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# Linear Integral Equations for a Certain Class of *H*-Functions Applicable to the Theory of Neutron Transport and Radiative Transfer*

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A matrix version of the classical Riemann–Hilbert problem defined on an open contour is discussed. The problem is reduced to a quasiregular integral equation for cases where the sufficient Hölder continuity condition is satisfied and the component indices are nonnegative. As an illustration of this procedure, linear integral equations (rather than the usual nonlinear forms), for Chandrasekhar's functions $H_l(\mu)$ and $H_r(\mu)$ are established in a form amenable to solution by numerical iteration.

## I. INTRODUCTION

In general, the use of the singular eigenfunction expansion technique, introduced by Case[1] for treating problems in neutron transport theory to solve boundary value problems in "particle" transport analysis, requires solutions to singular integral equations in order to establish the various expansion coefficients.[2–4] Once the appropriate singular equations are developed, the methods of Muskhelishvili[5] can be used to convert the boundary value problem to an equivalent Riemann–Hilbert problem, and in many cases[2] closed-form solutions may be obtained.

Following Case's original paper[1] on the subject of singular eigenfunction expansions, the method has been extended to include many different models in neutron transport theory and radiative transfer. For example, the degenerate kernel model for energy transfer has been discussed by Mika,[6] the case of anisotropic scattering in 1-speed neutron transport theory has been thoroughly investigated by McCormick and Kuščer,[3] and several studies in "multigroup" theory have been reported by Siewert and Zweifel,[4,7] Siewert and Fraley,[8] Mourad and Siewert,[9] and Shultis.[10]

The usual procedure,[1,2] once the normal modes of the considered equation of transfer are established, is first to attempt the proof of a full-range expansion theorem. In developing this proof, the singular integral equations encountered can normally[2] be reduced to a special case of the inhomogeneous Riemann–Hilbert problem which can then be solved straightforwardly even for the case of matrices.[9]

The considerably more interesting half-range expansion theorem[2] cannot, in general, be established quite so readily; in fact, no constructive proofs for the multigroup or matrix models considered by Shultis[10] or Mourad and Siewert,[9] for example, have been reported. Although for some cases the proof of the half-range theorem applicable to matrix models has been converted to the need to solve systems of regular integral equations,[10,11] there have been no rigorous proofs of the existence of solutions to these equations; in some instances, however, this approach has been shown to be feasible computationally for nonmultiplying media.[12] Another approach used for half-range applications has been exhibited by Metcalf and Zweifel[13] and Mourad,[14] who have shown it possible to solve by numerical iteration the *singular* integral equations encountered in two different matrix problems.

A more direct method for solving half-range problems with the singular eigenfunction expansion technique is to pursue the homogeneous Riemann–Hilbert problem.[5] However, as Leonard and Ferziger[15] and Kuščer[16] have illustrated, multigroup models normally lead to a matrix form of the Riemann–Hilbert problem, and closed-form solutions are not generally available. We should thus like to discuss the analysis required to reduce these analytically formidable problems to forms computationally more feasible.

## II. GENERAL ANALYSIS

For multigroup application of Case's method of normal modes,[1] the proof of the required half-range expansion theorem reduces to the need to solve a homogeneous Riemann–Hilbert problem for the normally called $\mathbf{X}$ matrix.[15] Here a matrix $\mathbf{X}(z)$, holomorphic in the complex plane cut from zero to one along the real line, is sought such that the boundary values from above (+) and below (−) the cut are related by

$$\mathbf{X}^+(\mu) = \mathbf{G}(\mu)\mathbf{X}^-(\mu), \quad \mu \in [0, 1], \tag{1}$$

where $\mathbf{G}(\mu)$ is a given matrix. We seek here the fundamental solution to Eq. (1), and thus $\mathbf{X}(z)$ and $\mathbf{X}^{\pm}(\mu)$ are required to be nonsingular in the finite plane.[5]

Since, in general, there exists no analytical solution to Eq. (1), we wish to make use of Muskhelishvili's theory[5] to convert Eq. (1) to a quasiregular linear integral equation for $\mathbf{X}^-(\mu)$, or alternatively for Chandrasekhar's **H**-matrix equivalent.[17]

