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ON THE NONCONSERVATIVE EQUATION OF TRANSFER FOR A COMBINATION OF RAYLEIGH AND ISOTROPIC SCATTERING

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ABSTRACT

The developed numerical results for the pertinent $\boldsymbol{H}$-matrix are used to evaluate semianalytical solutions to the albedo and Milne problems in context of the vector equation of transfer appropriate to a nonconservative mixture of Rayleigh- and isotropic-scattering laws. In addition to a tabulation of the $\boldsymbol{H}$-matrix for representative values of the considered parameters, computed results for the half-space albedo and the extrapolated endpoint, the laws of darkening, and the degree of polarization for the Milne problem are reported.

I. INTRODUCTION

We consider here the vector equation of transfer

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}(\tau, \mu) + \boldsymbol{I}(\tau, \mu) = \tfrac{1}{2}\omega \boldsymbol{Q}(\mu) \int_{-1}^{1} \boldsymbol{Q}^t(\mu')\boldsymbol{I}(\tau, \mu')d\mu \tag{1}$$

applicable to several studies of the scattering of polarized light (Chandrasekhar 1950). Relying principally on Chandrasekhar's (1950) formulation of this mathematical model, we denote by $\boldsymbol{I}(\tau, \mu)$ a vector whose two components $I_l(\tau, \mu)$ and $I_r(\tau, \mu)$ are the azimuth-independent angular intensities in the two states of polarization. Further, τ is the optical variable, and μ is the direction cosine (as measured from the positive τ-axis) of the propagating radiation.

The scattering process considered is characterized in equation (1) by the single-scattering albedo ω and the square matrix $\boldsymbol{Q}(\mu)$, with $\boldsymbol{Q}^t(\mu)$ denoting the transpose of $\boldsymbol{Q}(\mu)$. Although much of the analysis presented here is valid for a general $\boldsymbol{Q}$-matrix, we are concerned primarily with the form (Burniston and Siewert 1970)

$$\boldsymbol{Q}(\mu) = \frac{3(c+2)^{1/2}}{2(c+2)} \begin{vmatrix} c\mu^2 + \frac{2}{3}(1-c) & (2c)^{1/2}(1-\mu^2) \\ \frac{1}{3}(c+2) & 0 \end{vmatrix} ; \tag{2}$$

we thus allow the right-hand side of equation (1) to contain the two parameters ω and c so that the following special cases can be readily identified: for $c = 1$ and $\omega = 1$, equations (1) and (2) yield Chandrasekhar's (1950) conservative Rayleigh-scattering model and $\boldsymbol{Q}(\mu)$ reduces to Sekera's (1963) form for factoring the Rayleigh phase matrix, whereas for $\omega = 1$ and $c \in [0, 1]$ they yield Chandrasekhar's (1950) conservative model for a mixture of Rayleigh- and isotropic-scattering laws. Some discussion of the case $\omega = 1$ and $c = 0.5$ has been reported by Fymat (1967). Observing the choices $c = 1$ and $\omega \in [0, 1]$, we obtain the general Rayleigh-scattering problem, as considered, for example, by Simmons (1966), Mullikin (1969), Abhyankar and Fymat (1970) and Schnatz and Siewert (1970*a*, *b*). Here we allow the values $c \in [0, 1]$ and $\omega \in [0, 1]$ to study the general mixture of Rayleigh and isotropic scattering.

The singular-eigenfunction-expansion technique (Case and Zweifel 1967) can be used to develop analytical solutions to equation (1). Since the construction of the solutions required here follows closely the discussion given by Schnatz and Siewert (1970*a*), we simply summarize the relevant forms found for the considered problem. A rigorous general solution to equation (1) can be written as

$$I(\tau, \mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu) \exp(-\tau/\eta_0) + A(-\eta_0)\mathbf{\Phi}(-\eta_0, \mu) \exp(\tau/\eta_0) + \int_{-1}^{1} \mathbf{\Psi}(\eta, \mu) A(\eta) \exp(-\tau/\eta) d\eta , \quad (3)$$

where $A(\pm\eta_0)$ and the vector $A(\eta)$, with elements $A_1(\eta)$ and $A_2(\eta)$, are the arbitrary expansion coefficients to be determined once appropriate boundary conditions are specified. Here the discrete normal modes are written as

$$\mathbf{\Phi}(\pm\eta_0, \mu) = \tfrac{1}{2}\omega\eta_0 \frac{1}{\eta_0 \mp \mu} Q(\mu) M(\eta_0) , \quad (4)$$

where η_0 is the positive zero (there are only two zeros $\pm\eta_0$ in the complex plane cut from -1 to 1 along the real axis) of the dispersion function $\Lambda(z) = \det \mathbf{\Lambda}(z)$, with

$$\mathbf{\Lambda}(z) = I + z \int_{-1}^{1} \mathbf{\Psi}(\mu) \frac{d\mu}{\mu - z} \quad (5)$$

and the "characteristic function"

$$\mathbf{\Psi}(\mu) = \tfrac{1}{2}\omega Q^t(\mu) Q(\mu) . \quad (6)$$

In addition, we use here the normalization vector

$$M(\eta_0) = \int_{-1}^{1} Q^t(\mu)\mathbf{\Phi}(\pm\eta_0, \mu) d\mu = \begin{vmatrix} \Lambda_{22}(\eta_0) \\ -\Lambda_{12}(\eta_0) \end{vmatrix} , \quad (7)$$

where $\Lambda_{ij}(\eta_0)$ denotes the (ij)-component of $\mathbf{\Lambda}(\eta_0)$. The explicit forms follow readily from equation (5):

$$\Lambda_{11}(\eta_0) = 1 - \frac{9\omega}{4(c+2)} \left[\!\left[\left\{ \left[\frac{c+2}{3} - c(1-\eta_0^2)\right]^2 + \left(\frac{c+2}{3}\right)^2 \right\} \eta_0 \tanh^{-1}\frac{1}{\eta_0} + c^2\eta_0^2(1-\eta_0^2) - \tfrac{4}{3}c\eta_0^2 \right]\!\right] ,$$

$$\Lambda_{22}(\eta_0) = 1 - \frac{9\omega c}{2(c+2)} \left[(1-\eta_0^2)^2\eta_0 \tanh^{-1}\frac{1}{\eta_0} - \eta_0^2(\eta_0^2 - \tfrac{5}{3}) \right] ,$$

and

$$\Lambda_{12}(\eta_0) = \Lambda_{21}(\eta_0) = -\frac{9\omega(2c)^{1/2}}{4(c+2)} \left\{ \left[\frac{c+2}{3} - c(1-\eta_0^2)\right](1-\eta_0^2)\eta_0 \tanh^{-1}\frac{1}{\eta_0} + \eta_0^2\left[\frac{2-c}{3} - c(1-\eta_0^2)\right] \right\} .$$

