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ON THE TRANSFER OF POLARIZED LIGHT IN RAYLEIGH-SCATTERING HALF SPACES WITH TRUE ABSORPTION

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SUMMARY

The singular-eigenfunction-expansion technique, often employed for studies in neutron transport theory, is used to introduce a general solution to the vector equation describing the transfer of polarized light in a Rayleigh-scattering atmosphere with true absorption (imperfect scattering). Two discrete eigenvectors and two linearly independent, degenerate, continuum eigenvectors are presented and form the basis for subsequent analysis which is directed primarily towards the construction of solutions to typical half-space problems.

The eigenvectors are used to establish a singular integral equation and a linear integral constraint necessary and sufficient to specify the required **H**-matrix, a generalization of Chandrasekhar's H-function. Further, the analysis yields the non-linear integral equation, previously derived from the classical principles of invariance; this non-linear equation and the linear constraint are also shown to be sufficient to specify the **H**-matrix and are used for computational purposes.

After the general formalism is established, the method is used to construct complete solutions (valid at any optical depth) to the half-space albedo problem and the classical Milne problem. These solutions are written explicitly in terms of the normal modes, and all unknown expansion coefficients are then expressed in terms of the **H**-matrix. To illustrate the computational merits of the analysis, highly accurate numerical results for the half-space albedo and the Milne-problem extrapolated end-point are presented for a representative set of values of the single-scattering albedo.

1. INTRODUCTION

The influence of polarization phenomena on radiative transfer has been of fundamental interest to researchers in both neutron physics and astrophysics. Wigner (1) and later Bell & Goad (2) have discussed polarization effects in the context of neutron transport theory, while it is generally acknowledged that Chandrasekhar (3) first properly formulated the equations of transfer describing an arbitrarily polarized radiation field in a free-electron atmosphere.

Although most studies (4, 5, 6) of the scattering of polarized light have been based on Chandrasekhar's conservative model (3, 7), recent studies, by allowing the single-scattering albedo to be less than unity, have included the effects of true absorption for homogeneous (8, 9) as well as inhomogeneous atmospheres (10, 11). Abhyankar & Fymat (12) have used the principles of invariance (7) to evaluate the law of diffuse reflection for an imperfectly scattering semi-infinite atmosphere. These invariance principles have been successfully employed for the determination of surface distributions in many astrophysical problems (4, 5, 13), but yield no immediate information pertaining to the radiation field within the medium. In this regard, the singular-eigenfunction-expansion technique developed by Case (**14**) has been introduced to advantage by several authors to construct rigorous analytical solutions valid at any optical depth; Siewert & Fraley (**6**), for example, have solved the conservative Rayleigh-scattering problem in a semi-infinite atmosphere. Similar methods have also been used to establish full-range completeness and orthogonality theorems basic to the normal-mode approach for a conservative combination of Rayleigh and isotropic scattering (**15**), and numerical results have been obtained for the half-space Milne problem (**16**). More recently, Schnatz & Siewert (**17**) have established the normal modes and the required full-range theorems pertinent to the non-conservative problem considered herein.

Pahor (**18**), in his thorough treatment of the degenerate-kernel model in the theory of thermal-neutron scattering, has combined the principles of invariance and the method of normal modes to establish many results applicable here. It is the purpose of this paper to report semi-analytical solutions valid at any optical depth of various half-space problems. We base the analysis solely on normal-mode considerations and thus do not require the physical arguments usually associated with the principles of invariance.

We consider then the non-conservative Rayleigh-scattering equation of transfer in the form (**10**)

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \tfrac{1}{2}\tilde{\omega}\mathbf{Q}(\mu) \int_{-1}^{1} \tilde{\mathbf{Q}}(\mu')\mathbf{I}(\tau, \mu')\, d\mu', \tag{1}$$

where

$$\mathbf{Q}(\mu) = \frac{\sqrt{3}}{2}\begin{bmatrix} \mu^2 & \sqrt{2}(1-\mu^2) \\ 1 & 0 \end{bmatrix} \tag{2}$$

and $\tilde{\mathbf{Q}}(\mu)$ denotes the transpose of $\mathbf{Q}(\mu)$. Also, τ is the optical variable, μ is the direction cosine (as measured from the positive τ-axis) of the propagating radiation, and $\tilde{\omega} \in [0, 1]$ is the single-scattering albedo. Further, the components $I_l(\tau, \mu)$ and $I_r(\tau, \mu)$ are sufficient to determine the intensity and state of polarization of the azimuth-independent radiation field (**7**).

Since the analysis pertinent to the normal modes of equation (1) and the associated full-range expansion and orthogonality theorems have been reported (**17**), we should like to give only a brief review of the relevant results required in the present work.

Seeking soultions of the form

$$\mathbf{I}(\tau, \mu) = e^{-\tau/\eta}\, \mathbf{\Phi}(\eta, \mu), \tag{3}$$

we obtain the eigenvalue equation

$$(\eta - \mu)\mathbf{\Phi}(\eta, \mu) = \tfrac{1}{2}\tilde{\omega}\eta\mathbf{Q}(\mu)\mathbf{M}(\eta), \tag{4}$$

where the normalization vector

$$\mathbf{M}(\eta) = \int_{-1}^{1} \tilde{\mathbf{Q}}(\mu)\mathbf{\Phi}(\eta, \mu)\, d\mu \tag{5}$$

has been defined. The discrete solutions to equation (4) can be written as

$$\mathbf{\Phi}(\pm\eta_0, \mu) = \tfrac{1}{2}\tilde{\omega}\eta_0 \frac{1}{\eta_0 \mp \mu}\mathbf{Q}(\mu)\mathbf{M}(\eta_0), \tag{6}$$

where η_0 is the positive zero of the dispersion function

$$\Lambda(z) = 8 \det \mathbf{\Lambda}(z); \tag{7}$$

here the $\mathbf{\Lambda}$-matrix is given by

$$\mathbf{\Lambda}(z) = \mathbf{I} + z \int_{-1}^{1} \mathbf{\Psi}(\mu) \frac{d\mu}{\mu - z}, \tag{8}$$

where $\mathbf{I}$ is the identity matrix, and the ' characteristic ' function is

$$\mathbf{\Psi}(\mu) = \tfrac{1}{2}\tilde{\omega}\tilde{\mathbf{Q}}(\mu)\mathbf{Q}(\mu). \tag{9}$$

Equation (8) can be used in equation (7) to write the dispersion function more explicitly:

$$\Lambda(z) = \Lambda_1(z)\Lambda_2(z) + 12z^2(1-\tilde{\omega})\Lambda_0(z), \tag{10}$$

where

$$\Lambda_\alpha(z) = (-1)^\alpha + 3(1-z^2)\Lambda_0(z) - (-1)^\alpha 3z^2(1-\tilde{\omega}), \quad \alpha = 1 \text{ or } 2, \tag{11 a}$$

and

$$\Lambda_0(z) = 1 + \tfrac{1}{2}\tilde{\omega} z \int_{-1}^{1} \frac{d\mu}{\mu - z}. \tag{11 b}$$

Finally, the discrete normalization vector follows from the requirement

$$\mathbf{\Lambda}(\eta_0)\mathbf{M}(\eta_0) = \mathbf{0}; \tag{12}$$

we use the form

$$\mathbf{M}(\eta_0) = \sqrt{6}\begin{bmatrix} 2\sqrt{2}\eta_0^2(1-\tilde{\omega}) \\ \Lambda_2(\eta_0) + 2\eta_0^2(1-\tilde{\omega}) \end{bmatrix}. \tag{13}$$