If we now stipulate that $\mathbf{G}(\mu)$ obeys the Hölder condition[5] on the interval [0, 1] and further that $\mathbf{G}(0) = \mathbf{G}(1) = \mathbf{I}$, $\mathbf{I}$ being the unit matrix, then we can write [see Ref. 5, p. 386, Eq. (126.5)]

$$\mathbf{X}^-(\mu) = \mathbf{X}_\infty(\mu) + \frac{1}{2\pi i}\int_0^1 [\mathbf{G}^{-1}(\mu)\mathbf{G}(\nu) - \mathbf{I}]\mathbf{X}^-(\nu)\frac{d\nu}{\nu - \mu} + \frac{1}{2\pi i}[\mathbf{G}^{-1}(\mu) - \mathbf{I}]\int_{C_1} \mathbf{X}^-(\nu)\frac{d\nu}{\nu - \mu}, \quad (2)$$

where the arbitrary arc $C_1$ has been added (with the proviso that there be a continuously turning tangent) to the real-line segment [0, 1] to yield a closed contour $C$, as depicted in Fig. 1. Further, we have denoted the principal part of $\mathbf{X}(z)$ at infinity by $\mathbf{X}_\infty(z)$.

In order to establish Eq. (2), we have also defined $\mathbf{G}(\mu) \triangleq \mathbf{I}$ for $\mu \in C_1$. A similar procedure for closing the contour has been used by Leonard and Ferziger[15] and Kuščer,[16] though in the latter case a term due to the integral on $C_1$ appears to be missing [see Ref. 16, p. 267, Eq. (113)].

We note that Muskhelishvili's derivation[5] of the equation equivalent to our Eq. (2) was based on the proposition that the matrix $\mathbf{G}(\mu)$ was Hölder continuous on $C$, and that assumption is maintained here. Clearly, the fact that $\mathbf{G}(\mu)$ is taken to be a Hölder matrix is sufficient to ensure that Eq. (2) is quasiregular[5]; however, Leonard and Ferziger[15] applied Muskhelishvili's analysis without modification to a multigroup problem where the **G** matrix is not of the Hölder class, and the assertion that their equivalent to Eq. (2) is quasiregular is not immediately justified. [Clearly, simple continuity, as opposed to Hölder continuity, is not sufficient to ensure that Eq. (2) is quasiregular.]

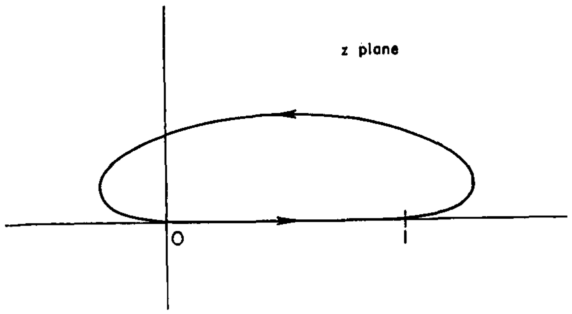


FIG. 1. The contour $C$ in the $z$ plane.

Since we are seeking to develop an integral equation for $\mathbf{X}^-(\mu)$ only on the real-line segment $\mu \in [0, 1]$, we rewrite Eq. (2) for the two cases $\mu \in [0, 1]$ and $\mu \in C_1$:

$$\mathbf{X}^-(\mu) = \mathbf{X}_\infty(\mu) + \frac{1}{2\pi i}\int_0^1 [\mathbf{G}^{-1}(\mu)\mathbf{G}(\nu) - \mathbf{I}]\mathbf{X}^-(\nu)\frac{d\nu}{\nu - \mu} + \frac{1}{2\pi i}[\mathbf{G}^{-1}(\mu) - \mathbf{I}]\int_{C_1} \mathbf{X}^-(\nu)\frac{d\nu}{\nu - \mu}, \quad \mu \in [0, 1], \quad (3a)$$

and

$$\mathbf{X}^-(\mu) = \mathbf{X}_\infty(\mu) + \frac{1}{2\pi i}\int_0^1 [\mathbf{G}(\nu) - \mathbf{I}]\mathbf{X}^-(\nu)\frac{d\nu}{\nu - \mu}, \quad \mu \in C_1. \quad (3b)$$

