \le \gamma \,. \tag{54}$$

Two additional conditions are that $I_x(\tau, \xi; \tau_0)$ must be bounded as τ tends to $\pm\infty$. (Note that only the total intensity need satisfy the boundary condition given by eq. [5], so the boundary condition for $\tau \to -\infty$ may be chosen for convenience.)

In order to separate the variables in equation (53), we follow the reasoning leading to equation (12) and write

$$\begin{aligned} I_x(\tau, \xi; \tau_0) &= (1 - \epsilon)\phi(x) G(\tau, \xi; \tau_0) + [1 - \Theta(\xi)] \\ &\times \exp[-(\tau - \tau_0)/\xi] \frac{1}{\xi} \left[\rho\beta + \epsilon\phi(x) - (1 - \epsilon)\phi(x) \frac{\Omega(\xi)}{\Psi(\xi)}\right] \,, \qquad \tau > \tau_0 \,, \end{aligned} \tag{55a}$$

and

$$\begin{aligned} I_x(\tau, \xi; \tau_0) &= (1 - \epsilon)\phi(x) G(\tau, \xi; \tau_0) \\ &- \Theta(\xi) \exp[-(\tau - \tau_0)/\xi] \frac{1}{\xi} \left[\rho\beta + \epsilon\phi(x) - (1 - \epsilon)\phi(x) \frac{\Omega(\xi)}{\Psi(\xi)}\right] \,, \qquad \tau < \tau_0 \,. \end{aligned} \tag{55b}$$

Here the second terms on the right-hand sides account for the contribution to $I_x(\tau, \xi; \tau_0)$ resulting from uncollided photons coming directly from the emission at τ_0, while the first account for scattered photons. Further, the function $\Omega(\xi)$ is to be determined such that the variables will separate. Substituting equations (55) into equation (53) yields the reduced equation for $G(\tau, \xi; \tau_0)$:

$$\xi \frac{\partial}{\partial \tau} G(\tau, \xi; \tau_0) + G(\tau, \xi; \tau_0) = \int_{-\gamma}^{\gamma} \Psi(\xi') G(\tau, \xi'; \tau_0) d\xi' \,, \qquad \tau \gtrless \tau_0 \,, \tag{56}$$

where $\Omega(\xi)$ is defined by the separation condition,

$$\int_{M_\xi(x)} \phi(x) \left[\rho\beta + \epsilon\phi(x) - (1 - \epsilon)\phi(x) \frac{\Omega(\xi)}{\Psi(\xi)}\right] dx = 0 \,, \qquad -\gamma \le \xi \le \gamma \,. \tag{57}$$

Clearly, the required definition of $\Omega(\xi)$ is obtained from equations (18) and (57):

$$\Omega(\xi) = \tfrac{1}{2}\int_{M_\xi(x)} \phi(x)[\rho\beta + \epsilon\phi(x)]dx\,, \qquad -\gamma \le \xi \le \gamma\,. \tag{58}$$

If we now substitute the solutions as expressed by equations (55) into equation (54), the discontinuity condition on $I_x(\tau, \xi; \tau_0)$ is seen to be ensured by the resulting "jump" boundary condition on $G(\tau, \xi; \tau_0)$:

$$\xi\Psi(\xi)[G(\tau_0^+, \xi; \tau_0) - G(\tau_0^-, \xi; \tau_0)] = \Omega(\xi)\,, \qquad -\gamma \le \xi \le \gamma\,. \tag{59}$$

Equation (59) is the expression for the discontinuity in the function $G(\tau, \xi; \tau_0)$ across the source plane. The remaining boundary conditions are that $G(\tau, \xi; \tau_0)$ must be bounded as τ tends to $\pm\infty$.

Using the formalism of § III, along with the knowledge that the eigenfunctions $\Phi(\pm\eta_0, \xi)$ and $\Phi(\eta, \xi)$, $-\gamma < \eta < \gamma$, form a complete set, we write

$$\begin{aligned} G(\tau, \xi; \tau_0) = {} & C(\eta_0)\Phi(\eta_0, \xi) \exp[-(\tau - \tau_0)/\eta_0] \\ & + \int_0^\gamma C(\eta)\Phi(\eta, \xi) \exp[-(\tau - \tau_0)/\eta]d\eta\,, \qquad \tau > \tau_0\,, \end{aligned} \tag{60a}$$