We note that, for the special case $\omega \to 1$ and thus $\eta_0 \to \infty$, the special forms of the discrete solutions, as reported by Mourad and Siewert (1969),

$$I_+(\tau, \mu) = \begin{vmatrix} 1 \\ 1 \end{vmatrix} \quad \text{and} \quad I_-(\tau, \mu) = (\tau - \mu)\begin{vmatrix} 1 \\ 1 \end{vmatrix} , \qquad \omega = 1 , \quad (8)$$

are required.

For the continuum $\eta \in (-1, 1)$, we combine the eigenvectors (Burniston and Siewert 1970)

$$\boldsymbol{\Phi}_\alpha(\eta, \mu) = \tfrac{1}{2}\omega\left[\eta \frac{P}{\eta - \mu} + \lambda^*_\alpha(\eta)\delta(\eta - \mu)\right] \boldsymbol{Q}(\mu)\boldsymbol{M}_\alpha(\eta) ,$$

$$\eta \in (-1, 1), \alpha = 1 \text{ and } 2 , \quad (9)$$

to form the 2×2 matrix

$$\boldsymbol{\Psi}(\eta, \mu) = |\,\boldsymbol{\Phi}_1(\eta, \mu) \quad \boldsymbol{\Phi}_2(\eta, \mu)\,| . \quad (10)$$

Here the symbol P is used to denote that all ensuing integrals are to be evaluated in the Cauchy principal-value sense, and the Dirac functional is represented by $\delta(x)$. Further, the scalar functions $\lambda^*_\alpha(\eta)$, $\alpha = 1$ and 2, are, in general, the two solutions of the quadratic equation

$$\det\,[\boldsymbol{\lambda}(\eta) - \lambda^*(\eta)\boldsymbol{\Psi}(\eta)] = 0 , \quad (11)$$

where

$$\boldsymbol{\lambda}(\eta) = \boldsymbol{I} + \eta P \int_{-1}^{1} \boldsymbol{\Psi}(\mu) \frac{d\mu}{\mu - \eta} , \quad (12)$$

and the normalization vectors $\boldsymbol{M}_\alpha(\eta)$ follow from

$$[\boldsymbol{\lambda}(\eta) - \lambda^*_\alpha(\eta)\boldsymbol{\Psi}(\eta)]\boldsymbol{M}_\alpha(\eta) = \boldsymbol{0} . \quad (13)$$

These normalization vectors $\boldsymbol{M}_\alpha(\eta)$ clearly have one degree of freedom, and since suitable forms are readily available through equations (11) and (13), we shall not require here any explicit choices (Bond 1970).

We should now like to establish two important full-range theorems regarding the eigenvectors $\boldsymbol{\Phi}(\xi, \mu)$, $\xi = \pm\eta_0$ or $\eta \in (-1, 1)$; since the analysis required here follows Case's (1960) method and is similar to that previously reported (Burniston and Siewert 1970; Schnatz and Siewert 1970*a*), we shall give only a brief sketch of the necessary proofs.

The full-range expansion theorem states that an arbitrary two-vector $\boldsymbol{I}(\mu)$ satisfying the Hölder condition (Muskhelishvili 1953) can be expanded in terms of the eigenvectors $\boldsymbol{\Phi}(\xi, \mu)$:

$$\boldsymbol{I}(\mu) = A(\eta_0)\boldsymbol{\Phi}(\eta_0, \mu) + A(-\eta_0)\boldsymbol{\Phi}(-\eta_0, \mu) + \int_{-1}^{1} \boldsymbol{\Psi}(\eta, \mu)\boldsymbol{A}(\eta)d\eta , \quad \mu \in (-1, 1). \quad (14)$$

We note that equation (9) can be pre-multiplied by $\boldsymbol{Q}^t(\mu)$ to yield, after use of equation (13),

$$\boldsymbol{Q}^t(\mu)\boldsymbol{\Psi}(\eta, \mu) = \left[\eta \frac{P}{\eta - \mu} \boldsymbol{\Psi}(\mu) + \delta(\eta - \mu)\boldsymbol{\lambda}(\eta)\right]\boldsymbol{V}(\eta) , \quad (15)$$

where $\boldsymbol{V}(\eta)$ is the normalization matrix:

$$\boldsymbol{V}(\eta) = \int_{-1}^{1} \boldsymbol{Q}^t(\mu)\boldsymbol{\Psi}(\eta, \mu)d\mu = |\,\boldsymbol{M}_1(\eta)\boldsymbol{M}_2(\eta)\,| . \quad (16)$$

If we now introduce the sectionally analytic function

$$\boldsymbol{N}(z) = \frac{1}{2\pi i} \int_{-1}^{1} \eta \boldsymbol{B}(\eta) \frac{d\eta}{\eta - z} , \quad (17)$$

with boundary values (Muskhelishvili 1953)

$$\boldsymbol{N}^{\pm}(\mu) = \frac{1}{2\pi i} P \int_{-1}^{1} \eta \boldsymbol{B}(\eta) \frac{1}{\eta - \mu} d\eta \pm \tfrac{1}{2}\mu \boldsymbol{B}(\mu) \qquad \mu \in (-1, 1) , \quad (18)$$

then equation (14), after pre-multiplication by $\boldsymbol{Q}^t(\mu)$, can be converted to the Riemann-Hilbert problem

$$\mu \boldsymbol{Q}^t(\mu)[\boldsymbol{I}(\mu) - A(\eta_0)\boldsymbol{\Phi}(\eta_0, \mu) - A(-\eta_0)\boldsymbol{\Phi}(-\eta_0, \mu)] = \boldsymbol{\Lambda}^+(\mu)\boldsymbol{N}^+(\mu) - \boldsymbol{\Lambda}^-(\mu)\boldsymbol{N}^-(\mu) . \tag{19}$$

