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Half-Range Expansion Theorems in Studies of Polarized Light*

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The established normal modes of the vector equation of transfer describing the transport of polarized light are used to construct solutions to typical half-space problems. The half-range completeness theorem required by this method is discussed in the context of systems of singular integral equations. Although the Riemann–Hilbert problem encountered here is defined in terms of continuous rather than Hölder-continuous functions, the existence of a canonical solution is established, and the developed properties of this canonical matrix are used to complete the proof of the necessary half-range expansion theorem.

I. INTRODUCTION

We consider here the vector equation of transfer

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \tfrac{1}{2}\omega \mathsf{Q}(\mu) \int_{-1}^{1} \tilde{\mathsf{Q}}(\mu')\mathbf{I}(\tau, \mu')\, d\mu' \tag{1}$$

applicable to several studies of the scattering of polarized light.[1] Relying principally on Chandrasekhar's formulation of this mathematical model,[1] we denote by $\mathbf{I}(\tau, \mu)$ a vector whose two components $I_l(\tau, \mu)$ and $I_r(\tau, \mu)$ are the angular intensities in the two states of polarization. Further, τ is the optical variable, and μ is the direction cosine (as measured from the positive τ axis) of the propagating radiation.

The scattering process considered here is characterized in Eq. (1) by the single-scatter albedo ω and the square matrix $\mathsf{Q}(\mu)$, with $\tilde{\mathsf{Q}}(\mu)$ denoting the transpose of $\mathsf{Q}(\mu)$. Although much of the analysis presented in this paper is valid for a general Q matrix of polynomials, we are concerned primarily with the form

$$\mathsf{Q}(\mu) = \frac{3(c+2)^{\frac{1}{2}}}{2(c+2)} \begin{bmatrix} c\mu^2 + \frac{2}{3}(1-c) & (2c)^{\frac{1}{2}}(1-\mu^2) \\ \frac{1}{3}(c+2) & 0 \end{bmatrix}; \tag{2}$$

we thus allow the right-hand side of Eq. (1) to contain the two parameters ω and c so that the following special cases can be readily identified:

For $c = 1$ and $\omega = 1$, Eq. (1) with Eq. (2) yields Chandrasekhar's conservative Rayleigh-scattering model[1]

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \tfrac{1}{2} \int_{-1}^{1} \mathsf{K}(\mu, \mu')\mathbf{I}(\tau, \mu')\, d\mu', \tag{3}$$

where

$$\mathsf{K}(\mu, \mu') = \tfrac{3}{4} \begin{bmatrix} 2(1-\mu^2)(1-\mu'^2) + \mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{bmatrix}. \tag{4}$$

For the case $c \in [0, 1]$ and $\omega = 1$, Eqs. (1) and (2) yield Chandrasekhar's conservative model[1] for a mixture of scattering laws,

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \tfrac{1}{2} \int_{-1}^{1} [c\mathsf{K}(\mu, \mu') + (1-c)\mathsf{E}]\mathbf{I}(\tau, \mu')\, d\mu', \tag{5}$$

where

$$\mathsf{E} = \tfrac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{6}$$

Finally, observing the choices $c = 1$ and $\omega \in [0, 1)$, we note that Eqs. (1) and (2) yield the nonconservative version of Eq. (3), as considered, for example, by Simmons,[2] Mullikin,[3] Abhyankar and Fymat,[4] and Schnatz and Siewert,[5] whereas, if we allow the values $c \in [0, 1]$ and $\omega \in [0, 1)$, we obtain the analogous nonconservative version of Eq. (5).

In order to establish the elementary solutions of Eq. (1), we introduce the proposed form

$$\mathbf{I}_\eta(\tau, \mu) = \mathbf{\Phi}(\eta, \mu) e^{-\tau/\eta} \tag{7}$$

to obtain

$$(\eta - \mu)\mathbf{\Phi}(\eta, \mu) = \tfrac{1}{2}\omega\eta \mathsf{Q}(\mu)\mathbf{M}(\eta), \tag{8}$$

where the normalization vector $\mathbf{M}(\eta)$ is given by

$$\mathbf{M}(\eta) = \int_{-1}^{1} \tilde{\mathsf{Q}}(\mu)\mathbf{\Phi}(\eta, \mu)\, d\mu. \tag{9}$$