and

$$\begin{aligned} G(\tau, \xi; \tau_0) = {} & -C(-\eta_0)\Phi(-\eta_0, \xi) \exp[(\tau - \tau_0)/\eta_0] \\ & - \int_{-\gamma}^0 C(\eta)\Phi(\eta, \xi) \exp[-(\tau - \tau_0)/\eta]d\eta\,, \qquad \tau < \tau_0\,, \end{aligned} \tag{60b}$$

after utilizing the boundary conditions that $G(\tau, \xi; \tau_0)$ be bounded. Entering equations (60) into equation (59), we note that the expansion coefficients $C(\pm\eta_0)$ and $C(\eta)$, $-\gamma < \eta < \gamma$, are to be determined from the equation

$$\begin{aligned} \xi\Psi(\xi)[C(\eta_0)\Phi(\eta_0, \xi) + C(-\eta_0)\Phi(-\eta_0, \xi) + \int_{-\gamma}^\gamma C(\eta)\Phi(\eta, \xi)d\eta] = \Omega(\xi)\,, \\ -\gamma \le \xi \le \gamma\,. \end{aligned} \tag{61}$$

Multiplying equation (61) by $\Phi(\tilde{\omega}, \xi)$ for $\tilde{\omega} = \pm\eta_0$ or $-\gamma < \tilde{\omega} < \gamma$ and applying equations (30), (33), and (36), we obtain the final analytical form for the expansion coefficients $C(\pm\eta_0)$ and $C(\eta)$, $-\gamma < \eta < \gamma$:

$$C(\tilde{\omega})N(\tilde{\omega}) = \int_{-\gamma}^\gamma \Phi(\tilde{\omega}, \xi)\Omega(\xi)d\xi\,. \tag{62}$$

The right-hand side of equation (62) may be explicitly calculated once $\phi(x)$ and β are specified for any problem.

Equation (62), along with equations (52), (55), and (60), completes the derivation of the quantity $I_x^p(\tau, \xi)$ needed for use in equation (45). The only other quantities needed to find $I_x(\tau, \xi)$ from equation (45) are the $A(\omega)$ of equation (44), which depend upon $\Gamma(\xi)$. Using equation (17), we note that $\Gamma(\xi)$ is eventually found to be

$$\begin{aligned} \Gamma(\xi) = \Psi(\xi)\int_0^\infty B(\tau_0)[C(-\eta_0)\Phi(-\eta_0, \xi) \exp(-\tau_0/\eta_0) \\ + \int_{-\gamma}^0 C(\eta)\Phi(\eta, \xi) \exp(\tau_0/\eta)d\eta]d\tau_0\,, \qquad 0 \le \xi \le \gamma\,, \end{aligned} \tag{63}$$

after use has been made of equations (18), (52), (55), and (60). Insertion of equation (63) into equation (44) and application of equation (31) then shows that

$$A(\omega)H(\omega)N(\omega) = \int_0^\infty B(\tau_0)\Big[\frac{C(-\eta_0)\eta_0\Phi(\omega, -\eta_0)e^{-\tau_0/\eta_0}}{H(\eta_0)} - \int_{-\gamma}^{0}\frac{C(\eta)\eta\Phi(\omega, \eta)e^{\tau_0/\eta}d\eta}{H(-\eta)}\Big]d\tau_0\,, \qquad \omega = \eta_0 \quad \text{or} \quad \in (0, \gamma)\,. \tag{64}$$

Thus it is now possible to obtain $I_x(\tau, \xi)$ everywhere in the stellar atmosphere.

V. PARTICULAR SOLUTIONS FOR SPECIAL PLANCK FUNCTIONS

The analysis of § IV permits calculation of the emerging distribution for general forms of $B(\tau)$. This general formalism can be circumvented by direct-substitution methods, however, if the Planck function is taken to be a linear or exponential function of the optical depth. Precise particular solutions for these two models are established in this section and used in the results of § III.

a) Case 1: Linear Planck Function

For

$$B(\tau) = a^{(0)} + a^{(1)}\tau\,, \tag{65}$$

where $a^{(0)}$ and $a^{(1)}$ are constants, the particular solution is found to be

$$I_x{}^p(\tau, \xi) = [a^{(0)} + a^{(1)}(\tau - \xi)][(1 - \epsilon)\phi(x)K + \rho\beta + \epsilon\phi(x)]\,, \tag{66}$$

where the constant K is defined as

$$K = \Big[1 - \int_{-\gamma}^{\gamma}\Psi(\xi)d\xi\Big]^{-1}\int_{-\gamma}^{\gamma}\Omega(\xi)d\xi\,, \tag{67}$$

and $\Omega(\xi)$ is given by equation (58). (The derivation of eq. [66] by means of the Green's function developed in § IV is discussed in the Appendix.)