Here we have defined

$$\boldsymbol{B}(\eta) = \boldsymbol{V}(\eta)\boldsymbol{A}(\eta) \tag{20}$$

and have made use of the Plemelj relations (Muskhelishvili 1953) obtained from the boundary values of $\boldsymbol{\Lambda}(z)$:

$$\boldsymbol{\Lambda}^+(\mu) - \boldsymbol{\Lambda}^-(\mu) = 2\pi i\mu\boldsymbol{\Psi}(\mu) , \quad \mu \in (-1, 1) , \tag{21a}$$

and

$$\boldsymbol{\Lambda}^+(\mu) + \boldsymbol{\Lambda}^-(\mu) = 2\boldsymbol{\lambda}(\mu) , \quad \mu \in (-1, 1) . \tag{21b}$$

Equation (19) can now be solved to yield

$$\boldsymbol{N}(z) = \boldsymbol{\Lambda}^{-1}(z)\frac{1}{2\pi i}\int_{-1}^{1}\mu \boldsymbol{Q}^t(\mu)[\boldsymbol{I}(\mu) - A(\eta_0)\boldsymbol{\Phi}(\eta_0, \mu) - A(-\eta_0)\boldsymbol{\Phi}(-\eta_0, \mu)]\frac{d\mu}{\mu - z} . \tag{22}$$

Since $\boldsymbol{\Lambda}^{-1}(\infty)$ is bounded ($\omega < 1$), the representation of $\boldsymbol{N}(z)$ given by equation (22) has the correct behavior as z tends to infinity (i.e., $z\boldsymbol{N}(z)$ is bounded as $z \to \infty$). This representation, however, is not consistent with the initial definition, equation (17), since $\boldsymbol{\Lambda}(z)$ is singular at $z = \pm\eta_0$. To complete the proof of the expansion theorem, these two apparent poles of $\boldsymbol{N}(z)$ can be removed by specifying the discrete coefficients $A(\eta_0)$ and $A(-\eta_0)$ to be solutions of the conditions

$$\int_{-1}^{1}\mu \boldsymbol{Q}^t(\mu)[\boldsymbol{I}(\mu) - A(\eta_0)\boldsymbol{\Phi}(\eta_0, \mu) - A(-\eta_0)\boldsymbol{\Phi}(-\eta_0, \mu)]\frac{d\mu}{\mu \pm \eta_0} = \boldsymbol{0} . \tag{23}$$

Though the expansion coefficients $\boldsymbol{A}(\eta)$ can now be established through equations (22), (20), and (18), the results may equally well be obtained from the full-range orthogonality theorem

$$\left(\frac{1}{\xi} - \frac{1}{\xi'}\right)\int_{-1}^{1}\mu\boldsymbol{\Phi}^t(\xi, \mu)\boldsymbol{\Phi}(\xi', \mu)d\mu = 0 . \tag{24}$$

In the usual manner (Case and Zweifel 1967), equation (24) may readily be established from the eigenvalue equation

$$(\xi - \mu)\boldsymbol{\Phi}(\xi, \mu) = \tfrac{1}{2}\omega\xi \boldsymbol{Q}(\mu)\boldsymbol{M}(\xi) ; \tag{25}$$

equation (25) is pre-multiplied by $\xi^{-1}\boldsymbol{\Phi}^t(\xi', \mu)$ and subtracted from the same resulting equation with ξ and ξ' interchanged.

For the full-range normalization integrals, we prefer the forms

$$\int_{-1}^{1}\mu\boldsymbol{\Psi}^t(\eta', \mu)\boldsymbol{\Psi}(\eta, \mu)d\mu = \boldsymbol{\Delta}\delta(\eta - \eta') , \tag{26a}$$

where

$$\Delta_{\alpha\beta} = \tfrac{1}{2}\omega\eta\boldsymbol{M}^t{}_\alpha(\eta)\boldsymbol{\Lambda}^+(\eta)\boldsymbol{\Psi}^{-1}(\eta)\boldsymbol{\Lambda}^-(\eta)\boldsymbol{M}_\beta(\eta) , \tag{26b}$$

and

$$\int_{-1}^{1}\mu\boldsymbol{\Phi}^t(\pm\eta_0, \mu)\boldsymbol{\Phi}(\pm\eta_0, \mu)d\mu = \pm M(\eta_0) , \tag{27a}$$

with

$$M(\eta_0) = \tfrac{1}{2}\omega\eta_0{}^2\Lambda_{22}(\eta_0)\frac{d}{dz}\Lambda(z)\Big|_{z=\eta_0} . \tag{27b}$$

II. HALF-SPACE PROBLEMS

Much of the half-space analysis required here has been reported by Pahor (1968) in a study relating to the transport of thermal neutrons. More explicitly and in a corresponding notation, Schnatz and Siewert (1970*b*, hereafter referred to as SS), have established the necessary formalism for the nonconservative Rayleigh-scattering model, $c = 1$, $\omega \in [0, 1)$. Since the problem considered here differs only in the form of the $\boldsymbol{Q}$-matrix, we shall borrow freely from SS, without further elaboration.