In the usual manner,[6] we first consider the discrete spectrum, $\eta \notin [-1, 1]$, and solve Eq. (8) to find

$$\mathbf{\Phi}(\pm\eta_0, \mu) = \tfrac{1}{2}\omega\eta_0 \frac{1}{\eta_0 \mp \mu} \mathsf{Q}(\mu)\mathbf{M}(\pm\eta_0), \tag{10}$$

where $\pm\eta_0$ are the two zeros (in the complex plane cut from -1 to 1 along the real axis) of the dispersion function

$$\Lambda(z) = \det \mathbf{\Lambda}(z), \tag{11}$$

where

$$\mathbf{\Lambda}(z) = \mathsf{I} + z \int_{-1}^{1} \mathbf{\Psi}(\mu) \frac{d\mu}{\mu - z}; \tag{12}$$

here I denotes the unit matrix, and the "characteristic" matrix is

$$\mathbf{\Psi}(\mu) = \tfrac{1}{2}\omega\tilde{\mathbf{Q}}(\mu)\mathbf{Q}(\mu). \tag{13}$$

Clearly since $\mathbf{\Psi}(\mu)$ is a symmetic matrix, $\mathbf{\Lambda}(z)$ is symmetric; further, we note that $\mathbf{\Lambda}(z) = \mathbf{\Lambda}(-z)$. Although the normalization vectors can be established from

$$\mathbf{\Lambda}(\eta_0)\mathbf{M}(\eta_0) = \mathbf{0}, \tag{14}$$

we do not require any explicit forms here.

Solving Eq. (8) now for $\eta \in (-1, 1)$, we write

$$\mathbf{\Phi}(\eta, \mu) = \tfrac{1}{2}\omega\left(\eta\frac{P}{\eta - \mu} + \lambda(\eta)\delta(\eta - \mu)\right)\mathbf{Q}(\mu)\mathbf{M}(\eta), \tag{15}$$

where the symbol P denotes that all ensuing integrals are to be evaluated in the Cauchy principal-value sense and $\delta(x)$ is the Dirac δ functional. If we multiply Eq. (15) by $\tilde{\mathbf{Q}}(\mu)$ and integrate over μ from -1 to 1, we find

$$[\mathbf{\lambda}(\eta) - \lambda(\eta)\mathbf{\Psi}(\eta)]\mathbf{M}(\eta) = \mathbf{0}; \tag{16}$$

and hence from

$$\det\,[\mathbf{\lambda}(\eta) - \lambda(\eta)\mathbf{\Psi}(\eta)] = 0, \tag{17}$$

where

$$\mathbf{\lambda}(\eta) = \mathbf{I} + \eta\int_{-1}^{1}\mathbf{\Psi}(\mu)\frac{P}{\mu - \eta}\,d\mu, \tag{18}$$

we obtain, in general, a quadratic equation in $\lambda(\eta)$, which yields two solutions $\lambda_1(\eta)$ and $\lambda_2(\eta)$. There is thus a twofold degeneracy for the continuum, $\eta \in (-1, 1)$; there result then two solutions to Eq. (8),

$$\mathbf{\Phi}_\alpha(\eta, \mu) = \tfrac{1}{2}\omega\left(\eta\frac{P}{\eta - \mu} + \lambda_\alpha(\eta)\delta(\eta - \mu)\right)\mathbf{Q}(\mu)\mathbf{M}_\alpha(\eta),$$
$$\eta \in (-1, 1), \quad \alpha = 1 \text{ and } 2. \tag{19}$$

Since the normal modes are now explicitly available, we write our general solution to Eq. (1) in the form

$$\mathbf{I}(\tau, \mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu)e^{-\tau/\eta_0} + A(-\eta_0)\mathbf{\Phi}(-\eta_0, \mu)e^{\tau/\eta_0} + \int_{-1}^{1}[A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]e^{-\tau/\eta}\,d\eta, \tag{20}$$

where $A(\pm\eta_0)$ and $A_\alpha(\eta)$, $\alpha = 1$ and 2, are the arbitrary expansion coefficients to be determined once the boundary conditions of a given problem are specified. The full-range expansion theorem for the eigenvectors considered here has been established by Schnatz and Siewert[5]; it is, however, the considerably more important half-range expansion theorem we wish to discuss.