Equation (3b) is clearly an explicit expression which relates $\mathbf{X}^-(\mu)$ for $\mu$ on $C_1$ to $\mathbf{X}_\infty(\mu)$ and values of $\mathbf{X}^-(\nu)$, where $\nu$ is confined to the real-line segment [0, 1]. This equation can thus be entered into the last term of Eq. (3a) to yield

$$\mathbf{X}^-(\mu) = \mathbf{X}_\infty(\mu) + \frac{1}{2\pi i}\int_0^1 [\mathbf{G}^{-1}(\mu)\mathbf{G}(\nu) - \mathbf{I}]\mathbf{X}^-(\nu)\frac{d\nu}{\nu - \mu} + \frac{1}{2\pi i}[\mathbf{G}^{-1}(\mu) - \mathbf{I}]\int_{C_1}\left(\mathbf{X}_\infty(\nu) + \frac{1}{2\pi i}\int_0^1 [\mathbf{G}(\nu') - \mathbf{I}]\mathbf{X}^-(\nu')\frac{d\nu'}{\nu' - \nu}\right)\frac{d\nu}{\nu - \mu}, \quad \mu \in [0, 1]. \quad (4)$$

We should now like to consider the repeated integral in the above equation and therefore introduce the definition

$$\mathbf{I}(\mu) \triangleq \int_{C_1}\left\{\int_0^1 [\mathbf{G}(\nu') - \mathbf{I}]\mathbf{X}^-(\nu')\frac{d\nu'}{\nu' - \nu}\right\}\frac{d\nu}{\nu - \mu}, \quad \mu \in (0, 1). \quad (5)$$

Since the inner integral in Eq. (5) is nonsingular [because **G** is Hölder on $C$ and $\mathbf{G}(0) = \mathbf{G}(1) = \mathbf{I}$], we invert the order of integration to obtain

$$\mathbf{I}(\mu) = \int_0^1 \left([\mathbf{G}(\nu') - \mathbf{I}]\mathbf{X}^-(\nu')\int_{C_1} \frac{1}{\nu' - \nu}\frac{1}{\nu - \mu}\,d\nu\right)d\nu', \quad \mu \in (0, 1). \quad (6)$$

Performing now the integration over $\nu$ in Eq. (6), we find

$$\mathbf{I}(\mu) = \int_0^1 [\mathbf{G}(\nu') - \mathbf{I}]\left[\ln\left(\frac{1}{\nu'} - 1\right) - \ln\left(\frac{1}{\mu} - 1\right)\right]\mathbf{X}^-(\nu')\frac{d\nu'}{\nu' - \mu}, \quad \mu \in (0, 1). \quad (7)$$

Equation (7) may now be substituted for the repeated integral in Eq. (4) to yield

$$\mathbf{X}^-(\mu) = \mathbf{X}_\infty(\mu) + \frac{1}{2\pi i}[\mathbf{G}^{-1}(\mu) - \mathbf{I}]\int_{C_1} \mathbf{X}_\infty(\nu)\frac{d\nu}{\nu - \mu} + \frac{1}{2\pi i}\int_0^1 \mathbf{K}(\nu,\mu)\mathbf{X}^-(\nu)\frac{d\nu}{\nu - \mu}, \quad \mu \in [0, 1], \tag{8}$$

where $\mathbf{K}(\nu, \mu)$ is given by

$$\mathbf{K}(\nu,\mu) = \mathbf{G}^{-1}(\mu)\mathbf{G}(\nu) - \mathbf{I} + \frac{1}{2\pi i}[\mathbf{G}^{-1}(\mu) - \mathbf{I}] \times [\mathbf{G}(\nu) - \mathbf{I}]\left[\ln\left(\frac{1}{\nu} - 1\right) - \ln\left(\frac{1}{\mu} - 1\right)\right]. \tag{9}$$

Finally, the integral defined on $C_1$ in Eq. (8) may be written as

$$\int_{C_1} \mathbf{X}_\infty(\nu)\frac{d\nu}{\nu - \mu} = -P\int_0^1 \mathbf{X}_\infty(\nu)\frac{d\nu}{\nu - \mu} + \pi i \mathbf{X}_\infty(\mu), \tag{10}$$

so that the desired equation in terms only of variables on the line segment [0, 1] is obtained:

$$\mathbf{X}^-(\mu) = \mathbf{X}_\infty(\mu) + [\mathbf{G}^{-1}(\mu) - \mathbf{I}] \times \left(\tfrac{1}{2}\mathbf{X}_\infty(\mu) - \frac{1}{2\pi i}P\int_0^1 \mathbf{X}_\infty(\nu)\frac{d\nu}{\nu - \mu}\right) + \frac{1}{2\pi i}\int_0^1 \mathbf{K}(\nu,\mu)\mathbf{X}^-(\nu)\frac{d\nu}{\nu - \mu}, \quad \mu \in [0, 1]. \tag{11}$$

Equation (11) represents the basic version of the classical result [see Ref. 5, p. 386, Eq. (126.5)] modified for an open contour and is based on the proposition that the given **G**-matrix is Hölder continuous on the interval [0, 1] and further that $\mathbf{G}(0) = \mathbf{G}(1) = \mathbf{I}$.