For the Milne problem we seek a diverging (as $\tau \to \infty$) solution $\boldsymbol{I}_M(\tau, \mu)$ of equation (1) such that

$$\boldsymbol{I}_M(0, \mu) = \boldsymbol{0}\,, \qquad \mu \in (0, 1)\,, \tag{28}$$

and

$$\lim_{\tau\to\infty} \boldsymbol{I}_M(\tau, \mu) \exp(-\tau/\eta_0) < \infty\,. \tag{29}$$

The desired solution follows from equation (3):

$$\boldsymbol{I}_M(\tau, \mu) = A(\eta_0)\boldsymbol{\Phi}(\eta_0, \mu) \exp(-\tau/\eta_0) + \boldsymbol{\Phi}(-\eta_0, \mu) \exp(\tau/\eta_0) + \int_0^1 \boldsymbol{\Psi}(\eta, \mu)\boldsymbol{A}(\eta) \exp(-\tau/\eta) d\eta\,, \tag{30}$$

where the normalization $A(-\eta_0) = 1$ has been imposed. Equation (30) clearly satisfies the second of the Milne-problem boundary conditions, equation (29); the expansion coefficients $A(\eta_0)$ and $\boldsymbol{A}(\eta)$, $\eta \in (0, 1)$, must then follow from equation (28):

$$-\boldsymbol{\Phi}(\eta_0, \mu) = A(\eta_0)\boldsymbol{\Phi}(\eta_0, \mu) + \int_0^1 \boldsymbol{\Psi}(\eta, \mu)\boldsymbol{A}(\eta) d\eta\,, \qquad \mu \in (0, 1)\,. \tag{31}$$

Burniston and Siewert (1970), through the proof of a general half-range expansion theorem, have shown that equation (31) is soluble when the left-hand side is an arbitrary two-vector satisfying the Hölder condition (Muskhelishvili 1953); the vector $-\boldsymbol{\Phi}(\eta_0, \mu)$ is clearly of this class. We note, however, that the exit distribution $\boldsymbol{I}_M(0, -\mu)$, $\mu \in (0, 1)$, can actually be evaluated (SS) without having to find $A(\eta_0)$ or $\boldsymbol{A}(\eta)$:

$$\boldsymbol{I}_M(0, -\mu) = \tfrac{1}{2}\omega\eta_0 \frac{1}{\eta_0 - \mu} \boldsymbol{Q}(\mu)\boldsymbol{H}(\mu)\boldsymbol{H}^{-1}(\eta_0)\boldsymbol{M}(\eta_0)\,, \qquad \mu \in (0, 1)\,, \tag{32}$$

where the $\boldsymbol{H}$-matrix is sufficiently specified by (Pahor 1968)

$$\boldsymbol{H}(\mu) = \boldsymbol{I} + \mu\boldsymbol{H}(\mu)\int_0^1 \boldsymbol{H}^t(\mu')\boldsymbol{\Psi}(\mu') \frac{d\mu'}{\mu' + \mu} \tag{33a}$$

and

$$\boldsymbol{0} = \left[\boldsymbol{I} + \eta_0\int_0^1 \boldsymbol{H}^t(\mu')\boldsymbol{\Psi}(\mu') \frac{d\mu'}{\mu' - \eta_0}\right]\boldsymbol{M}(\eta_0) \tag{33b}$$

or, alternatively, by the singular integral equation

$$\boldsymbol{H}^t(\eta)\boldsymbol{\lambda}(\eta) = \boldsymbol{I} + \eta P\int_0^1 \boldsymbol{H}^t(\mu)\boldsymbol{\Psi}(\mu) \frac{d\mu}{\mu - \eta}\,, \qquad \eta \in (0, 1)\,, \tag{34}$$

and the linear constraint, equation (33b).

In addition to the law of darkening given by equation (32) and the thus-available degree of polarization (Chandrasekhar 1950), a quantity of interest here (and in finite-media applications) is the Milne-problem extrapolated endpoint τ_0 defined by

$$\int_{-1}^1 \boldsymbol{I}_{M\,\mathrm{as}}(-\tau_0, \mu) d\mu = \boldsymbol{0}\,. \tag{35}$$

Here the asymptotic solution $I_{M_{as}}(\tau, \mu)$ is obtained by neglecting the continuum $\eta \in (0, 1)$ in equation (30). We find (SS)

$$\tau_0 = -\tfrac{1}{2}\eta_0 \ln [-A(\eta_0)] , \tag{36}$$

where

$$A(\eta_0) = -\tfrac{1}{4}\omega\eta_0 \frac{1}{M(\eta_0)} M^t(\eta_0)[(H^t)(\eta_0)]^{-1}H^{-1}(\eta_0)M(\eta_0) . \tag{37}$$

We note that the proper evaluation of the H-matrix is of primary importance to any computations. There are two possible formulations which may be used to determine the H-matrix, namely, the nonlinear integral equation, plus the constraint, equations (33), or the singular integral equation, equation (34), plus the constraint. However, since the numerical solution of singular integral equations generally requires the numerical evaluation of derivatives, a practice surely to be avoided, we prefer to solve equations (33).

The computations were performed in double-precision arithmetic on an IBM 360/75 digital computer by using an improved Gaussian-quadrature (Kronrod 1965) representation of the integration process. Essentially, equation (33a), written as

$$H(\mu) = \left[I - \mu \int_0^1 H^t(\mu')\Psi(\mu') \frac{d\mu'}{\mu' + \mu}\right]^{-1} \tag{38a}$$

or

$$H(\mu) = \left[I - H^t{}_0 + \int_0^1 H^t(\mu')\Psi(\mu')\mu' \frac{d\mu'}{\mu' + \mu}\right]^{-1} , \tag{38b}$$

where

$$H^t{}_0 = \int_0^1 H^t(\mu)\Psi(\mu)d\mu , \tag{39}$$

was solved iteratively. The iteration process was continued until successive calculations yielded identical results to fourteen significant figures; the solution converged rapidly for low values of ω, while the rate of convergence decreased markedly as ω approached unity. Further, for strictly conservative cases, $\omega = 1$, the rate of convergence decreased with increasing c, and thus the H-matrix for the conservative Rayleigh-scattering model was found to be the most difficult to compute in this manner.