II. ANALYSIS

In order to illustrate the need for the half-range expansion theorem, we consider a typical half-space problem: We seek a bounded solution to Eq. (1) for $\tau \in [0, \infty)$ such that the radiation incident at the surface may be specified, i.e.,

$$\mathbf{I}(0, \mu) = \mathbf{I}(\mu), \quad \mu \in (0, 1), \tag{21}$$

where $\mathbf{I}(\mu)$ is given. Clearly, a bounded solution can be readily obtained from Eq. (20) by requiring $A(-\eta_0) \equiv 0$ and $A_1(\eta) = A_2(\eta) \equiv 0$, $\eta < 0$. Thus the desired solution can be written as

$$\mathbf{I}(\tau, \mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu)e^{-\tau/\eta_0} + \int_{0}^{1}[A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]^{-\tau/\eta}\,d\eta, \tag{22}$$

where the expansion coefficients must be chosen such that Eq. (21) is satisfied. We must solve, therefore, the system of singular integral equations

$$\mathbf{I}(\mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu) + \int_{0}^{1}[A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]\,d\eta, \quad \mu \in (0, 1). \tag{23}$$

A statement to the effect that Eq. (23) admits a solution for an arbitrary Hölder[7] vector $\mathbf{I}(\mu)$ is the required half-range expansion theorem; it is this statement we wish to establish.

For the sake of notational convenience, we now introduce the matrix

$$\mathbf{\Psi}(\eta, \mu) = [\mathbf{\Phi}_1(\eta, \mu)\;\mathbf{\Phi}_2(\eta, \mu)] \tag{24}$$

and a vector $\mathbf{A}(\eta)$, with elements $A_1(\eta)$ and $A_2(\eta)$, in order to write Eq. (23) as

$$\mathbf{I}'(\mu) = \int_{0}^{1}\mathbf{\Psi}(\eta, \mu)\mathbf{A}(\eta)\,d\eta, \quad \mu \in (0, 1), \tag{25}$$

where temporarily we have taken the discrete term to the left-hand side of the equation and defined

$$\mathbf{I}'(\mu) = \mathbf{I}(\mu) - A(\eta_0)\mathbf{\Phi}(\eta_0, \mu). \tag{26}$$

We note that, when Eq. (15) is premultiplied by $\tilde{\mathbf{Q}}(\mu)$ and Eq. (16) is used, there results

$$\tilde{\mathbf{Q}}(\mu)\mathbf{\Psi}(\eta, \mu) = \left(\eta\frac{P}{\eta - \mu}\mathbf{\Psi}(\mu) + \delta(\eta - \mu)\mathbf{\lambda}(\eta)\right)\mathbf{V}(\eta), \tag{27}$$

where $\mathsf{V}(\eta)$ is the normalization matrix:

$$\mathsf{V}(\eta) = \int_{-1}^{1} \tilde{\mathsf{Q}}(\mu)\mathbf{\Psi}(\eta, \mu)\, d\mu. \quad (28)$$

A more convenient form of Eq. (25) can now be established by premultiplying that equation by $\tilde{\mathsf{Q}}(\mu)$ and using Eq. (27):

$$\tilde{\mathsf{Q}}(\mu)\mathbf{I}'(\mu) = \mathbf{\lambda}(\mu)\mathbf{B}(\mu) + \mathbf{\Psi}(\mu)\int_0^1 \eta\mathbf{B}(\eta)\frac{P}{\eta-\mu}\,d\eta, \quad \mu \in (0,1), \quad (29)$$

where we have defined

$$\mathbf{B}(\eta) = \mathsf{V}(\eta)\mathbf{A}(\eta). \quad (30)$$

In the usual manner, Eq. (29) can be converted to an equivalent inhomogeneous Riemann–Hilbert problem.[5–7] To this end, we introduce the sectionally holomorphic function

$$\mathbf{N}(z) = \frac{1}{2\pi i}\int_0^1 \eta\mathbf{B}(\eta)\frac{d\eta}{\eta - z}, \quad (31)$$

with boundary values, from above (+) and below (−) the cut, given by the Plemelj formulas[7]

$$\mathbf{N}^{\pm}(\mu) = \frac{1}{2\pi i}\int_0^1 \eta\mathbf{B}(\eta)\frac{P}{\eta-\mu}\,d\eta \pm \tfrac{1}{2}\mu\mathbf{B}(\mu), \quad \mu \in (0,1); \quad (32)$$

and thus

$$\pi i[\mathbf{N}^+(\mu) + \mathbf{N}^-(\mu)] = \int_0^1 \eta\mathbf{B}(\eta)\frac{P}{\eta-\mu}\,d\eta, \quad \mu \in (0,1), \quad (33a)$$

and

$$\mathbf{N}^+(\mu) - \mathbf{N}^-(\mu) = \mu\mathbf{B}(\mu), \quad \mu \in (0,1). \quad (33b)$$