Solving equation (4) now for $\eta \in (-1, 1)$, we write

$$\mathbf{\Phi}(\eta, \mu) = \frac{\tilde{\omega}}{2}\left[\eta \frac{\mathscr{P}}{\eta - \mu} + \bar{\lambda}(\eta)\delta(\eta - \mu)\right] \mathbf{Q}(\mu)\mathbf{M}(\eta), \tag{14}$$

where the symbol $\mathscr{P}$ denotes that all ensuing integrals are to be evaluated in the Cauchy principal-value sense, and $\delta(\eta)$ is the Dirac delta function. If we multiply equation (14) by $\tilde{\mathbf{Q}}(\mu)$ and integrate over μ from -1 to 1, we find

$$[\boldsymbol{\lambda}(\eta) - \bar{\lambda}(\eta)\mathbf{\Psi}(\eta)]\mathbf{M}(\eta) = \mathbf{0}; \tag{15}$$

and hence from

$$\det [\boldsymbol{\lambda}(\eta) - \bar{\lambda}(\eta)\mathbf{\Psi}(\eta)] = 0, \tag{16}$$

where

$$\boldsymbol{\lambda}(\eta) = \mathbf{I} + \eta \mathscr{P} \int_{-1}^{1} \mathbf{\Psi}(\mu) \frac{d\mu}{\mu - \eta}, \tag{17}$$

we obtain a quadratic equation in $\bar{\lambda}(\eta)$, which yields two solutions $\bar{\lambda}_1(\eta)$ and $\bar{\lambda}_2(\eta)$. There is thus a two-fold degeneracy for the continuum, $\eta \in (-1, 1)$, resulting then in two linearly independent solutions to equation (4):

$$\mathbf{\Phi}_\alpha(\eta, \mu) = \frac{\tilde{\omega}}{2}\left[\eta \frac{\mathscr{P}}{\eta - \mu} + \bar{\lambda}_\alpha(\eta)\delta(\eta - \mu)\right] \mathbf{Q}(\mu)\mathbf{M}_\alpha(\eta), \quad \eta \in (-1, 1), \quad \alpha = 1 \text{ and } 2. \tag{18}$$

The general solution of equation (1) can now be written as a linear sum of the normal modes:

$$\mathbf{I}(\tau, \mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu)\, e^{-\tau/\eta_0} + A(-\eta_0)\mathbf{\Phi}(-\eta_0, \mu)\, e^{\tau/\eta_0} + \int_{-1}^{1} \mathbf{\Psi}(\eta, \mu)\mathbf{A}(\eta)\, e^{-\tau/\eta}\, d\eta \quad (19)$$

where $\mathbf{\Psi}(\eta, \mu)$ denotes the 2×2 matrix

$$\mathbf{\Psi}(\eta, \mu) = [\mathbf{\Phi}_1(\eta, \mu) \quad \mathbf{\Phi}_2(\eta, \mu)], \quad (20)$$

and

$$\mathbf{A}(\eta) = \begin{bmatrix} A_1(\eta) \\ A_2(\eta) \end{bmatrix}. \quad (21)$$

Here, $A(\pm\eta_0)$, $A_1(\eta)$ and $A_2(\eta)$ are the arbitrary coefficients to be determined from the boundary conditions of a given problem.

It has been shown (**17**) that the eigenvectors $\mathbf{\Phi}(\pm\eta_0, \mu)$ and $\mathbf{\Phi}_\alpha(\eta, \mu)$, $\alpha = 1$ and 2, are complete on the full-range, $\mu \in (-1, 1)$, in the sense that an arbitrary two-component vector $\mathbf{I}(\mu)$ satisfying the Hölder condition (**19**) can be expanded in the form

$$\mathbf{I}(\mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu) + A(-\eta_0)\mathbf{\Phi}(-\eta_0, \mu) + \int_{-1}^{1} \mathbf{\Psi}(\eta, \mu)\mathbf{A}(\eta)\, d\eta, \quad \mu \in (-1, 1). \quad (22)$$

Although we have yet to specify the normalization vectors $\mathbf{M}_\alpha(\eta)$, $\alpha = 1$ and 2, and the resulting explicit forms of the continuum eigenvectors, the following orthogonality condition follows immediately from equation (4):

$$\int_{-1}^{1} \mu\tilde{\mathbf{\Phi}}(\xi', \mu)\mathbf{\Phi}(\xi, \mu)\, d\mu = 0, \quad \xi' \neq \xi. \quad (23)$$

In addition, the associated full-range discrete normalization integrals,

$$M(\pm\eta_0) = \int_{-1}^{1} \mu\tilde{\mathbf{\Phi}}(\pm\eta_0, \mu)\mathbf{\Phi}(\pm\eta_0, \mu)\, d\mu, \quad (24a)$$

can be written as

$$M(\pm\eta_0) = \pm\frac{\tilde{\omega}}{2}\eta_0^2\tilde{\mathbf{M}}(\eta_0)\frac{d}{dz}\mathbf{\Lambda}(z)\Big|_{z=\eta_0}\mathbf{M}(\eta_0). \quad (24b)$$

Clearly, the analogous normalization integrals for the continuum solutions depend on the forms used; for the previously reported (**17**)

$$\mathbf{\Phi}_1(\eta, \mu) = \begin{bmatrix} \frac{3\tilde{\omega}\eta}{2}(1-\eta^2)(1-\mu^2)\frac{\mathscr{P}}{\eta-\mu} + [(1-\eta^2)\lambda_1(\eta) + 2\eta^2(1-\tilde{\omega})]\delta(\eta-\mu) \\ -2\eta^2(1-\tilde{\omega})\delta(\eta-\mu) \end{bmatrix}, \quad \eta \in (-1, 1), \quad (25a)$$

and

$$\mathbf{\Phi}_2(\eta, \mu) = \begin{bmatrix} \frac{3\tilde{\omega}\eta}{2}(1-\eta^2)\frac{\mathscr{P}}{\eta-\mu} + \lambda_1(\eta)\delta(\eta-\mu) \\ \frac{3\tilde{\omega}\eta}{2}(1-\eta^2)\frac{\mathscr{P}}{\eta-\mu} + \lambda_2(\eta)\delta(\eta-\mu) \end{bmatrix}, \quad \eta \in (-1, 1), \quad (25b)$$

where

$$\lambda_\alpha(\eta) = (-1)^\alpha + 3(1-\eta^2)\lambda_0(\eta) - (-1)^\alpha 3\eta^2(1-\tilde{\omega}), \quad \alpha = 1 \text{ or } 2, \tag{26a}$$

and

$$\lambda_0(\eta) = 1 - \tilde{\omega}\eta \tanh^{-1}\eta, \tag{26b}$$

we find

$$\int_{-1}^{1} \mu \tilde{\mathbf{\Phi}}_\alpha{}^\dagger(\eta', \mu)\mathbf{\Phi}_\beta(\eta, \mu)\, d\mu = M(\eta)\delta(\eta-\eta')\delta_{\alpha\beta}, \tag{27}$$

where adjoint vectors $\mathbf{\Phi}_\alpha{}^\dagger(\eta, \mu)$ are the linear combinations of $\mathbf{\Phi}_1(\eta, \mu)$ and $\mathbf{\Phi}_2(\eta, \mu)$ given in (**17**). Further,

$$M(\eta) = \eta(1-\eta^2)^2\Lambda^+(\eta)\Lambda^-(\eta), \tag{28}$$

and $\Lambda^\pm(\eta)$ denote the boundary values (**19**) of $\Lambda(z)$ as z approaches the branch cut $[-1, 1]$ from above $(+)$ or below $(-)$.

The continuum solutions given by equations (25) clearly are not the only choices; in fact, a mutually orthogonal set can be defined simply by selecting appropriate linear combinations of equations (25).