For the case $\omega = 1$, we find that the free term in equation (38b) can be written as

$$I - H^t{}_0 = \begin{vmatrix} (\tfrac{1}{2}c)^{1/2}\beta & -\beta \\ -[\tfrac{1}{2}c(\alpha^2 - \beta^2)]^{1/2} & (\alpha^2 - \beta^2)^{1/2} \end{vmatrix} , \tag{40}$$

where

$$\alpha^2 = \frac{2}{c + 2}(1 - \tfrac{7}{10}c) , \tag{41}$$

and the only unknown β has the limiting values

$$\beta = 0 \quad \text{for} \quad c = 0 , \quad \text{and} \quad \beta = \tfrac{1}{10}10^{1/2}q \quad \text{for} \quad c = 1 , \tag{42}$$

where $q = 0.68989 \ldots$ is the constant introduced by Chandrasekhar (1950). For conservative cases, equation (40) has been used in equation (38b) to accelerate the convergence of our iterative computation of the H-matrix.

As expected, the calculated H-matrix satisfied to fourteen significant figures the moment relation (Pahor 1968)

$$(H_0 - I)(H^t{}_0 - I) = I - 2\int_0^1 \Psi(\mu)d\mu . \tag{43}$$

However, since similar moment relations have not satisfactorily indicated the accuracy of scalar H-function calculations, several additional checks on the computed $\boldsymbol{H}$-matrix are desirable. In fact, since equation (43) admits an infinite number of solutions for $\boldsymbol{H}_0$, that check is of very little practical or theoretical value. We should, in addition, like to remark that any solution of equation (33a) also satisfies equation (43); this moment relation then is not, as reported by Abhyankar and Fymat (1970) for the case $c = 1$, a sufficient condition for determining the physical significance of the computed result. Equation (43) is not a generalization of the scalar moment relation of Chandrasekhar (1950, p. 106, eq. [11]), but rather is the quadratic form (p. 106, eq. [10]) and thus is satisfied by all solutions, both physical and otherwise.

The linear constraint given by equation (33b) and rewritten here as

$$\boldsymbol{M}(\eta_0) = \eta_0 \int_0^1 \boldsymbol{H}^t(\mu)\boldsymbol{\Psi}^r(\mu) \frac{d\mu}{\eta_0 - \mu} \boldsymbol{M}(\eta_0) \qquad (44)$$

provides the first and probably the most important numerical check on the computation of the $\boldsymbol{H}$-matrix. Upon substituting the calculated $\boldsymbol{H}$-matrix into equation (44), we found that the fractional difference between the right- and left-hand components was less than 2×10^{-14} for all cases considered. Further, the $\boldsymbol{H}$-matrix was substituted into equation (34), and that singular integral equation was integrated numerically over η from 0 to 1; although the fractional difference in each component of the resulting equation was less than 2×10^{-8}, this check is considered extremely conservative, due primarily to the logarithmic nature of the integrands involved. Notwithstanding, the number of significant figures reported here is within the limit of this difference. As discussed in the Appendix, an exact analytical expression (or alternatively, a nonlinear H-type equation) is available for det $\boldsymbol{H}(\mu)$. We have evaluated det $\boldsymbol{H}(\mu)$ from the formulation in the Appendix, and the fractional difference between that result and the values computed from the $\boldsymbol{H}$-matrix was consistently less than 10^{-8}.

Finally, all calculations were performed by using 21- and 81-point improved Gaussian schemes in order to ensure that the results reported herein were insensitive to further refinements of the integration process.

We list in Tables 1, 2, and 3 the $\boldsymbol{H}$-matrix for representative values of the two parameters c and ω. (At the time of this writing, our code is available upon request.) For the Milne problem, our numerical results for the free-surface calculation, in the form of the laws of darkening $D_\alpha(\mu)$, $\alpha = l$ or r, and the degree of polarization $\delta(\mu)$, are presented in Table 4, while the extrapolated endpoint τ_0 and the discrete eigenvalue η_0 are exhibited in Table 5. For the cases $\omega = 1$ reported, our results are in agreement with the calculation of the conservative Milne problem by Mourad (1970).

We note from our calculations for fixed ω or c that the degree of polarization at the limb is a monotonically increasing function of c, and a monotonically decreasing function of ω. These effects can be observed in Figure 1.

For the albedo problem, we seek a bounded solution to equation (1) such that the incident distribution may be specified. Thus neglecting $A(-\eta_0)$ and $\boldsymbol{A}(-\eta)$, $\eta \in (0, 1)$, in equation (3), we write the desired solution as

$$\boldsymbol{I}_a(\tau, \mu) = A(\eta_0)\boldsymbol{\Phi}(\eta_0, \mu) \exp(-\tau/\eta_0) + \int_0^1 \boldsymbol{\Psi}^r(\eta, \mu)\boldsymbol{A}(\eta) \exp(-\tau/\eta) d\eta \, . \qquad (45)$$

We now constrain the albedo solution above to meet a specified incident distribution $\boldsymbol{I}_{\text{inc}}(\mu)$:

$$\boldsymbol{I}_a(0, \mu) = \boldsymbol{I}_{\text{inc}}(\mu) \qquad \mu \in (0, 1) \, . \qquad (46)$$

TABLE 1

THE *H*-MATRIC FOR $\omega = 1.0$

c = 0.4				
μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0.0	0.10000(1)	0.0	0.0	0.10000(1)
0.1	0.12023(1)	0.83819(-1)	0.86555(-1)	0.10959(1)
0.2	0.13716(1)	0.15239	0.16026	0.11528(1)
0.3	0.15329(1)	0.21693	0.23111	0.11990(1)
0.4	0.16902(1)	0.27935	0.30057	0.12395(1)
0.5	0.18452(1)	0.34045	0.36922	0.12765(1)
0.6	0.19987(1)	0.40064	0.43733	0.13112(1)
0.7	0.21510(1)	0.46018	0.50506	0.13442(1)
0.8	0.23026(1)	0.51923	0.57251	0.13759(1)
0.9	0.24536(1)	0.57789	0.63975	0.14067(1)
1.0	0.26041(1)	0.63626	0.70682	0.14367(1)