The boundary values of the $\mathbf{\Lambda}$ matrix, as given by Eq. (12), are related by

$$\mathbf{\Lambda}^+(\mu) + \mathbf{\Lambda}^-(\mu) = 2\mathbf{\lambda}(\mu), \quad \mu \in (-1,1), \quad (34a)$$

and

$$\mathbf{\Lambda}^+(\mu) - \mathbf{\Lambda}^-(\mu) = 2\pi i\mu\mathbf{\Psi}(\mu), \quad \mu \in (-1,1); \quad (34b)$$

these relations can be used with Eqs. (33) to write Eq. (29) in the form

$$\mu\tilde{\mathsf{Q}}(\mu)\mathbf{I}'(\mu) = \mathbf{\Lambda}^+(\mu)\mathbf{N}^+(\mu) - \mathbf{\Lambda}^-(\mu)\mathbf{N}^-(\mu), \quad \mu \in (0,1). \quad (35)$$

The general solution to the inhomogeneous Eq. (35) may be written as[7]

$$\mathbf{N}(z) = \mathsf{X}^{-1}(z)\left(\frac{1}{2\pi i}\int_0^1 \mathbf{\Gamma}(\mu)\mathbf{I}'(\mu)\frac{d\mu}{\mu - z} + \mathbf{P}(z)\right), \quad (36)$$

where $\mathbf{P}(z)$ is a vector with polynomial elements,

$$\mathbf{\Gamma}(\mu) = \mu\mathsf{X}^+(\mu)[\mathbf{\Lambda}^+(\mu)]^{-1}\tilde{\mathsf{Q}}(\mu), \quad (37)$$

and $\mathsf{X}(z)$ is the canonical solution to the homogeneous problem

$$\mathsf{X}^+(\mu) = \mathsf{X}^-(\mu)[\mathbf{\Lambda}^-(\mu)]^{-1}\mathbf{\Lambda}^+(\mu), \quad \mu \in (0,1). \quad (38)$$

Clearly, then, to complete the desired proof,we must argue that a matrix $\mathsf{X}(z)$, analytic in the complex plane cut from 0 to 1 along the real axis, exists and has properties such that $\mathbf{N}(z)$ as given by Eq. (36) can be made consistent with the original definition introduced by Eq. (31).

In order to be consistent with the notational convention established by Muskhelishvili[7] and Vekua,[8] we define $\mathbf{\Phi}(z)$ to be the transpose of $\mathsf{X}(z)$ and thus write the transpose of Eq. (38) as

$$\mathbf{\Phi}^+(\mu) = \mathsf{G}(\mu)\mathbf{\Phi}^-(\mu), \quad \mu \in (0,1), \quad (39a)$$

where the symmetry properties of $\mathbf{\Lambda}(z)$ allow us to write

$$\mathsf{G}(\mu) = \mathbf{\Lambda}^+(\mu)[\mathbf{\Lambda}^-(\mu)]^{-1}, \quad \mu \in (0,1). \quad (39b)$$

It is clear that,by adding an arbitrary arc C_1 to the real-line segment [0, 1], we need only deal with a closed Lyapanov contour C. On C_1 we define $\mathsf{G}(\mu) = \mathsf{I}$; thus, since Eq. (12) yields continuous boundary values on (0, 1) while at the end points of the line segment $\lim_{\mu\to 0+} \mathsf{G}(\mu) = \lim_{\mu \to 1-} \mathsf{G}(\mu) = \mathsf{I}$, the matrix $\mathsf{G}(\mu)$ is continuous for all $\mu \in C$. This function, however, fails to be Hölder continuous at $\mu = 1$, as can be seen from the special case $c = 0$. Here $\mathsf{G}(\mu)$ becomes

$$\mathsf{G}_0(\mu) = \begin{bmatrix} g(\mu) & 0 \\ 0 & 1 \end{bmatrix}, \quad \mu \in (0,1), \quad c = 0, \quad (40)$$

with $g(\mu)$ being equivalent to the one-speed result discussed by Case and Zweifel[9]:

$$g(\mu) = \left[1 + \tfrac{1}{2}\omega\mu\ln\left(\frac{1-\mu}{1+\mu}\right) + \tfrac{1}{2}\omega\mu\pi i\right] \times \left[1 + \tfrac{1}{2}\omega\mu\ln\left(\frac{1-\mu}{1+\mu}\right) - \tfrac{1}{2}\omega\mu\pi i\right]^{-1}. \quad (41)$$