2. GENERAL HALF-RANGE ANALYSIS

Although the general solution as given by equation (19) and the foregoing analysis are sufficient for problems defined by full-range boundary conditions, the considerably more important and more interesting analysis deals with problems where half-range boundary conditions are appropriate. An albedo problem is typical of these half-range situations; here we seek a bounded solution of equation (1) for $\tau \in (0, \infty)$ such that an arbitrary incident distribution $\mathbf{I}(\mu)$ may be specified. Hence the general solution given by equation (19) must be constrained such that

$$\lim_{\tau\to\infty} \mathbf{I}(\tau, \mu) < \infty \tag{29a}$$

and

$$\mathbf{I}(0, \mu) = \mathbf{I}(\mu), \quad \mu \in (0, 1). \tag{29b}$$

Equation (29 a) clearly requires that we specify $A(-\eta_0)$ and $\mathbf{A}(\eta)$, $\eta < 0$, in equation (19) to be zero; we thus write

$$\mathbf{I}(\tau, \mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu)\, e^{-\tau/\eta_0} + \int_0^1 \mathbf{\Psi}(\eta, \mu)\mathbf{A}(\eta)\, e^{-\tau/\eta}\, d\eta. \tag{30}$$

Substituting equation (30) into the surface boundary conditions, equation (29b), we observe that $\mathbf{I}(\tau, \mu)$ will be the desired solution only if $A(\eta_0)$ and $\mathbf{A}(\eta)$, $\eta \in (0, 1)$, exist such that the half-range expansion

$$\mathbf{I}(\mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu) + \int_0^1 \mathbf{\Psi}(\eta, \mu)\mathbf{A}(\eta)\, d\eta, \quad \mu \in (0, 1), \tag{31}$$

is valid. Equation (31) is a coupled set of singular integral equations which must yield a solution for the expansion coefficients. Burniston & Siewert (**20**) have recently proved that the eigenvectors $\mathbf{\Phi}(\eta_0, \mu)$ and $\mathbf{\Phi}_\alpha(\eta, \mu)$, $\alpha = 1$ and 2, $\eta \in (0, 1)$, are complete on the half-range in the sense that an arbitrary two-vector $\mathbf{I}(\mu)$ satisfying the Hölder condition may be expanded in the form of equation (31). The essence of the proof is to reduce equation (31) to a matrix version of the Riemann-Hilbert problem for which the theory of Muskhelishvili (**19**) and Mandžavidze &

Hvedelidze (**21**) can be used to ensure the existence of a solution. However, in contrast to similar completeness proofs for scalar equations of transfer (**22**), the analysis of Burniston & Siewert (**20**) does not yield an analytical result for the required **X**-matrix, and thus closed-form expressions for $A(\eta_0)$ and $\mathbf{A}(\eta)$ were not obtained. Although the half-range completeness proof can be pursued to yield similar results, we prefer here to make use, as have Pahor & Zweifel (**23**), of the **S**-matrix concept discussed by Chandrasekhar (**7**).

We consider then the generalized albedo problem and thus seek for $\tau \in [0, \infty)$ a bounded solution $\mathbf{I}(\tau, \mu, \mu_0)$ of equation (1) such that

$$\mathbf{I}(0, \mu, \mu_0) = \delta(\mu-\mu_0)\mathbf{F}, \quad \mu \text{ and } \mu_0 \in (0, 1), \tag{32}$$

where **F** is a constant. The resulting emergent distribution can be defined in terms of the scattering matrix introduced by Chandrasekhar (**7**) and discussed by Sekera (**10**):

$$\mathbf{I}(0, -\mu, \mu_0) = \frac{1}{2\mu}\mathbf{S}(\mu, \mu_0)\mathbf{F}, \quad \mu \text{ and } \mu_0 \in (0, 1). \tag{33}$$

Clearly, once the scattering matrix is determined, the reflected radiation resulting from an arbitrary incident distribution with axial symmetry follows immediately:

$$\mathbf{I}(0, -\mu) = \frac{1}{2\mu}\int_0^1 \mathbf{S}(\mu, \mu_0)\mathbf{I}(\mu_0)\, d\mu_0, \quad \mu \in (0, 1). \tag{34}$$

Thus, $\mathbf{I}(0, \mu)$ becomes known for all $\mu \in (-1, 1)$, and hence the full-range theory (**17**) can be used to construct $\mathbf{I}(\tau, \mu)$ for $\tau \in [0, \infty)$ and $\mu \in (-1, 1)$.

From the fact that a sufficiently general incident distribution can be expanded as in equation (31), and since from equation (30) the exit solution resulting from such incident radiation is given by

$$\mathbf{I}(0, -\mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, -\mu) + \int_0^1 \mathbf{\Psi}(\eta, -\mu)\mathbf{A}(\eta)\, d\eta, \quad \mu \in (0, 1), \tag{35}$$

we conclude, in view of equation (34), that necessary and sufficient conditions required to specify the scattering matrix are that the equations

$$\mathbf{\Psi}(\eta, -\mu) = \frac{1}{2\mu}\int_0^1 \mathbf{S}(\mu, \mu')\mathbf{\Psi}(\eta, \mu')\, d\mu', \quad \eta \text{ and } \mu \in (0, 1), \tag{36a}$$

and

$$\mathbf{\Phi}(\eta_0, -\mu) = \frac{1}{2\mu}\int_0^1 \mathbf{S}(\mu, \mu')\mathbf{\Phi}(\eta_0, \mu')\, d\mu', \quad \mu \in (0, 1), \tag{36b}$$

be satisfied. We note that equations (36) are directly analogous to those derived by Mullikin (**24**) for scalar cases.

Though equations (36) are the basic equations for $\mathbf{S}(\mu, \mu')$, we prefer the more tractable forms obtained by pre-multiplying each equation by $\tilde{\mathbf{Q}}(\mu)$ and integrating over μ from zero to one; we thus write

$$\mathbf{G}(\eta) = \int_0^1 \tilde{\mathbf{N}}(\mu)\mathbf{\Psi}(\eta, \mu)\, d\mu, \quad \eta \in (0, 1), \tag{37a}$$

and

$$\mathbf{M}(\eta_0) = \int_0^1 \tilde{\mathbf{N}}(\mu)\mathbf{\Phi}(\eta_0, \mu)\, d\mu. \tag{37b}$$

Here, to be consistent with the notation (**12**) resulting from an application of Chandrasekhar's invariance principles (**7**), we have denoted the matrix $\mathbf{N}(\mu)$ by

$$\mathbf{N}(\mu) = \mathbf{Q}(\mu)+\tfrac{1}{2}\int_0^1 \mathbf{S}(\mu,\mu')\mathbf{Q}(\mu')\frac{d\mu'}{\mu'} \tag{38}$$

and have invoked a statement of the reciprocity theorem (**7, 10**)

$$\mathbf{S}(\mu,\mu') = \tilde{\mathbf{S}}(\mu',\mu). \tag{39}$$

Further, $\mathbf{M}(\eta_0)$ is the discrete normalization vector given by equation (13), and we have introduced the 2×2 matrix

$$\mathbf{G}(\eta) = \int_{-1}^1 \tilde{\mathbf{Q}}(\mu)\boldsymbol{\Psi}(\eta,\mu)\,d\mu = [\mathbf{M}_1(\eta) \quad \mathbf{M}_2(\eta)]. \tag{40}$$

If equation (20) is pre-multiplied by $\tilde{\mathbf{Q}}(\mu)$, we find

$$\tilde{\mathbf{Q}}(\mu)\boldsymbol{\Psi}(\eta,\mu) = \left[\eta\frac{\mathscr{P}}{\eta-\mu}\boldsymbol{\Psi}(\mu)+\delta(\eta-\mu)\boldsymbol{\lambda}(\eta)\right]\mathbf{G}(\eta), \tag{41}$$

where equation (15), written collectively for both $\lambda_1(\eta)$ and $\lambda_2(\eta)$ as

$$\boldsymbol{\lambda}(\eta)\mathbf{G}(\eta) = \boldsymbol{\Psi}(\eta)\mathbf{G}(\eta)\begin{bmatrix}\lambda_1(\eta) & 0\\ 0 & \lambda_2(\eta)\end{bmatrix}, \tag{42}$$

and equation (18) have been used. We now introduce the **H**-matrix (**18, 12**)

$$\mathbf{N}(\mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu), \tag{43}$$

make the substitution in equations (37) and use equations (6) and (41) to obtain

$$\tilde{\mathbf{H}}(\eta)\boldsymbol{\lambda}(\eta) = \mathbf{I}+\eta\int_0^1 \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\frac{\mathscr{P}}{\mu-\eta}\,d\mu, \quad \eta\in(0,1), \tag{44a}$$

and

$$\mathbf{0} = \left[\mathbf{I}+\eta_0\int_0^1 \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\frac{1}{\mu-\eta_0}\,d\mu\right]\mathbf{M}(\eta_0). \tag{44b}$$

Equation (44 a) is a singular integral equation for the matrix $\mathbf{H}(\mu)$, $\mu\in(0,1)$, whereas equation (44 b) represents an additional integral constraint that must be satisfied by the required $\mathbf{H}(\mu)$. These equations are analogous to Mullikin's singular integral equation and linear constraint (**24**), and are remarkably similar to those derived by Pahor & Zweifel (**23**) for the case of the one-speed isotropic-scattering neutron transport equation, and by Pahor (**18**) for a problem in the theory of thermal-neutron transport. While for the scalar theory a closed-form solution for the H-function was readily established (**23**), here we are assured only that equations (44) are necessary and sufficient conditions for determining the required **H**-matrix.