Since $I_x{}^p(\tau, \xi)$ is now known, all that remain unknown in equation (45) for the special case being considered are the expansion coefficients as given by equation (44). We first observe that $\Gamma(\xi)$ of equation (17) is given by

$$\Gamma(\xi) = [K\Psi(\xi) + \Omega(\xi)](a^{(1)}\xi - a^{(0)})\,, \qquad 0 \le \xi \le \gamma\,. \tag{68}$$

Then utilizing equation (68) in equation (44) shows that

$$A(\omega)H(\omega)N(\omega) = -\omega K a^{(1)}h_1 + \omega K(a^{(1)}\omega - a^{(0)})(1 - h_0) + I(\omega)\,, \qquad \omega = \eta_0 \quad \text{or} \quad \in (0, \gamma)\,, \tag{69}$$

where we have used equation (40) and have defined h_j and $I(\eta)$ by the equations

$$h_j = \int_0^\gamma \xi^j\Psi(\xi)H(\xi)d\xi\,, \qquad j = 0, 1\,; \tag{70}$$

$$I(\eta) = \int_0^\gamma (a^{(1)}\xi - a^{(0)})\Omega(\xi)\Phi(\eta, \xi)\xi H(\xi)d\xi\,. \tag{71}$$

The use of equations (66) and (69) in equation (45) yields a solution for the radiation intensity at an arbitrary depth.

The emerging intensity is found most easily by entering equation (68) into equation (47) to obtain

$$I_x(0,-\xi) = (a^{(0)} + a^{(1)}\xi)[\rho\beta + \epsilon\phi(x) + (1-\epsilon)\phi(x)K(1-h_0)H(\xi)] \\ + (1-\epsilon)\phi(x)H(\xi)[Ka^{(1)}h_1 + \xi^{-1}I(-\xi)]\,, \qquad 0 \le \xi \le \gamma\,, \tag{72}$$

where use has been made of equations (40), (66), and (71).

For the source function, we find

$$[\phi(x)+\beta]S_x(\tau) = [\rho\beta + \epsilon\phi(x)](a^{(0)} + a^{(1)}\tau) \\ + (1-\epsilon)\phi(x)\left[A(\eta_0)\exp(-\tau/\eta_0) + \int_0^\gamma A(\eta)e^{-\tau/\eta}d\eta + (a^{(0)} + a^{(1)}\tau)K\right], \tag{73}$$

after substituting equations (65) and (66) into equation (48) and using equations (18) and (58) and the equations

$$\int_{-\gamma}^{\gamma} \xi\Psi(\xi)d\xi = \int_{-\gamma}^{\gamma} \xi\Omega(\xi)d\xi = 0\,. \tag{74}$$

Finally, we note that

$$[\phi(x)+\beta]S_x(0) = [\rho\beta + \epsilon\phi(x)]a^{(0)} \\ + (1-\epsilon)\phi(x)\left[Ka^{(1)}h_1 + Ka^{(0)}(1-h_0) + \int_0^\gamma (a^{(1)}\xi - a^{(0)})\Omega(\xi)H(\xi)d\xi\right], \tag{75}$$

after use of equations (50), (71), and (72).

We have thus constructed an exact solution for a model atmosphere in which the temperature was known and led to a linear Planck function, $a^{(0)} + a^{(1)}\tau$, for the continuous emission. Invoking the assumptions that $\epsilon = 0$ and $\rho = 1$ and that $\phi(x)$ is independent of frequency leads to the results obtained earlier by Siewert and McCormick (1967).

b) Case 2: Exponential Planck Function

A specialized Planck function considered by Ivanov (1963) was

$$B(\tau) = e^{-m\tau}\,, \tag{76}$$

where m^{-1} is a constant $\notin (-\gamma, \gamma)$. The particular solution corresponding to this Planck function can be established by elementary methods or, alternatively, found from the Green's function (see Appendix):

$$I_x^p(\tau,\xi) = \frac{e^{-m\tau}}{1-m\xi}[\rho\beta + \epsilon\phi(x) + (1-\epsilon)\phi(x)L]\,, \tag{77}$$

where the constant L is defined as

$$L = \left[1 - \int_{-\gamma}^{\gamma} \frac{\Psi(\xi)d\xi}{1-m\xi}\right]^{-1} \int_{-\gamma}^{\gamma} \frac{\Omega(\xi)d\xi}{1-m\xi} = \left[\Lambda\left(\frac{1}{m}\right)\right]^{-1} \int_{-\gamma}^{\gamma} \frac{\Omega(\xi)d\xi}{1-m\xi}\,. \tag{78}$$