It is clear that once $\mathbf{X}^-(\mu)$ is determined, as say from Eq. (11), $\mathbf{X}(z)$ follows immediately through the appropriate Cauchy integral; however, there remains the task of proving the equivalence between the original problem and Eq. (11), the ordinary integral equation for $\mathbf{X}^-(\mu)$. Furthermore, the solubility of Eq. (11) needs to be established in order to ensure that a solution to the original problem exists. It is stated by Muskhelishvili (Ref. 5, p. 389) that conditions sufficient for proving the required equivalence and solubility are that neither the accompanying problem,

$$\mathbf{\Psi}^+(\mu) = \mathbf{G}^{-1}(\mu)\mathbf{\Psi}^-(\mu), \quad \mu \in C, \tag{12}$$

nor the associate problem,

$$\mathbf{\Phi}^+(\mu) = [\mathbf{G}^{\mathrm{T}}(\mu)]^{-1}\mathbf{\Phi}^-(\mu), \quad \mu \in C, \tag{13}$$

has a nontrivial solution vanishing at infinity.[5] Here the transpose of $\mathbf{G}(\mu)$ is denoted by $\mathbf{G}^{\mathrm{T}}(\mu)$.

Although there seems to be no general method for calculating the so-called component indices[5] for the original Riemann–Hilbert problem given by Eq. (1), it appears that, for problems normally encountered in neutron transport theory and radiative transfer, the **G** matrix is such that its indices are nonnegative.[16] We thus restrict our attention to those problems for which the **G** matrix leads to nonnegative component indices. It now follows that the boundary value problems defined by $\mathbf{G}^{-1}$ and $[\mathbf{G}^T]^{-1}$ will have nonpositive component indices, and therefore the only solution of the accompanying or associate problems which vanishes at infinity must be the trivial solution.

## III. QUASIREGULAR FREDHOLM EQUATIONS FOR $H_l(\mu)$ AND $H_r(\mu)$

We should like to apply the analysis of the previous section to two special cases pertinent to the study of polarized light in a free-electron atmosphere.[8,17] We thus seek solutions $X_1(z)$ and $X_2(z)$ of the Riemann–Hilbert problem given by Eq. (1) for the two scalar cases:

$$G_\alpha(\mu) = \frac{\Lambda_\alpha^+(\mu)}{\Lambda_\alpha^-(\mu)}, \quad \mu \in [0, 1], \quad \alpha = 1 \text{ or } 2, \tag{14}$$

with $\Lambda_\alpha^\pm(\mu)$ being the boundary values of the function

$$\Lambda_\alpha(z) = (-1)^\alpha + 3(1 - z^2)\left(1 + \tfrac{1}{2}z\int_{-1}^1 \frac{d\mu}{\mu - z}\right). \tag{15}$$

Since we require here the canonical solutions, i.e., solutions which are nonvanishing in the finite plane and which yield nonvanishing boundary values $X_\alpha^\pm(\mu)$, $\alpha = 1$ or 2, on the cut $\mu \in [0, 1]$, we follow Muskhelishvili[5] and first calculate the required indices $\aleph_1$ and $\aleph_2$ of the two problems:

$$\aleph_\alpha = \frac{1}{2\pi i}[\arg G_\alpha(\mu)]_C, \quad \alpha = 1 \text{ or } 2, \tag{16}$$

where $[\ ]_C$ is used to denote the increase of the function in brackets as the contour $C$ is traversed in the positive direction. It is a simple matter[8] to show for the functions $G_\alpha(\mu)$ considered here that

$$\aleph_1 = 1 \tag{17a}$$

and

$$\aleph_2 = 0. \tag{17b}$$

Consequently, from the remarks of the previous section, it follows that the integral equation for $X_\alpha^-(\mu)$ is equivalent to Eq. (1) and, furthermore, that Eq. (11) is soluble, the solution being unique to within a multiplicative constant.