Since the **H**-matrix will henceforth be the principal quantity of interest, we should like to reconstruct the solution for the scattering matrix $\mathbf{S}(\mu,\mu')$. By analogy with the case of isotropic scattering (**23**), we propose the form

$$\mathbf{S}(\mu,\mu') = \frac{\mu\mu'}{\mu+\mu'}\mathbf{R}(\mu,\mu')\tilde{\mathbf{Q}}(\mu'), \tag{45}$$

where $\mathbf{R}(\mu,\mu')$ is to be determined. If equation (45) is substituted into equation

32

(36), the results

$$\mathbf{R}(\mu,\eta)\boldsymbol{\lambda}(\eta) = \tilde{\omega}\mathbf{Q}(\mu)\mathbf{H}(\mu)+\eta\int_0^1 \mathbf{R}(\mu,\mu')\boldsymbol{\Psi}(\mu')\frac{\mathscr{P}}{\mu'-\eta}\,d\mu', \quad \eta,\quad \mu\in(0,1), \tag{46a}$$

and

$$\mathbf{0} = \left[\tilde{\omega}\mathbf{Q}(\mu)\mathbf{H}(\mu)+\eta_0\int_0^1 \mathbf{R}(\mu,\mu')\boldsymbol{\Psi}(\mu')\frac{1}{\mu'-\eta_0}\,d\mu'\right]\mathbf{M}(\eta_0), \quad \mu\in(0,1), \tag{46b}$$

are readily established after use of the definitions of the eigenvectors, equations (6) and (41), partial fraction analysis, and equations (38) and (43). Comparing equations (46) with equations (44), we make the immediate identification, through $\mathbf{R}(\mu,\mu')$, that

$$\mathbf{S}(\mu,\mu') = \tilde{\omega}\,\frac{\mu\mu'}{\mu+\mu'}\,\mathbf{Q}(\mu)\mathbf{H}(\mu)\tilde{\mathbf{H}}(\mu')\tilde{\mathbf{Q}}(\mu'). \tag{47}$$

Influenced by the form of equation (44 a), we now extend the definition of the **H**-matrix to the entire complex plane by

$$\tilde{\mathbf{H}}(z)\boldsymbol{\Lambda}(z) = \mathbf{I}+z\int_0^1 \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\frac{d\mu}{\mu-z}. \tag{48}$$

Since $\boldsymbol{\Lambda}(z)$ is analytic in the complex plane cut from -1 to 1 along the real line and is singular only at $z=\pm\eta_0$, $\tilde{\mathbf{H}}(z)$ is at least meromorphic in the same cut plane with possible poles at $z=\pm\eta_0$. If we now restrict our attention to Re $z>0$ and take boundary values of equation (48) as z approaches the real-line segment $(0,1)$, the Plemelj formulae (19) yield

$$\tilde{\mathbf{H}}^+(\eta)\boldsymbol{\Lambda}^+(\eta)+\tilde{\mathbf{H}}^-(\eta)\boldsymbol{\Lambda}^-(\eta) = 2\tilde{\mathbf{H}}(\eta)\boldsymbol{\lambda}(\eta), \qquad \eta\in(0,1), \tag{49a}$$

and

$$\tilde{\mathbf{H}}^+(\eta)\boldsymbol{\Lambda}^+(\eta)-\tilde{\mathbf{H}}^-(\eta)\boldsymbol{\Lambda}^-(\eta) = 2\pi i\eta\tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta), \quad \eta\in(0,1), \tag{49b}$$

where equation (44 a) has been invoked. Applying the Plemelj formulae to equation (8) and using the definition given by equation (17), we note that

$$\boldsymbol{\Lambda}^+(\eta)+\boldsymbol{\Lambda}^-(\eta) = 2\boldsymbol{\lambda}(\eta) \tag{50a}$$

and

$$\boldsymbol{\Lambda}^+(\eta)-\boldsymbol{\Lambda}^-(\eta) = 2\pi i\eta\boldsymbol{\Psi}(\eta). \tag{50b}$$

Solving equations (49) simultaneously and utilizing equations (50) reveals that $\mathbf{H}(z)$ has no discontinuity across the line segment $(0,1)$; in fact, there results

$$\mathbf{H}^+(\eta) = \mathbf{H}^-(\eta) = \mathbf{H}(\eta), \quad \eta\in(0,1). \tag{51}$$

If we now change z to $-z$ in equation (48) and introduce (simply for notational convenience)

$$\mathbf{Y}(z) = \mathbf{H}(-z), \tag{52}$$

we can evaluate boundary values of the resulting equation for Re $z>0$ to find

$$\mathbf{Y}^+(\eta) = [\boldsymbol{\Lambda}^+(\eta)]^{-1}\left[\mathbf{I}-\eta\int_0^1 \boldsymbol{\Psi}(\mu)\mathbf{H}(\mu)\frac{d\mu}{\mu+\eta}\right], \quad \eta\in(0,1), \tag{53a}$$

and

$$\mathbf{Y}^-(\eta) = [\mathbf{\Lambda}^-(\eta)]^{-1}\left[\mathbf{I}-\eta\int_0^1 \mathbf{\Psi}(\mu)\mathbf{H}(\mu)\frac{d\mu}{\mu+\eta}\right], \quad \eta\in(0, 1). \tag{53b}$$

The matrix $\mathbf{Y}(z)$, or alternatively $\mathbf{H}(-z)$, must therefore be a solution of the homogeneous matrix Riemann-Hilbert problem (**19**),

$$\mathbf{Y}^+(\eta) = [\mathbf{\Lambda}^+(\eta)]^{-1}\mathbf{\Lambda}^-(\eta)\mathbf{Y}^-(\eta), \quad \eta\in(0, 1). \tag{54}$$

The paper by Burniston & Siewert (**20**) can be used here to ensure that a solution to equation (54) exists. Also, by writing equation (48) as

$$\tilde{\mathbf{H}}(z) = \frac{8}{\Lambda(z)}\left[\mathbf{I}+z\int_0^1 \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu)\frac{d\mu}{\mu-z}\right]\mathbf{\Lambda}_a(z), \tag{55}$$

where $\mathbf{\Lambda}_a(z) = \Lambda(z)\mathbf{\Lambda}^{-1}(z)/8$ is the appropriate adjoint matrix, and observing from equation (12) that $\mathbf{\Lambda}_a(\eta_0)$ can be written as

$$\mathbf{\Lambda}_a(\eta_0) = \mathbf{M}(\eta_0)\mathbf{K}(\eta_0) \tag{56}$$

where $\mathbf{K}(\eta_0)$ is a constant row vector, we conclude in view of equation (44 b) that $\mathbf{H}(z)$ is analytic at $z = \eta_0$ since the apparent singularity is removable.