(The second form of eq. [78] follows from eq. [28].) In a manner similar to that employed to treat the linear Planck function, solutions may be obtained for the quantities $A(\omega)$, $\omega = \eta_0$ or $0 \le \omega \le \gamma$, $I_x(0, -\xi)$, $S_x(\tau)$, and $S_x(0)$. These solutions follow after using equations (17) and (77) to show that

$$\Gamma(\xi) = -(1-m\xi)^{-1}[\Omega(\xi) + L\Psi(\xi)]\,, \qquad 0 \le \xi \le \gamma\,. \tag{79}$$

As an example, we evaluate $I_x(0, -\xi)$ from equations (47) and (79) and the identity

$$\frac{1}{\xi}\int_0^\gamma \frac{\Psi(\xi')\Phi(-\xi, \xi')\xi' H(\xi')d\xi'}{1 - m\xi'} = \frac{1}{1 + m\xi}\left[\frac{1}{H(\xi)} - H\left(\frac{1}{m}\right)\Lambda\left(\frac{1}{m}\right)\right], \quad 0 \leq \xi \leq \gamma. \tag{80}$$

Equation (80) is derived by using partial fractions, equation (32), and the identity (Chandrasekhar 1950)

$$H(-z)H(z)\Lambda(z) = 1. \tag{81}$$

Using equations (78)–(80) in equation (47) shows that

$$I_x(0, -\xi) = \frac{1}{1 + m\xi}\left[\rho\beta + \epsilon\phi(x) + (1 - \epsilon)\phi(x)H(\xi)H\left(\frac{1}{m}\right)\int_{-\gamma}^{\gamma}\frac{\Omega(\xi')d\xi'}{1 - m\xi'}\right] - \frac{(1 - \epsilon)\phi(x)H(\xi)}{\xi}\int_0^\gamma \frac{\Omega(\xi')\Phi(-\xi, \xi')\xi' H(\xi')d\xi'}{1 - m\xi'}, \quad 0 \leq \xi \leq \gamma. \tag{82}$$

From equations (50) and (82),

$$[\phi(x) + \beta]S_x(0) = \rho\beta + \epsilon\phi(x) + (1 - \epsilon)\phi(x)\int_{-\gamma}^{\gamma}\frac{\Omega(\xi)[H(m^{-1}) - H(\xi)]d\xi}{1 - m\xi}. \tag{83}$$

In the event that $\beta = 0$, the results in equations (82) and (83) are greatly simplified since

$$\Omega(\xi) = \epsilon\Psi(\xi)/(1 - \epsilon), \quad \beta = 0. \tag{84}$$

We then find that equation (80) may be used to reduce equation (82) to the form

$$I_x(0, -\xi) = \frac{\epsilon\phi(x)H(\xi)H(m^{-1})}{1 + m\xi}, \quad 0 \leq \xi \leq \gamma, \quad \beta = 0. \tag{85}$$

After a change of variables, this is further simplified to a result consistent with equation (49) of Ivanov (1963),

$$I_x(0, -\mu) = \frac{\epsilon H[\mu/\phi(x)]H(m^{-1})}{1 + m\mu/\phi(x)}, \quad 0 \leq \mu \leq 1, \quad \beta = 0. \tag{86}$$

Finally, use of equations (50) and (85) yields

$$S_x(0) = \epsilon H(m^{-1}), \quad \beta = 0. \tag{87}$$

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APPENDIX

ILLUSTRATIONS OF THE USE OF THE GREEN'S FUNCTION TO OBTAIN PARTICULAR SOLUTIONS

In order to illustrate the procedure for more general Planck functions, we use the Green's function developed in § IV to derive particular solutions for the two special Planck functions considered in § V. Equation (52) is the general form for the required particular solution and becomes, by using equations (55) and (60),