c = 0.6				
μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0.0	0.10000(1)	0.0	0.0	0.10000(1)
0.1	0.11907(1)	0.81444(-1)	0.84669(-1)	0.11380(1)
0.2	0.13496(1)	0.15256	0.16198	0.12213(1)
0.3	0.15007(1)	0.22111	0.23826	0.12888(1)
0.4	0.16481(1)	0.28832	0.31419	0.13479(1)
0.5	0.17931(1)	0.35469	0.38995	0.14015(1)
0.6	0.19367(1)	0.42047	0.46564	0.14515(1)
0.7	0.20792(1)	0.48583	0.54130	0.14989(1)
0.8	0.22210(1)	0.55088	0.61693	0.15442(1)
0.9	0.23622(1)	0.61567	0.69255	0.15881(1)
1.0	0.25030(1)	0.68026	0.76817	0.16306(1)

c = 0.8				
μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0.0	0.10000(1)	0.0	0.0	0.10000(1)
0.1	0.11854(1)	0.70815(-1)	0.74339(-1)	0.11791(1)
0.2	0.13379(1)	0.13823	0.14872	0.12895(1)
0.3	0.14821(1)	0.20534	0.22467	0.13792(1)
0.4	0.16223(1)	0.27238	0.30181	0.14574(1)
0.5	0.17600(1)	0.33941	0.37983	0.15282(1)
0.6	0.18961(1)	0.40645	0.45853	0.15939(1)
0.7	0.20311(1)	0.47350	0.53775	0.16557(1)
0.8	0.21653(1)	0.54055	0.61739	0.17146(1)
0.9	0.22988(1)	0.60761	0.69737	0.17712(1)
1.0	0.24318(1)	0.67468	0.77762	0.18260(1)

TABLE 2

THE *H*-MATRIX FOR $\omega = 0.9$

$c = 0.4$				
μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0.0	0.10000(1)	0.0	0.0	0.10000(1)
0.1	0.11396(1)	0.57614(-1)	0.58883(-1)	0.10775(1)
0.2	0.12386(1)	0.97329(-1)	0.10072	0.11180(1)
0.3	0.13223(1)	0.13037	0.13611	0.11475(1)
0.4	0.13957(1)	0.15901	0.16715	0.11709(1)
0.5	0.14612(1)	0.18438	0.19487	0.11903(1)
0.6	0.15204(1)	0.20713	0.21990	0.12068(1)
0.7	0.15744(1)	0.22774	0.24268	0.12211(1)
0.8	0.16238(1)	0.24653	0.26354	0.12337(1)
0.9	0.16693(1)	0.26377	0.28275	0.12450(1)
1.0	0.17114(1)	0.27966	0.30050	0.12551(1)

$c = 0.6$				
μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0.0	0.10000(1)	0.0	0.0	0.10000(1)
0.1	0.11322(1)	0.53399(-1)	0.54839(-1)	0.11113(1)
0.2	0.12258(1)	0.92798(-1)	0.96693(-1)	0.11707(1)
0.3	0.13048(1)	0.12644	0.13309	0.12141(1)
0.4	0.13741(1)	0.15607	0.16555	0.12485(1)
0.5	0.14360(1)	0.18258	0.19486	0.12770(1)
0.6	0.14919(1)	0.20655	0.22154	0.13012(1)
0.7	0.15429(1)	0.22838	0.24598	0.13222(1)
0.8	0.15895(1)	0.24838	0.26847	0.13407(1)
0.9	0.16325(1)	0.26679	0.28926	0.13572(1)
1.0	0.16723(1)	0.28382	0.30854	0.13719(1)

$c = 0.8$				
μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0.0	0.10000(1)	0.0	0.0	0.10000(1)
0.1	0.11299(1)	0.43007(-1)	0.44499(-1)	0.11438(1)
0.2	0.12210(1)	0.77923(-1)	0.82016(-1)	0.12223(1)
0.3	0.12976(1)	0.10883	0.11588	0.12802(1)
0.4	0.13647(1)	0.13664	0.14677	0.13262(1)
0.5	0.14245(1)	0.16189	0.17508	0.13643(1)
0.6	0.14784(1)	0.18498	0.20115	0.13967(1)
0.7	0.15276(1)	0.20618	0.22523	0.14248(1)
0.8	0.15726(1)	0.22573	0.24755	0.14494(1)
0.9	0.16140(1)	0.24383	0.26829	0.14713(1)
1.0	0.16523(1)	0.26064	0.28760	0.14909(1)

TABLE 3

THE *H*-MATRIX FOR ω = 0.8

c = 0.4

μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0.0	0.10000(1)	0.0	0.0	0.10000(1)
0.1	0.11125(1)	0.45844(-1)	0.46651(-1)	0.10657(1)
0.2	0.11873(1)	0.75193(-1)	0.77286(-1)	0.10983(1)
0.3	0.12477(1)	0.98484(-1)	0.10195	0.11211(1)
0.4	0.12989(1)	0.11792	0.12274	0.11387(1)
0.5	0.13433(1)	0.13458	0.14070	0.11528(1)
0.6	0.13823(1)	0.14911	0.15645	0.11646(1)
0.7	0.14169(1)	0.16194	0.17042	0.11745(1)
0.8	0.14481(1)	0.17337	0.18292	0.11831(1)
0.9	0.14761(1)	0.18365	0.19420	0.11906(1)
1.0	0.15017(1)	0.19295	0.20443	0.11973(1)

c = 0.6

μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0.0	0.10000(1)	0.0	0.0	0.10000(1)
0.1	0.11070(1)	0.41645(-1)	0.42541(-1)	0.10939(1)
0.2	0.11780(1)	0.70152(-1)	0.72501(-1)	0.11413(1)
0.3	0.12354(1)	0.93368(-1)	0.97283(-1)	0.11748(1)
0.4	0.12841(1)	0.11305	0.11852	0.12006(1)
0.5	0.13263(1)	0.13010	0.13707	0.12213(1)
0.6	0.13634(1)	0.14509	0.15347	0.12385(1)
0.7	0.13965(1)	0.15840	0.16811	0.12531(1)
0.8	0.14261(1)	0.17033	0.18128	0.12657(1)
0.9	0.14529(1)	0.18109	0.19321	0.12767(1)
1.0	0.14772(1)	0.19086	0.20406	0.12864(1)