It thus follows that $\mathbf{H}(z)$ is meromorphic in the complex plane cut from -1 to 0 along the real axis and can possibly have a pole only at $z = -\eta_0$. We note, therefore, that the quantity

$$\mathbf{F}(z) = \tilde{\mathbf{H}}(z)\mathbf{\Lambda}(z)\mathbf{H}(-z) \tag{57}$$

is meromorphic in the plane cut from -1 to 1 and possibly has poles only at $z = \pm\eta_0$. In view of equation (54), $\mathbf{F}(z)$ is continuous across the cut for Re $z>0$, whereas equation (48) ensures that $\mathbf{F}(z)$ is also continuous across the cut for Re $z<0$. Thus $\mathbf{F}(z)$ is meromorphic in the entire complex plane. Further, since from equation (48) it is clear that $\tilde{\mathbf{H}}(z)\mathbf{\Lambda}(z)$ cannot have a pole at $z=-\eta_0$ and since $\mathbf{F}(-z) = \tilde{\mathbf{F}}(z)$, it follows that $\mathbf{F}(z)$ is actually analytic in the entire complex plane. Noting that $\mathbf{F}(\infty)$ is bounded, we conclude from Liouville's theorem (**25**) that $\mathbf{F}(z)$ is a constant. We therefore evaluate equation (57) at $z = 0$ and use $\mathbf{H}(0) = \mathbf{\Lambda}(0) = \mathbf{I}$ to find $\mathbf{F}(z) = \mathbf{I}$; we obtain then the very useful identity (**18**)

$$\tilde{\mathbf{H}}(z)\mathbf{\Lambda}(z)\mathbf{H}(-z) = \mathbf{I}. \tag{58}$$

From this result, it is clear that $\mathbf{H}(z)$ does, indeed, have a pole at $z = -\eta_0$.

If we now make use of equation (58) in equation (48) there results the non-linear definition of the **H**-matrix in the complex plane:

$$\mathbf{H}(z) = \mathbf{I}+z\mathbf{H}(z)\int_0^1 \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu)\frac{d\mu}{\mu+z}. \tag{59}$$

Restricting z to the real-line segment (0, 1), we find, alternative to equations (44), that the desired **H**-matrix is sufficiently specified by

$$\mathbf{H}(\mu) = \mathbf{I}+\mu\mathbf{H}(\mu)\int_0^1 \tilde{\mathbf{H}}(\mu')\mathbf{\Psi}(\mu')\frac{d\mu'}{\mu'+\mu} \tag{60a}$$

and

$$\mathbf{0} = \left[\mathbf{I}+\eta_0\int_0^1 \tilde{\mathbf{H}}(\mu')\mathbf{\Psi}(\mu')\frac{d\mu'}{\mu'-\eta_0}\right]\mathbf{M}(\eta_0). \tag{60b}$$

Equation (60 a) above is the obvious non-conservative version of Chandrasekhar's equation for the **H**-matrix and can be readily derived from the principles of invariance (**26**). We note, however, that equation (60a) is by no means sufficient to specify $\mathbf{H}(\mu)$ and that equation (60b) is the additional required constraint (**18**).

In summary, we have used the appropriate half-range completeness theorem (**20**) to develop the singular integral equation and the integral constraint, equations (44), necessary and sufficient to specify the required fundamental matrix $\mathbf{H}(\mu)$. Further, through the use of the basic identity given by equation (58), we have derived the alternative regular, non-linear integral equation for $\mathbf{H}(\mu)$, which as will be discussed, is useful for computing the **H**-matrix. We note also that the procedure of Pahor & Shultis (**27**) can be used to establish a Fredholm equation for $\mathbf{H}(\mu)$, and it can be shown that the solution of this Fredholm equation exists and is unique (**28**). Finally, we note that an analytical expression for the det $\mathbf{H}(\mu)$ can be developed (see the Appendix), as is usual for scalar Riemann-Hilbert problems (**19**, **22**), and that this exact result may be used as a numerical check of any iterative solutions of the non-linear **H**-equation.

Since all the necessary analysis and formalism is established, the solution to any half-space problem can be constructed in a straightforward manner: the desired solution may be written as

$$\mathbf{I}(\tau,\mu) = \mathbf{I}_B(\tau,\mu) + A(-\eta_0)\mathbf{\Phi}(-\eta_0,\mu)\,e^{\tau/\eta_0} + \int_0^1 \mathbf{\Psi}(-\eta,\mu)\mathbf{A}(-\eta)\,e^{\tau/\eta}\,d\eta + \mathbf{I}_p(\tau,\mu), \quad \mu\in(-1,1), \tag{61}$$

where $\mathbf{I}_B(\tau,\mu)$ is the bounded portion of the general solution,

$$\mathbf{I}_B(\tau,\mu) = A(\eta_0)\mathbf{\Phi}(\eta_0,\mu)\,e^{-\tau/\eta_0} + \int_0^1 \mathbf{\Psi}(\eta,\mu)\mathbf{A}(\eta)\,e^{-\tau/\eta}\,d\eta, \tag{62}$$

and a particular solution $\mathbf{I}_p(\tau,\mu)$ is included to account for any inhomogeneous terms that might appear in equation (1). The expansion coefficients $A(-\eta_0)$ and $\mathbf{A}(-\eta)$, $\eta\in(0,1)$, are to be determined from a given boundary condition on $\mathbf{I}(\tau,\mu)$ as τ tends to infinity (as, for example, in the Milne problem). Thus, to meet the specified surface boundary condition,

$$\mathbf{I}(0,\mu) = \mathbf{I}_{\text{inc}}(\mu), \quad \mu\in(0,1), \tag{63}$$

we obtain the half-range expansion

$$\mathbf{I}(\mu) = A(\eta_0)\mathbf{\Phi}(\eta_0,\mu) + \int_0^1 \mathbf{\Psi}(\eta,\mu)\mathbf{A}(\eta)\,d\eta, \quad \mu\in(0,1), \tag{64a}$$

where the 'expansion function' $\mathbf{I}(\mu)$ is given by

$$\mathbf{I}(\mu) = \mathbf{I}_{\text{inc}}(\mu) - A(-\eta_0)\mathbf{\Phi}(-\eta_0,\mu) - \int_0^1 \mathbf{\Psi}(-\eta,\mu)\mathbf{A}(-\eta)\,d\eta - \mathbf{I}_p(0,\mu), \quad \mu\in(0,1). \tag{64b}$$

We observe from equations (31), (34) and (35) that if the right-hand side of equation (64 a) is formally extended to $\mu\in(-1,0)$ the 'switched result' follows (**29**):

$$\mathbf{I}_B(0,-\mu) = \frac{1}{2\mu}\int_0^1 \mathbf{S}(\mu,\mu')\mathbf{I}(\mu')\,d\mu' \tag{65}$$

where $\mathbf{S}(\mu,\mu')$ is given by equation (47).

Since $A(-\eta_0)$ and $\mathbf{A}(-\eta)$ are fixed from boundary conditions at infinity, equation (61), through equation (65), establishes $\mathbf{I}(0, \mu)$ for $\mu \in (-1, 0)$; further, since $\mathbf{I}(0, \mu)$ is prescribed by equation (63) for $\mu \in (0, 1)$, the surface solution $\mathbf{I}(0, \mu)$ is available for all $\mu \in (-1, 1)$. Thus, we can obtain $A(\eta_0)$ and $\mathbf{A}(\eta)$, needed to complete the solution, by setting $\tau = 0$ in equation (61) and making use of the full-range orthogonality theorem (17); we find

$$A(\eta_0) = \frac{1}{M(\eta_0)} \int_{-1}^{1} \mu \tilde{\mathbf{\Phi}}(\eta_0, \mu)[\mathbf{I}(0, \mu) - \mathbf{I}_p(0, \mu)]\, d\mu \tag{66a}$$

and

$$\mathbf{A}(\eta) = \frac{1}{M(\eta)} \int_{-1}^{1} \mu \tilde{\mathbf{\Psi}}^{\dagger}(\eta, \mu)[\mathbf{I}(0, \mu) - \mathbf{I}_p(0, \mu)]\, d\mu, \quad \eta \in (0, 1), \tag{66b}$$

where

$$\mathbf{\Psi}^{\dagger}(\eta, \mu) = [\mathbf{\Phi}_1{}^{\dagger}(\eta, \mu) \;\; \mathbf{\Phi}_2{}^{\dagger}(\eta, \mu)]. \tag{67}$$

3. HALF-SPACE PROBLEMS

In this section, we should like to specialize the foregoing half-space analysis to construct solutions to the classical Milne and albedo problems. Further, in order to illustrate the computational merits of the method, numerical results for several basic quantities of interest are reported.