$$\begin{aligned} I_x{}^p(\tau, \xi) = {} & (1-\epsilon)\phi(x)\int_0^\tau B(\tau_0)\Big\{C(\eta_0)\Phi(\eta_0, \xi) \exp[-(\tau-\tau_0)/\eta_0] \\ & + \int_0^\gamma C(\eta)\Phi(\eta, \xi) \exp[-(\tau-\tau_0)/\eta]d\eta\Big\}d\tau_0 \\ & - (1-\epsilon)\phi(x)\int_\tau^\infty B(\tau_0)\Big\{C(-\eta_0)\Phi(-\eta_0, \xi) \exp[(\tau-\tau_0)/\eta_0] \\ & + \int_{-\gamma}^0 C(\eta)\Phi(\eta, \xi) \exp[-(\tau-\tau_0)/\eta]d\eta\Big\}d\tau_0 \\ & + \frac{1}{\xi}\left[\rho\beta + \epsilon\phi(x) - (1-\epsilon)\phi(x)\frac{\Omega(\xi)}{\Psi(\xi)}\right]\Big\{[1-\Theta(\xi)]\int_0^\tau B(\tau_0) \exp[-(\tau-\tau_0)/\xi]d\tau_0 \\ & - \Theta(\xi)\int_\tau^\infty B(\tau_0) \exp[-(\tau-\tau_0)/\xi]d\tau_0\Big\}. \end{aligned} \tag{A1}$$

a) Case 1: Linear Planck Function

In order to derive equation (66) from equation (A1), we need the identities

$$\begin{aligned} & \eta_0{}^{j+1}C(\eta_0)\Phi(\eta_0, \xi) + (-\eta_0)^{j+1}C(-\eta_0)\Phi(-\eta_0, \xi) \\ & + \int_{-\gamma}^{\gamma}\eta^{j+1}C(\eta)\Phi(\eta, \xi)d\eta = \xi^j\left[K + \frac{\Omega(\xi)}{\Psi(\xi)}\right], \qquad j = 0, 1, \end{aligned} \tag{A2}$$

which follow from equation (61) after the use of partial fraction analysis and equations (39), (67), and (74). Integration of equation (A1) with $B(\tau_0) = a^{(0)} + a^{(1)}\tau_0$, along with the use of equation (A2), gives

$$\begin{aligned} I_x{}^p(\tau, \xi) = {} & [a^{(0)} + a^{(1)}(\tau - \xi)][(1-\epsilon)\phi(x)K + \rho\beta + \epsilon\phi(x)] \\ & - (1-\epsilon)\phi(x)\Big[\eta_0(a^{(0)} - a^{(1)}\eta_0)C(\eta_0)\Phi(\eta_0, \xi) \exp(-\tau/\eta_0) \\ & + \int_0^\gamma \eta(a^{(0)} - a^{(1)}\eta)C(\eta)\Phi(\eta, \xi)e^{-\tau/\eta}d\eta\Big] \\ & - [1-\Theta(\xi)]e^{-\tau/\xi}(a^{(0)} - a^{(1)}\xi)\left[\rho\beta + \epsilon\phi(x) - (1-\epsilon)\phi(x)\frac{\Omega(\xi)}{\Psi(\xi)}\right]. \end{aligned} \tag{A3}$$

Since the terms proportional to $\Phi(\omega, \xi)e^{-\tau/\omega}$, $\omega = \eta_0$ or $\eta \in (0, 1)$, are solutions of equation (10), and since equation (57) guarantees that the term proportional to $e^{-\tau/\xi}$ is a solution of equation (10), a particular solution for the linear Planck function is obtained by omitting these spurious terms; hence, equation (A3) agrees with equation (66).

b) Case 2: Exponential Planck Function

The case of an exponential Planck function can be treated in the above manner; this time we require the identity

$$\frac{\eta_0 C(\eta_0)\Phi(\eta_0, \xi)}{1 - m\eta_0} - \frac{\eta_0 C(-\eta_0)\Phi(-\eta_0, \xi)}{1 + m\eta_0} + \int_{-\gamma}^{\gamma}\frac{\eta C(\eta)\Phi(\eta, \xi)d\eta}{1 - m\eta} = \frac{L + \Omega(\xi)[\Psi(\xi)]^{-1}}{1 - m\xi}. \tag{A4}$$

Equation (A4) is derived by use of partial fraction analysis, equation (A2) for $j = 0$, and the integral of equation (A2) for $j = 0$ over ξ from $-\gamma$ to γ. In addition, the identity

$$\frac{\eta_0 C(\eta_0)}{1 - m\eta_0} - \frac{\eta_0 C(-\eta_0)}{1 + m\eta_0} + \int_{-\gamma}^{\gamma} \frac{\eta C(\eta) d\eta}{1 - m\eta} = L \tag{A5}$$

is needed to prove equation (A4) and is derived by dividing equation (61) by $(1 - m\xi)$, integrating over ξ from $-\gamma$ to γ, and using equations (39) and (78).

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