It follows from Eq. (17a) that $X_1(z)$ will have a simple zero at infinity, and thus $X_{1\infty}(\nu) \equiv 0$. In a similar manner, $X_{2\infty}(\nu)$ must be a constant, which we arbitrarily choose equal to unity in order to normalize our results in the established manner.[8] Since the principal parts of the functions $X_\alpha(\mu)$ have been determined, we can now write the forms of Eq. (11) appropriate here:

$$X_1^-(\mu) = \frac{1}{2\pi i}\int_0^1 K_1(\nu, \mu) X_1^-(\nu)\frac{d\nu}{\nu - \mu}, \quad \mu \in [0, 1], \tag{18a}$$

and

$$X_2^-(\mu) = 1 + \frac{G_2^{-1}(\mu) - 1}{2\pi i}\left[\pi i - \ln\left(\frac{1}{\mu} - 1\right)\right] + \frac{1}{2\pi i}\int_0^1 K_2(\nu, \mu) X_2^-(\nu)\frac{d\nu}{\nu - \mu}, \quad \mu \in [0, 1], \tag{18b}$$

where

$$K_\alpha(\nu, \mu) = G_\alpha^{-1}(\mu)G_\alpha(\nu) - 1 + \frac{1}{2\pi i}[G_\alpha^{-1}(\mu) - 1] \times [G_\alpha(\nu) - 1]\left[\ln\left(\frac{1}{\nu} - 1\right) - \ln\left(\frac{1}{\mu} - 1\right)\right]. \tag{19}$$

Equation (18a) is clearly a homogeneous equation for $X_1^-(\mu)$, and thus we wish to select a normalization consistent with that used previously, since the desired canonical solution is fixed only to within an arbitrary multiplicative constant. In the process of establishing the exact form of $X_1(z)$, Siewert and Fraley[8] normalized their solution such that

$$X_1(0) = \sqrt{5}. \tag{20}$$

Now setting $\mu = 0$ in Eq. (18a), we find

$$X_1^-(0) = X_1(0) = \frac{1}{2\pi i}\int_0^1 [G_1(\nu) - 1]X_1^-(\nu)\frac{d\nu}{\nu}, \tag{21}$$

which, when the explicit forms of $G_1(\nu)$ and Eq. (20) are used, yields the identity

$$\int_0^1 \Psi_l(\nu)\frac{X_1^-(\nu)}{\Lambda_1^-(\nu)}d\nu = \frac{\sqrt{5}}{2}, \tag{22}$$

where we have introduced Chandrasekhar's characteristic function[17]

$$\Psi_l(\nu) = \tfrac{3}{4}(1 - \nu^2). \tag{23}$$

We thus seek a solution to the homogeneous Eq. (18a) such that $X_1(0) = \sqrt{5}$ or, alternatively, such that it is subject to the integral constraint given by Eq. (22).

Rather than pursue the analysis for $X_1^-(\mu)$, we prefer to write our equations in terms of Chandrasekhar's function[17] $H_l(\mu)$,

$$H_l(\mu) = \frac{2\sqrt{5}}{5}\frac{X_1^-(\mu)}{\Lambda_1^-(\mu)}, \quad \mu \in [0, 1], \tag{24}$$

and thus convert Eq. (18a) to the equivalent

$$H_l(\mu) = \frac{1}{2\pi i}\int_0^1 \frac{\Lambda_1^-(\nu)}{\Lambda_1^-(\mu)}K_1(\nu, \mu)H_l(\nu)\frac{d\nu}{\nu - \mu}. \tag{25}$$

Furthermore, the normalization $H_l(0) = 1$ follows from Eqs. (15), (20), and (24), whereas the alternative integral constraint follows from Eq. (22):

$$\int_0^1 \Psi_l(\nu)H_l(\nu)\,d\nu = 1. \tag{26}$$

In the process of simplifying the algebra once the explicit form of $\Lambda_1(z)$ is used in Eq. (25), it becomes possible to split off a term proportional to the integral given in Eq. (26), and thus by using that identity we are able to convert the homogeneous integral equation for $H_l(\mu)$ to an inhomogeneous form, the solution of which necessarily is normalized in the desired manner. We find finally that $H_l(\mu)$ is the solution to the Fredholm equation

$$H_l(\mu) = 2g_1(\mu)(2 - 3\mu^2) + \frac{9g_1(\mu)}{4}\int_0^1\left(\frac{\nu\mu(1 - \nu^2)(1 - \mu^2)\Delta(\mu, \nu)}{\nu - \mu} + \frac{2\mu(\nu + \mu)}{3}\right)H_l(\nu)\,d\nu, \tag{27}$$