In order for $g(\mu)$ to be Hölder continuous at $\mu = 1$, we require $|g(\mu) - g(1)|/|\mu - 1|^\alpha$ to be bounded for some $\alpha \in [0, 1)$. However,

$$\frac{|g(\mu) - 1|}{|\mu - 1|^\alpha} = \frac{\omega\mu\pi}{|\mu-1|^\alpha}\left|1 + \tfrac{1}{2}\omega\mu\ln\left(\frac{1-\mu}{1+\mu}\right) - \tfrac{1}{2}\omega\mu\pi i\right| \quad (42)$$

is clearly unbounded for all appropriate α. For the same reason, the matrix $\mathsf{G}(\mu)$ fails to be piecewise Hölder continuous on C. Thus, without modification, it is apparent that neither the theory given by Muskhelishvili[7] nor that of Vekua[8] is sufficient for the solution of

$$\mathbf{\Phi}^+(\mu) = \mathsf{G}(\mu)\mathbf{\Phi}^-(\mu), \quad \mu \in C, \tag{43}$$

and so we base our reasoning for the existence of a canonical solution on the theory given by Mandžavidze and Hvedelidze.[10] These authors prove that if $\mathsf{G}(\mu)$ is a nonsingular continuous matrix and C a simple closed Lyapanov curve, then there exists a so-called canonical matrix $\mathbf{\Phi}_0(z)$ such that: (i) The matrices $\mathbf{\Phi}_0(z)$ and $[\mathbf{\Phi}_0(z)]^{-1}$ are representable by Cauchy integrals with polynomial principal parts at infinity; (ii) the matrix $\mathbf{\Phi}_0(z)$ has normal form at infinity; (iii) the boundary values on C of $\mathbf{\Phi}_0(z)$ are L_p functions ($p > 1$) and those of $[\mathbf{\Phi}_0(z)]^{-1}$ are L_q functions, p and q being conjugate indices; in addition, these boundary values satisfy

$$\mathsf{G}(\mu) = \mathbf{\Phi}_0^+(\mu)[\mathbf{\Phi}_0^-(\mu)]^{-1} \tag{44}$$

almost everywhere on C. The procedure reported by Mandžavidze and Hvedelidze is not concerned with the solubility and equivalence of a certain quasi-Fredholm equation as is the theory given by Muskhelishvili (see Eq. 126.5, p. 386 of Ref. 7). Basically their method is to show that the problem

$$\mathbf{\Theta}^+(\mu) - \mathbf{\Theta}^-(\mu) = \mathsf{G}_1(\mu)\mathbf{\Theta}^-(\mu) + \mathsf{E}(\mu), \quad \mu \text{ on } C, \tag{45}$$

where each component of the matrix $\mathsf{G}_1(\mu)$ is sufficiently small and $\mathsf{E}(\mu)$ is a given matrix of L_p functions, can be solved by the following sequence:

$$\mathbf{\Theta}_0^-(\mu) = \mathbf{0} \tag{46a}$$

and

$$\mathbf{\Theta}_m(z) = \frac{1}{2\pi i}\int_C \mathsf{G}_1(\mu)\mathbf{\Theta}_{m-1}^-(\mu)\frac{d\mu}{\mu - z} + \frac{1}{2\pi i}\int_C \mathsf{E}(\mu)\frac{d\mu}{\mu - z}, \quad m = 1, 2, 3, \cdots. \tag{46b}$$

This sequence has been shown to be Cauchy in the L_p norm[10] and hence convergent to an L_p function which satisfies Eq. (45) almost everywhere.

In order to establish the required properties of this canonical solution, we need to determine the index of $\mathsf{G}(\mu)$, namely

$$\kappa = \frac{1}{2\pi i}[\arg \det \mathsf{G}(\mu)]_C$$

which is easily seen, for the case of $\mathsf{Q}(\mu)$ as given by Eq. (2), to be unity. Thus the partial indices[7,10] satisfy

$$\kappa_1 + \kappa_2 = 1. \tag{47}$$

In actual fact, these partial indices turn out to be zero and unity. A proof of this may be modeled on one given by Kuščer.[11] Let $\mathbf{\Phi}_0(z)$ be the canonical solution to Eq. (43); then it is easily shown[7,10] that any other solution of finite degree at infinity can be expressed as

$$\mathbf{\Phi}(z) = \mathbf{\Phi}_0(z)\mathsf{P}(z), \tag{48}$$

where $\mathsf{P}(z)$ is a matrix of polynomials. Considering now the function

$$\mathbf{\Psi}(z) = \mathbf{\Lambda}(z)[\tilde{\mathbf{\Phi}}_0(-z)]^{-1}, \tag{49}$$