c = 0.8

μ	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{21}(\mu)$	$H_{22}(\mu)$
0.0	0.10000(1)	0.0	0.0	0.10000(1)
0.1	0.11056(1)	0.32526(-1)	0.33428(-1)	0.11204(1)
0.2	0.11753(1)	0.57066(-1)	0.59456(-1)	0.11827(1)
0.3	0.12315(1)	0.77786(-1)	0.81800(-1)	0.12270(1)
0.4	0.12791(1)	0.95740(-1)	0.10138	0.12611(1)
0.5	0.13203(1)	0.11153	0.11874	0.12887(1)
0.6	0.13566(1)	0.12556	0.13426	0.13116(1)
0.7	0.13888(1)	0.13813	0.14824	0.13310(1)
0.8	0.14177(1)	0.14947	0.16090	0.13477(1)
0.9	0.14439(1)	0.15976	0.17243	0.13622(1)
1.0	0.14676(1)	0.16914	0.18297	0.13751(1)

TABLE 4

THE MILNE-PROBLEM DARKENING AND DEGREE OF POLARIZATION

$$D_\alpha(\mu) = \frac{I_\alpha(0,-\mu)}{I_\ell(0,-1) + I_r(0,-1)} \quad \text{and} \quad \delta(\mu) = \frac{I_r(0,-\mu) - I_\ell(0,-\mu)}{I_r(0,-\mu) + I_\ell(0,-\mu)}$$

μ	ω = 1.0 and c = 0.4			ω = 1.0 and c = 0.8		
	$D_\ell(\mu)$	$D_r(\mu)$	$\delta(\mu)$	$D_\ell(\mu)$	$D_r(\mu)$	$\delta(\mu)$
0.0	0.16277	0.17562	0.03797	0.15138	0.17988	0.08604
0.1	0.20763	0.21678	0.02157	0.19793	0.21980	0.05234
0.2	0.24374	0.25110	0.01487	0.23529	0.25347	0.03721
0.3	0.27773	0.28375	0.01073	0.27042	0.28564	0.02738
0.4	0.31066	0.31557	0.00785	0.30445	0.31708	0.02032
0.5	0.34295	0.34689	0.00571	0.33781	0.34807	0.01496
0.6	0.37482	0.37787	0.00406	0.37073	0.37877	0.01073
0.7	0.40639	0.40862	0.00274	0.40334	0.40927	0.00729
0.8	0.43775	0.43920	0.00166	0.43572	0.43961	0.00445
0.9	0.46894	0.46965	0.00076	0.46793	0.46985	0.00205
1.0	0.50000	0.50000	0.0	0.50000	0.50000	0.0

μ	ω = 0.9 and c = 0.4			ω = 0.9 and c = 0.8		
	$D_\ell(\mu)$	$D_r(\mu)$	$\delta(\mu)$	$D_\ell(\mu)$	$D_r(\mu)$	$\delta(\mu)$
0.0	0.11772	0.13137	0.05479	0.10249	0.13495	0.13669
0.1	0.14883	0.16082	0.03873	0.13270	0.16367	0.10449
0.2	0.17542	0.18684	0.03154	0.15865	0.18933	0.08817
0.3	0.20234	0.21334	0.02646	0.18524	0.21556	0.07564
0.4	0.23086	0.24138	0.02227	0.21382	0.24337	0.06465
0.5	0.26193	0.27180	0.01850	0.24544	0.27358	0.05423
0.6	0.29652	0.30548	0.01489	0.28121	0.30705	0.04393
0.7	0.33582	0.34350	0.01130	0.32247	0.34481	0.03347
0.8	0.38132	0.38721	0.00766	0.37096	0.38820	0.02271
0.9	0.43508	0.43849	0.00390	0.42903	0.43907	0.01157
1.0	0.50000	0.50000	0.0	0.50000	0.50000	0.0

μ	ω = 0.8 and c = 0.4			ω = 0.8 and c = 0.8		
	$D_\ell(\mu)$	$D_r(\mu)$	$\delta(\mu)$	$D_\ell(\mu)$	$D_r(\mu)$	$\delta(\mu)$
0.0	0.08058	0.09343	0.07385	0.06599	0.09692	0.18988
0.1	0.10089	0.11336	0.05821	0.08454	0.11660	0.15940
0.2	0.11919	0.13187	0.05050	0.10153	0.13508	0.14180
0.3	0.13889	0.15179	0.04437	0.12026	0.15502	0.12629
0.4	0.16128	0.17426	0.03870	0.14209	0.17754	0.11091
0.5	0.18771	0.20054	0.03304	0.16852	0.20386	0.09491
0.6	0.22001	0.23231	0.02718	0.20154	0.23563	0.07798
0.7	0.26089	0.27208	0.02099	0.24413	0.27529	0.06000
0.8	0.31475	0.32396	0.01442	0.30112	0.32684	0.04097
0.9	0.38937	0.39519	0.00743	0.38095	0.39724	0.02094
1.0	0.50000	0.50000	0.0	0.50000	0.50000	0.0

TABLE 5

THE DISCRETE EIGENVALUE AND THE MILNE-PROBLEM EXTRAPOLATED ENDPOINT

ω	c	η_0	τ_0	ω	c	η_0	τ_0
1.0....	0.4	∞	0.710878	0.9....	0.8	1.915598	0.795906
1.0....	0.6	∞	0.711174	0.9....	1.0	1.924622	0.799784
1.0....	0.8	∞	0.711561	0.8....	0.4	1.413024	0.892416
1.0....	1.0	∞	0.712110	0.8....	0.6	1.417259	0.894772
0.9....	0.4	1.907192	0.791777	0.8....	0.8	1.423479	0.897832
0.9....	0.6	1.910483	0.793466	0.8....	1.0	1.433478	0.901861