For the Milne problem, we seek a solution to equation (1) which satisfies the boundary conditions

$$\mathbf{I}_{\mathrm{M}}(0, \mu) = \mathbf{0}, \quad \mu \in (0, 1) \tag{68a}$$

and

$$\mathbf{I}_{\mathrm{M}}(\tau, \mu) \sim \mathbf{\Phi}(-\eta_0, \mu)\, \mathrm{e}^{\tau/\eta_0}, \quad \tau \to \infty. \tag{68b}$$

In reference to the general result given by equation (61), we see that a properly diverging solution follows by setting $\mathbf{I}_p(\tau, \mu)$ and $\mathbf{A}(-\eta)$, $\eta \in (0, 1)$, equal to zero:

$$\mathbf{I}_{\mathrm{M}}(\tau, \mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu)\, \mathrm{e}^{-\tau/\eta_0} + \mathbf{\Phi}(-\eta_0, \mu)\, \mathrm{e}^{\tau/\eta_0} + \int_0^1 \mathbf{\Psi}(\eta, \mu)\mathbf{A}(\eta)\, \mathrm{e}^{-\tau/\eta}\, d\eta, \tag{69}$$

where we have arbitrarily normalized the solution by taking $A(-\eta_0) = 1$. Since there is no radiation incident here, the expansion function is, as follows from equation (64b), simply $\mathbf{I}(\mu) = -\mathbf{\Phi}(-\eta_0, \mu)$. We now observe that equations (62), (64), (65) and (69) yield the exit distribution

$$\mathbf{I}_{\mathrm{M}}(0, -\mu) = \mathbf{I}_{\mathrm{B}}(0, -\mu) + \mathbf{\Phi}(\eta_0, \mu), \quad \mu \in (0, 1), \tag{70a}$$

where

$$\mathbf{I}_{\mathrm{B}}(0, -\mu) = -\frac{1}{2\mu} \int_0^1 \mathbf{S}(\mu, \mu')\mathbf{\Phi}(-\eta_0, \mu')\, d\mu', \quad \mu \in (0, 1). \tag{70b}$$

If we now write $\mathbf{S}(\mu, \mu')$ as in equation (47), the integral appearing in equation (70b) can be expressed in terms of the $\mathbf{H}$-matrix by using the non-linear equation to define $\mathbf{H}(\eta_0)$; equation (70a) thus can be written more explicitly as

$$\mathbf{I}_{\mathrm{M}}(0, -\mu) = \tfrac{1}{2}\tilde{\omega}\eta_0 \frac{1}{\eta_0 - \mu} \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}^{-1}(\eta_0)\mathbf{M}(\eta_0), \quad \mu \in (0, 1). \tag{71}$$

Having established, through equations (68 a) and (71), $\mathbf{I}(0, \mu)$ for all $\mu \in (-1, 1)$, we note that the expansion coefficients $A(\eta_0)$ and $\mathbf{A}(\eta)$, $\eta \in (0, 1)$, follow immediately from equations (66). We find, after use of equations (6), (59), (58), and (12) and partial-fraction analysis, the tractable forms

$$A(\eta_0) = -\tfrac{1}{4}\tilde{\omega}\eta_0 \frac{1}{M(\eta_0)} \tilde{\mathbf{M}}(\eta_0)\tilde{\mathbf{H}}^{-1}(\eta_0)\mathbf{H}^{-1}(\eta_0)\mathbf{M}(\eta_0) \tag{72a}$$

and

$$\mathbf{A}(\eta) = -\tfrac{1}{2}\tilde{\omega}\eta\eta_0 \frac{1}{M(\eta)(\eta_0+\eta)} \tilde{\mathbf{G}}^{\dagger}(\eta)\tilde{\mathbf{H}}^{-1}(\eta)\mathbf{H}^{-1}(\eta_0)\mathbf{M}(\eta_0), \quad \eta \in (0, 1), \tag{72b}$$

where (**17**)

$$\mathbf{G}^{\dagger}(\eta) = \sqrt{6}(1-\eta^2)\begin{bmatrix} -\sqrt{2}M_{12}(\eta) & \sqrt{2}M_{11}(\eta) \\ M_{22}(\eta)-M_{12}(\eta) & M_{11}(\eta)-M_{21}(\eta) \end{bmatrix}, \tag{73}$$

with

$$M_{12}(\eta) = M_{21}(\eta) = (1-\eta^2)\Lambda_1^{+}(\eta)\Lambda_1^{-}(\eta) - 4\eta^2(1-\tilde{\omega})[1-3\eta^2(1-\tilde{\omega})], \tag{74a}$$

$$M_{11}(\eta) = (1-\eta^2)^2\Lambda_1^{+}(\eta)\Lambda_1^{-}(\eta) + 4\eta^2(1-\tilde{\omega})[(1-\eta^2)\lambda_1(\eta)+2\eta^2(1-\tilde{\omega})], \tag{74b}$$

and

$$M_{22}(\eta) = \Lambda_1^{+}(\eta)\Lambda_1^{-}(\eta) + \Lambda_2^{+}(\eta)\Lambda_2^{-}(\eta); \tag{74c}$$

here, $\Lambda_\alpha^{\pm}(\eta)$, $\alpha = 1$ and 2, denote the boundary values of $\Lambda_\alpha(z)$. Since equations (72) are exact expressions for the required expansion coefficients and depend only on an accurate evaluation of the **H**-matrix, they may be entered into equation (69) to yield the complete solution to the Milne problem.

A parameter of interest here (and in finite-media applications) is the Milne-problem extrapolated end-point defined by

$$\boldsymbol{\rho}_{\text{Mas}}(-\tau_0) = \mathbf{0}, \tag{75}$$

where

$$\boldsymbol{\rho}_{\text{Mas}}(\tau) = \int_{-1}^{1} \mathbf{I}_{\text{Mas}}(\tau, \mu)\, d\mu, \tag{76}$$

with $\mathbf{I}_{\text{Mas}}(\tau, \mu)$ denoting the asymptotic solution obtained by neglecting the continuum, $\eta \in (0, 1)$, in equation (69). It follows from equations (75) and (69) that

$$\tau_0 = -\tfrac{1}{2}\eta_0 \log[-A(\eta_0)]. \tag{77}$$

We note from equations (71) and (72) that proper evaluation of the **H**-matrix is of primary importance to any computation. There are two possible formulations which may be used to determine the **H**-matrix: namely, the singular integral equation, plus the constraint, equations (44), or the non-linear integral equation plus the constraint, equations (60). However, since the numerical solution of singular integral equations generally requires the numerical evaluation of derivatives, a practice surely to be avoided, we prefer to solve equations (60).

The computations were performed in double-precision arithmetic on an IBM 360/75 digital computer using the improved Gaussian-quadrature (**30**) representation of the integration process. Equation (60a) written in an equivalent and more rapidly convergent form,

$$\mathbf{H}(\mu) = \left[\mathbf{I} - \mu \int_0^1 \tilde{\mathbf{H}}(\mu')\mathbf{\Psi}(\mu') \frac{d\mu'}{\mu'+\mu}\right]^{-1}, \tag{78}$$

was solved iteratively. This method, of course, is the matrix equivalent of that reported by Chandrasekhar (**7**). The iteration procedure was continued until successive calculations yielded identical results to 14 significant figures; the solution converged rapidly for low values of $\tilde{\omega}$ (8 iterations for $\tilde{\omega} = 0{\cdot}1$) while the rate of convergence decreased significantly as $\tilde{\omega}$ approached unity (over 200 iterations for $\tilde{\omega} = 0{\cdot}999$). A more expeditious method for calculating $\mathbf{H}(\mu)$ has recently been reported by Kriese & Siewert (**31**).