we note that the boundary values of $\mathbf{\Psi}(z)$ on C clearly satisfy Eq. (43) almost everywhere. If we now change z to $-z$ in Eq. (49), we can take boundary values of the resulting equation to obtain

$$\mathbf{\Psi}^-(-\mu) = \mathbf{\Lambda}^+(\mu)[\tilde{\mathbf{\Phi}}_0^+(\mu)]^{-1}, \quad \mu \in (0, 1), \tag{50a}$$

while

$$\mathbf{\Psi}^+(-\mu) = \mathbf{\Lambda}^-(\mu)[\tilde{\mathbf{\Phi}}_0^-(\mu)]^{-1}, \quad \mu \in (0, 1), \tag{50b}$$

since, as noted previously, $\mathbf{\Lambda}(z) = \mathbf{\Lambda}(-z)$. Equation (43) and the fact that $\mathbf{\Lambda}(z) = \tilde{\mathbf{\Lambda}}(z)$ can now be used to show that Eqs. (50) yield $\mathbf{\Psi}^-(-\mu) = \mathbf{\Psi}^+(-\mu)$, $\mu \in (0, 1)$, almost everywhere. Clearly, then, $\mathbf{\Psi}(z)$ is analytic in the plane cut from 0 to 1 on the real axis and of finite degree at infinity, and thus $\mathbf{\Psi}(z)$ is a solution to Eq. (43). Consequently, it can be expressed in the form of Eq. (48). We now suppose that one of the partial indices is negative, say κ_1 [note from Eq. (47) that only one can be negative]; then the first element of the first column of $\mathbf{\Phi}_0(z)$ has a pole at infinity. This implies, however, from Eq. (49) that the first column of $\mathbf{\Psi}(z)$ vanishes at infinity. But, recalling Eq. (48), we note that this is impossible. Thus the partial indices are zero and unity.

Since the existence of a canonical matrix $\mathbf{\Phi}_0(z)$ has been established, it is a simple matter to complete the proof of half-range completeness. The fact that $\mathbf{\Phi}_0(z)$ must be of normal form at infinity requires that

$$\lim_{z\to\infty} \mathbf{\Phi}_0(z)\begin{bmatrix} z^{\kappa_1} & 0 \\ 0 & z^{\kappa_2} \end{bmatrix} = \mathsf{I}, \tag{51}$$

and, since we have shown that the partial indices must be nonnegative and sum to unity, without loss of generality, we select $\kappa_1 = 0$ and $\kappa_2 = 1$.

We note that the analytic properties of $\mathbf{N}(z)$, as given by Eq. (36), are correct if we make the identification

$$\mathsf{X}(z) = \tilde{\mathbf{\Phi}}_0(z); \tag{52}$$

on the other hand, we observe from Eq. (31) that $z\mathbf{N}(z)$ must be bounded as z tends to infinity. The required behavior of $\mathsf{X}^{-1}(z)$ for large z can be deduced from Eqs. (51) and (52):

$$\mathsf{X}^{-1}(z) \sim z \begin{bmatrix} \frac{1}{z} + \cdots & \frac{a}{z} + \cdots \\ \frac{b}{z^2} + \cdots & 1 + \cdots \end{bmatrix}. \quad (53)$$

Considering now Eq. (36), we find that, in order for $z\mathbf{N}(z)$ to be bounded at infinity, we must take $\mathbf{P}(z) \equiv \mathbf{0}$; the behavior of $\mathsf{X}^{-1}(z)$ for large z, as given by Eq. (53), indicates that $z\mathbf{N}(z)$ will not be bounded at infinity unless we impose on $\mathbf{I}'(\mu)$ the constraint

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\mathrm{T}} \int_0^1 \mathbf{\Gamma}(\mu)\mathbf{I}'(\mu)\, d\mu = 0, \quad (54)$$

where the superscript T denotes transpose. Recalling Eq. (26), we see that Eq. (54) can be satisfied for all appropriate $\mathbf{I}'(\mu)$ simply by choosing the correct discrete coefficient $A(\eta_0)$:

$$A(\eta_0)\begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\mathrm{T}} \int_0^1 \mathbf{\Gamma}(\mu)\mathbf{\Phi}(\eta_0, \mu)\, d\mu = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\mathrm{T}} \int_0^1 \mathbf{\Gamma}(\mu)\mathbf{I}(\mu)\, d\mu. \quad (55)$$

The desired expansion theorem (23) is thus established.

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