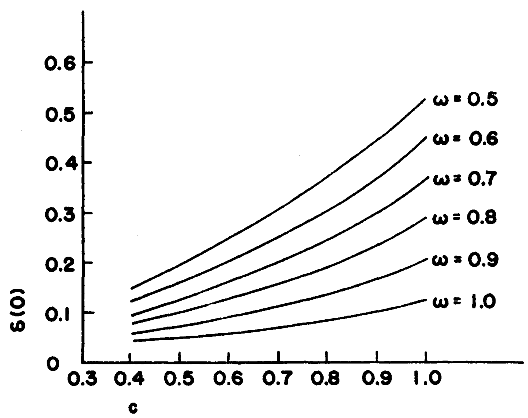

FIG. 1.—Degree of polarization at the limb

The exit distribution can now be written in terms of the S-matrix (Chandrasekhar 1950):

$$I_a(0, -\mu) = \frac{1}{2\mu} \int_0^1 S(\mu, \mu') I_{\text{inc}}(\mu') d\mu' , \qquad \mu \in (0, 1) , \quad (47)$$

where, after Pahor (1968) and Schnatz and Siewert (1970*b*), we find

$$S(\mu, \mu') = \omega \frac{\mu\mu'}{\mu + \mu'} Q(\mu) H(\mu) H^t(\mu') Q^t(\mu') . \quad (48)$$

The albedo β, defined as the fraction of incident radiation reflected from the half-space, has been computed for the case of a normally incident beam,

$$\beta(\mu_0) \Rightarrow I_{\text{inc}}(\mu) = \delta(\mu - \mu_0) F , \qquad \mu_0 \to 1 , \quad (49)$$

with F being a constant, and for three cases of isotropic incident radiation:

$$\beta(l, r) \Rightarrow I_{\text{inc}}(\mu) = \begin{vmatrix} l \\ r \end{vmatrix} , \qquad (l, r) = (0, 1), (1, 0), \text{ and } (1, 1) . \quad (50)$$

The results of our "exact" calculations of the albedo for representative values of ω and c are given in Table 6.

TABLE 6

THE HALF-SPACE ALBEDO

ω	c	Isotropic Incidence $\beta(0, 1)$	$\beta(1, 1)$	$\beta(1, 0)$	Beam, $\beta(\mu_0 \to 1)$
0.9........	0.4	0.47933	0.47832	0.47731	0.41654
0.9........	0.6	0.48040	0.47855	0.47670	0.41779
0.9........	0.8	0.48199	0.47888	0.47577	0.41961
0.9........	1.0	0.48455	0.47940	0.47425	0.42251
0.8........	0.4	0.34408	0.34234	0.34060	0.28763
0.8........	0.6	0.34569	0.34268	0.33966	0.28934
0.8........	0.8	0.34787	0.34313	0.33840	0.29164
0.8........	1.0	0.35096	0.34378	0.33660	0.29491

NOTE.

$$\beta = \int_0^1 \begin{vmatrix} 1 \\ 1 \end{vmatrix}^t I(0, -\mu)\mu d\mu \Big/ \int_0^1 \begin{vmatrix} 1 \\ 1 \end{vmatrix}^t I_{\rm inc}(\mu)\mu d\mu \, .$$

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APPENDIX

Since the analysis is identical with that reported previously (Schnatz and Siewert 1970*b*), we simply state the appropriate forms required here. An exact analytical result for $H(z) = \det \boldsymbol{H}(z)$ is

$$H(z) = \frac{\Delta(1+z)}{\eta_0 + z} \exp\left[-\frac{1}{\pi}\int_0^1 \arg \Lambda^+(\mu) \frac{d\mu}{\mu + z}\right], \qquad \omega \neq 1\,, \quad \text{(A1a)}$$

which in the limit $\omega \to 1$ reduces to

$$H(z) = \left(\frac{30}{10 - 7c}\right)^{1/2} (1+z) \exp\left[-\frac{1}{\pi}\int_0^1 \arg \Lambda^+(\mu) \frac{d\mu}{\mu + z}\right], \qquad \omega = 1\,. \quad \text{(A1b)}$$

Here

$$\Delta = [(1-\omega)(1 - \tfrac{7}{10}\omega c)]^{-1/2}\,, \quad \text{(A2)}$$

and $\Lambda^+(\mu)$ denotes the boundary value (as z approaches the branch cut $[-1, 1]$ from above) of

$$\Lambda(z) = \det \boldsymbol{\Lambda}(z) = \tfrac{1}{8}c\Lambda_1(z)\Lambda_2(z) + [(1-c) + \tfrac{3}{2}c(1-\omega)z^2]\Lambda_0(z)\,, \quad \text{(A3)}$$

where

$$\Lambda_\alpha(z) = (-1)^\alpha + 3(1-z^2)\Lambda_0(z) - (-1)^\alpha 3(1-\omega)z^2\,, \qquad \alpha = 1 \text{ and } 2\,, \quad \text{(A4)}$$

with

$$\Lambda_0(z) = 1 + \tfrac{1}{2}\omega z \int_{-1}^1 \frac{d\mu}{\mu - z}\,. \quad \text{(A5)}$$

Although equations (A1) are rigorous solutions for $H(z)$, a nonlinear equation, with computational merits, may be derived by making use of Cauchy's integral representation of $H(z)$:

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \Psi(\mu') H(\mu') \frac{d\mu'}{\mu' + \mu}, \qquad \mu \in (0, 1), \quad \text{(A6)}$$

where

$$\Psi(\mu) = \frac{1}{2\pi i \mu} [\Lambda^+(\mu) - \Lambda^-(\mu)] \quad \text{(A7)}$$

or, more explicitly,

$$\Psi(\mu) = \tfrac{1}{8}\omega[9c(1 - \mu^2)^2\lambda_0(\mu) + 6c\mu^2(1 - \omega) + 4(1 - c)], \quad \text{(A8)}$$

where

$$\lambda_0(\mu) = 1 - \omega\mu \tanh^{-1} \mu. \quad \text{(A9)}$$

We note that equation (A6) is of the usual form (Chandrasekhar 1950) except, of course, here the characteristic function $\Psi(\mu)$ is not a polynomial. While many of the identities normally associated with H-functions can be developed here, we list only the expression

$$\Delta = \left[1 - \int_0^1 \Psi(\mu) H(\mu) d\mu\right]^{-1} \quad \text{(A10)}$$

often used in H-function calculations.

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