where

$$g_\alpha(\mu) = [\Lambda_\alpha^+(\mu)\Lambda_\alpha^-(\mu)]^{-1}, \quad \alpha = 1 \text{ or } 2, \tag{28a}$$

$$\Lambda_\alpha^\pm(\mu) = (-1)^\alpha + 3(1 - \mu^2)(1 - \mu\tanh^{-1}\mu) \pm \tfrac{3}{2}\pi i\mu(1 - \mu^2), \quad \alpha = 1 \text{ or } 2, \tag{28b}$$

and

$$\Delta(\mu, \nu) = \ln\left(\frac{(1 - \mu)^2}{\mu(1 + \mu)}\right) - \ln\left(\frac{(1 - \nu)^2}{\nu(1 + \nu)}\right). \tag{28c}$$

For the sake of brevity, we state simply that the expressions

$$X_{2\infty}(\nu) = 1, \tag{29a}$$

$$X_2(0) = \sqrt{2}, \tag{29b}$$

$$H_r(\nu) = 2\sqrt{2}\frac{X_2^+(\nu)}{\Lambda_2^+(\nu)}, \tag{29c}$$

and

$$\int_0^1 \Psi_r(\nu)H_r(\nu)\,d\nu = 1 - \tfrac{1}{2}\sqrt{2}, \tag{29d}$$

with

$$\Psi_r(\nu) = \tfrac{3}{8}(1 - \nu^2), \tag{29e}$$

can be used in a manner analogous to that used for the $\alpha = 1$ case to find the resulting version of Eq. (18b) for $H_r(\mu)$:

$$H_r(\mu) = g_2(\mu)\left[3\sqrt{2}\,\mu(1-\mu^2)\ln\left(\frac{(1-\mu)^2}{\mu(1+\mu)}\right) + 4(4-3\mu^2)\right] + \frac{9g_2(\mu)}{4}\int_0^1\left(\frac{\nu\mu(1-\nu^2)(1-\mu^2)\Delta(\mu,\nu)}{\nu-\mu} - \frac{2\mu(\nu+\mu)}{3}\right)H_r(\nu)\,d\nu. \quad (30)$$

Equations (27) and (30) clearly are not so concise as Chandrasekhar's nonlinear equations[17]:

$$H_\alpha(\mu) = 1 + \mu H_\alpha(\mu)\int_0^1 \Psi_\alpha(\nu)H_\alpha(\nu)\frac{d\nu}{\nu+\mu}, \quad \alpha = l \text{ or } r. \quad (31)$$

Furthermore, for the case of scalar Riemann–Hilbert problems, exact analytical solutions are available; however, the extension to matrices cannot be made analytically, whereas it is felt that the method employed here may be used to advantage for certain classes of matrix Riemann–Hilbert problems.

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[1] K. M. Case, Ann. Phys. (N.Y.) **9**, 1 (1960).
[2] K. M. Case and P. F. Zweifel, *Linear Transport Theory* (Addison-Wesley, Reading, Mass., 1967).
[3] N. J. McCormick and I. Kuščer, J. Math. Phys. **6**, 1939 (1965).
[4] C. E. Siewert and P. F. Zweifel, Ann. Phys. (N.Y.) **36**, 61 (1966).
[5] N. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen, The Netherlands, 1953).
[6] J. Mika, Nucl. Sci. Eng. **22**, 235 (1965).
[7] C. E. Siewert and P. F. Zweifel, J. Math. Phys. **7**, 2092 (1966).
[8] C. E. Siewert and S. K. Fraley, Ann. Phys. (N.Y.) **43**, 338 (1967).
[9] S. A. Mourad and C. E. Siewert, Astrophys. J. **155**, 555 (1969).
[10] J. K. Shultis, Nucl. Sci. Eng. **38**, 83 (1969).
[11] R. Zelazny and A. Kuszell, Ann. Phys. (N.Y.) **16**, 81 (1961).
[12] J. K. Shultis, Ph.D. thesis, University of Michigan, 1968.
[13] D. R. Metcalf and P. F. Zweifel, Nucl. Sci. Eng. **33**, 307 (1968).
[14] S. A. Mourad, Ph.D. thesis, North Carolina State University, 1970.
[15] A. Leonard and J. H. Ferziger, Nucl. Sci. Eng. **26**, 181 (1966).
[16] I. Kuščer, in *Developments In Transport Theory*, edited by E. Inönü and P. F. Zweifel (Academic, New York, 1967).
[17] S. Chandrasekhar, *Radiative Transfer* (Oxford U.P., London, 1950).