As expected the **H**-matrix satisfied to 14 significant figures the moment relation (**26, 18**)

$$(\mathbf{H}_0 - \mathbf{I})(\tilde{\mathbf{H}}_0 - \mathbf{I}) = \mathbf{I} - 2\int_0^1 \mathbf{\Psi}(\mu)\, d\mu \tag{79a}$$

where we have defined

$$\mathbf{H}_0 = \int_0^1 \mathbf{\Psi}(\mu)\mathbf{H}(\mu)\, d\mu; \tag{79b}$$

however, since similar moment relations have not satisfactorily indicated the accuracy of scalar H-function calculations (**32**), several additional checks on the computed **H**-matrix are desirable. In fact, since equation (79a) admits an infinite number of solutions for $\mathbf{H}_0$, that check is of very little practical or theoretical value.

The linear constraint given by equation (60b) and rewritten here as

$$\mathbf{M}(\eta_0) = \eta_0 \int_0^1 \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu) \frac{d\mu}{\eta_0 - \mu} \mathbf{M}(\eta_0), \tag{80}$$

provides the first and probably the most important numerical check on the **H**-matrix computation. Upon substituting the calculated **H**-matrix into equation (80), we found that the fractional difference (fd) between the right- and left-hand components was less than 5×10^{-11} for all cases considered. Further, the **H**-matrix was substituted into equation (44a) and that singular integral equation was integrated numerically over η from 0 to 1; although the fd in each component of the resulting equation was less than 5×10^{-7}, this check is considered extremely conservative, due primarily to the logarithmic nature of the integrands involved. Notwithstanding, the number of significant figures reported here are within the limit of this difference. As discussed in the Appendix, an analytical expression (or alternatively, a non-linear H-type equation) is available for det $\mathbf{H}(\mu)$. We have evaluated det $\mathbf{H}(\mu)$ from the formulation in the Appendix, and the fd between that result and the values computed from the **H**-matrix was consistently less than 10^{-8}.

Finally, all calculations were performed using successively higher-order quadrature schemes (21-, 41-, and 81-point improved Gaussian) in order to ensure that the results reported herein were insensitive to further refinements of the integration process.

Sample calculations of $\mathbf{H}(\mu)$ and $\mathbf{N}(\mu)$ have been performed, and upon comparing $\mathbf{N}(\mu)$ for $\tilde{\omega} = 0{\cdot}5$ with the results of Abhyankar & Fymat (**12**), we find disagreement by as many as 2 units in the 4th significant figure. Abhyankar & Fymat (**33**) have recently revised their calculational procedure for determining $\mathbf{N}(\mu)$, and our results are in agreement for low and intermediate values of the single-scattering albedo $\tilde{\omega}$. However, their calculations for values of $\tilde{\omega}$ near unity are still of questionable accuracy; in fact, their results for $\tilde{\omega} = 1$ differ from Chandrasekhar's (**7**) accepted values by as many as 3 units in the 3rd significant figure. These discrepancies may be attributed to a different computational scheme or less stringent convergence criteria, but, in any event, we believe there is substantial evidence to justify confidence in our calculations.

We should, in addition, like to remark that any solution of equation (60a) also satisfies equation (79a); this moment relation then is not, as reported by Abhyankar & Fymat (**12**), a sufficient condition for determining the ' physical ' significance of the computed result. Equation (79a) is not a generalization of the scalar moment relation of Chandrasekhar ((**7**), p. 106, equation (11)), but rather is the quadratic form ((**7**), p. 106, equation (10)) and thus is satisfied by all solutions both physical and otherwise.

For the Milne problem, we have made free-surface calculations in the form of the laws of darkening and the degree of polarization (**28**). The extrapolated end-point, the discrete eigenvalue and det $\mathbf{H}^{-1}(\eta_0) = H^{-1}(\eta_0)$ are exhibited in Table I. In order to ensure that accuracy was maintained during these claculations, two additional numerical checks were incorporated. We verify that the solution given by equation (69) satisfies the moment integrals of the free-surface boundary conditions, equation (68 a); thus for

$$\int_0^1 \mu^\alpha \mathbf{\Phi}(-\eta_0, \mu)\, d\mu = -\int_0^1 \mu^\alpha \mathbf{I}_B(0, \mu)\, d\mu, \quad \alpha = 0, 1, 2, 3 \text{ and } 4, \tag{81}$$

we found the *fd* in all cases to be less than 10^{-8}. Finally, upon comparing the emergent flux

$$j(0) = \int_0^1 \mu \widetilde{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \mathbf{I}_M(0, -\mu)\, d\mu, \tag{82}$$

with $\mathbf{I}_M(0, -\mu)$ computed from equation (70a), to the emergent flux with $\mathbf{I}_M(0, -\mu)$ computed from equation (71) we found a *fd* of less than 10^{-10}.

It is interesting to note that in order to resolve the rapidly varying details of the continuum coefficient $\mathbf{A}(\eta)$, $\eta \in (0, 1)$, and thereby maintain the required accuracy, we found it advantageous to divide the interval $\eta \in (0, 1)$ into several sub-intervals and to place a high concentration of quadrature nodal points near the end-point

TABLE I

The discrete eigenvalue, extrapolated end-point, and $H^{-1}(\eta_0)$

$\tilde{\omega}$	η_0	τ_0	$H^{-1}(\eta_0)$
0·10	1·000001	6·059782	0·93829
0·20	1·000709	3·084199	0·87576
0·30	1·007230	2·147636	0·81199
0·40	1·025904	1·677234	0·74612
0·50	1·062363	1·384414	0·67700
0·60	1·125231	1·179429	0·60296
0·70	1·232743	1·024846	0·52132
0·80	1·433478	0·901861	0·42691
0·90	1·924622	0·799784	0·30611
0·95	2·651351	0·754441	0·22071
0·99	5·804339	0·720357	0·10322
1·00	∞	0·712110	0·0

unity. Though we do not report here any numerical values, we find this procedure adequate to evaluate $\mathbf{A}(\eta)$, and thus the complete numerical solution to the Milne problem is readily available.

For the albedo problem, we seek a bounded solution to equation (1) such that the incident distribution may be specified. Thus neglecting $A(-\eta_0)$, $\mathbf{A}(-\eta)$, $\eta \in (0, 1)$, and $\mathbf{I}_p(\tau, \mu)$ in equation (61), we write the required solution as

$$\mathbf{I}_a(\tau, \mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu)\, e^{-\tau/\eta_0} + \int_0^1 \mathbf{\Psi}(\eta, \mu)\mathbf{A}(\eta)\, e^{-\tau/\eta}\, d\eta. \tag{83}$$

For this application, the expansion function $\mathbf{I}(\mu)$ is simply the distribution of incident radiation

$$\mathbf{I}(\mu) = \mathbf{I}_a(0, \mu) = \mathbf{I}_{inc}(\mu), \quad \mu \in (0, 1). \tag{84}$$

Establishing first the exit radiation, we observe that equations (61), (64b) and (65) yield

$$\mathbf{I}_a(0, -\mu) = \frac{1}{2\mu}\int_0^1 \mathbf{S}(\mu, \mu')\mathbf{I}_{inc}(\mu')\, d\mu', \quad \mu \in (0, 1), \tag{85}$$

and thus the expansion coefficients follow immediately from equations (66):

$$A(\eta_0) = \frac{1}{M(\eta_0)}\int_0^1 \mu[\tilde{\mathbf{\Phi}}(\eta_0, \mu)\mathbf{I}_{inc}(\mu) - \tilde{\mathbf{\Phi}}(-\eta_0, \mu)\mathbf{I}_a(0, -\mu)]\, d\mu \tag{86a}$$

and

$$\mathbf{A}(\eta) = \frac{1}{M(\eta)}\int_0^1 \mu[\tilde{\mathbf{\Psi}}^\dagger(\eta, \mu)\mathbf{I}_{inc}(\mu) - \tilde{\mathbf{\Psi}}^\dagger(-\eta, \mu)\mathbf{I}_a(0, -\mu)]\, d\mu. \tag{86b}$$

Equations (83) and (86) thus affect the complete solution to the most general albedo problem.

We should now like to consider the case for which the incident radiation is represented by

$$\mathbf{I}_{inc}(\mu) = \delta(\mu - \mu_0)\mathbf{F}, \quad \mu, \mu_0 \in (0, 1), \tag{87}$$

where $\mathbf{F}$ is a constant vector. Here, equation (85) reduces at once to

$$\mathbf{I}_a(0, -\mu) = \frac{1}{2\mu}\mathbf{S}(\mu, \mu_0)\mathbf{F}. \tag{88}$$

Further, the expressions for the expansion coefficients, equations (86), reduce to the more concise forms

$$A(\eta_0) = \frac{\mu_0}{M(\eta_0)}\tilde{\mathbf{\Phi}}(\eta_0, \mu_0)\tilde{\mathbf{Q}}^{-1}(\mu_0)\tilde{\mathbf{H}}^{-1}(\eta_0)\tilde{\mathbf{H}}(\mu_0)\tilde{\mathbf{Q}}(\mu_0)\mathbf{F} \tag{89a}$$

and

$$\mathbf{A}(\eta) = \frac{\mu_0}{M(\eta)}\tilde{\mathbf{\Psi}}^\dagger(\eta, \mu_0)\tilde{\mathbf{Q}}^{-1}(\mu_0)\tilde{\mathbf{H}}^{-1}(\eta)\tilde{\mathbf{H}}(\mu_0)\tilde{\mathbf{Q}}(\mu_0)\mathbf{F}. \tag{89b}$$

The albedo, defined as the fraction of incident flux of radiation reflected from the half space, has been computed for the case of a normally incident beam, $\mu_0 \to 1$

and for isotropic incident radiation in the three states of polarization $\delta = 1$, $\delta = 0$ and $\delta = -1$, or alternatively for

$$\mathbf{I}_{\text{inc}}(\mu) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{I}_{\text{inc}}(\mu) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{I}_{\text{inc}}(\mu) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{90}$$

In Table II we compare our 'exact' results to those of Pomraning (**34**) who used a variational method for this problem. Finally, for the case of the incident beam, we observe that the albedo is independent of the polarization state of the incoming radiation. Clearly this is the case, as can be seen by noting from the Rayleigh-scattering kernel, $\mathbf{Q}(\mu)\tilde{\mathbf{Q}}(\mu')$, that scattered radiation resulting from an incident beam depends only on the total intensity of the beams $F_l + F_r$ and not on the relative intensity of the components.

TABLE II

The Rayleigh-scattering albedo

$$\int_0^1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \mathbf{I}(0, -\mu)\mu \, d\mu \Big/ \int_0^1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \mathbf{I}_{\text{inc}}(\mu)\mu \, d\mu$$

	Isotropic incident radiation						
	$\delta = 1$		$\delta = 0$		$\delta = -1$		
$\tilde{\omega}$	Exact	Ref. **34**	Exact	Ref. **34**	Exact	Ref. **34**	Beam
0·10	0·02330	—	0·02206	—	0·02081	—	0·01784
0·20	0·04948	0·049	0·04698	0·037	0·04448	0·026	0·03817
0·30	0·07926	—	0·07552	—	0·07177	—	0·06168
0·40	0·11368	0·117	0·10873	0·094	0·10378	0·071	0·08934
0·50	0·15429	—	0·14824	—	0·14218	—	0·12267
0·60	0·20359	0·214	0·19664	0·182	0·18968	0·151	0·16415
0·70	0·26605	—	0·25859	—	0·25112	—	0·21832
0·80	0·35096	0·367	0·34378	0·335	0·33660	0·303	0·29491
0·90	0·48455	0·499	0·47940	0·475	0·47425	0·452	0·42251
0·95	0·60022	0·611	0·59747	0·596	0·59473	0·581	0·54023
0·99	0·79438	0·799	0·79472	0·794	0·79506	0·790	0·75324

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APPENDIX

THE DETERMINANT PROBLEM

As observed by Muskhelishvili (**19**), we obtain a scalar boundary-value problem by 'taking determinants' of the matrix Riemann-Hilbert problem defined by equation (54):

$$Y^+(\eta) = \frac{\Lambda^-(\eta)}{\Lambda^+(\eta)}\, Y^-(\eta), \quad \eta \in (0, 1), \tag{A1}$$

where $Y(z)$ denotes det $\mathbf{Y}(z)$ and the boundary values of $\Lambda(z)$ follow from equations (10) and (11). We seek then a function $Y(z)$ meromorphic in the plane cut from 0 to 1 and the boundary values of which satisfy equation (A1). In fact, we note from equations (48) and (44b) that $Y(z)$ has a pole only at $z = \eta_0$. Further, for $\tilde{\omega} \neq 1$ we see from equation (48) that $Y(z)$ is bounded as z tends to infinity, while equation (58) can be used to write

$$\lim_{z\to\infty} Y(z) = \pm\sqrt{8/\Lambda(\infty)} = \pm\Delta \tag{A2}$$

where, more explicitly (17),

$$\Delta = [(1-\tilde{\omega})(1-\tfrac{7}{10}\tilde{\omega})]^{-1/2}. \tag{A3}$$

In accordance with Muskhelishvili's theory (19) and Case's method for the one-speed neutron-transport problem (22), the most general bounded meromorphic (with a single pole at $z = \eta_0$) solution to equation (A 1) is

$$Y(z) = \frac{A(1-z)}{\eta_0 - z} \exp\left[-\frac{1}{\pi}\int_0^1 \arg\Lambda^+(\mu)\frac{d\mu}{\mu - z}\right], \tag{A4}$$

where A is a non-zero constant, and, as discussed previously (17), $0 \leq \arg\Lambda^+(\mu) \leq \pi$, for $\mu \in [0, 1]$. Noting equation (48) in the limit as z tends to zero, we find that $Y(0) = 1$ and thus equation (A4) can be used to resolve the choice of signs in equation (A 2). We find, therefore, $A = \Delta$, and hence

$$H(z) = \frac{\Delta(1+z)}{\eta_0 + z} \exp\left[-\frac{1}{\pi}\int_0^1 \arg\Lambda^+(\mu)\frac{d\mu}{\mu + z}\right], \quad \tilde{\omega} < 1, \tag{A5a}$$

which in the limit as $\tilde{\omega} \to 1$ reduces to

$$H(z) = \sqrt{10}(1+z) \exp\left[-\frac{1}{\pi}\int_0^1 \arg\Lambda^+(\mu)\frac{d\mu}{\mu + z}\right], \quad \tilde{\omega} = 1. \tag{A5b}$$

Although equations (A5) are exact analytical solutions for $H(z)$, a non-linear equation, with computational merits, may be derived by making use of Cauchy's integral representation of $H(z)$:

$$H(\mu) = 1 + \mu H(\mu)\int_0^1 \Psi(\mu')H(\mu')\frac{d\mu'}{\mu' + \mu}, \quad \mu \in (0, 1), \tag{A6}$$

where

$$\Psi(\mu) = \frac{1}{16\pi i\mu}[\Lambda^+(\mu) - \Lambda^-(\mu)], \tag{A7}$$

or more explicitly,

$$\Psi(\mu) = \frac{\tilde{\omega}}{8}[9(1-\mu^2)^2\lambda_0(\mu) + 6\mu^2(1-\tilde{\omega})]. \tag{A8}$$

We note that equation (A6) is of the usual (7) form except, of course, here the characteristic function $\Psi(\mu)$ is not a polynomial. Many of the identities normally associated with H-functions can be developed here; however, we list only the expression

$$\Delta = \left[1 - \int_0^1 \Psi(\mu)H(\mu)\,d\mu\right]^{-1} \tag{A9}$$

which is often used in H-function calculations.