Radiative Transfer. II*

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This paper presents an exact solution to the equations of radiative transfer for a generalization of the Uniform-Picket-Fence model discussed in a previous work. Here the absorption coefficient is allowed to take N different values over the frequency spectrum. Case's method is used to construct the normal mode solutions to the set of N coupled integral equations. Then half-range completeness and orthogonality theorems are proved that enable one to solve typical half-space problems. Explicitly, the asymptotic solution to the Milne problem is developed, including the extrapolated end point, while implicitly the complete solution is available.

I. INTRODUCTION

In a previous paper[1] (hereafter referred to as I) we presented exact solutions for the equations of radiative transfer with local thermodynamic equilibrium for a particular model of the absorption coefficient K_ν (uniform picket-fence model). In this model, K_ν was assumed to take only two values K_1 and K_2. In the present paper, we generalize the results of I to the case where K_ν can take on N values. The derivation of the basic (matrix) transport equation follows similar lines of reasoning as that of I. However, we briefly give the cogent points in the derivation below with a slight generalization of the case treated in I; namely, we include a scattering term in the transport equation which we neglect in order to obtain an explicit solution.

We begin, then, with the equation for radiative transfer

$$\mu \frac{\partial \psi_\nu(z, \mu)}{\partial z} + \rho(z)(K_\nu + S_\nu)\psi_\nu(z, \mu) = \rho(z)K_\nu\beta_\nu[T(z)] + \rho(z)S_\nu \int \psi_\nu(z, \mu')f_\nu(\mathbf{\Omega}'\cdot\mathbf{\Omega})\, d\mathbf{\Omega}'. \qquad (1)$$

Here, as in I, we assume plane symmetry (coordinate z), where μ is the cosine of the angle between the photon velocity vector and the z axis; ψ_ν is the angular energy density of frequency ν; $\rho(z)$ is the material density, and

$$\beta_\nu[T(z)] = (2h\nu^3/c^2)[\exp h\nu/kT(z) - 1]^{-1}$$

is the Planck black body function for the "local temperature" $T(z)$. This equation is identical to Eq. (1) of I except that here a monochromatic scattering term is included; S_ν is the scattering coefficient, and

$$\int f(\mathbf{\Omega}'\cdot\mathbf{\Omega})\, d\mathbf{\Omega}' = \int f(\mathbf{\Omega}'\cdot\mathbf{\Omega})\, d\mathbf{\Omega} = 1. \qquad (2)$$

We now assume that the frequency spectrum can be divided into ranges $\Delta\nu_i$ in each of which K_ν and S_ν take on the same N different values ($K_1, K_2, \cdots, K_N$; $S_1, S_2, \cdots, S_N$) and that the fractional width $w_n(i)$ of $\Delta\nu_i$ over which K_ν and S_ν have the same values is the same for all $\Delta\nu_i$.[2] Further, we must assume that $B_\nu[T(z)]$ can be taken independent of ν over the range $\Delta\nu_i$. The meaning of these assumptions may be clarified by examining Fig. 1. If these assumptions are not reasonable *in detail* (uniform model) they may be so on *the average* (random model).

Keeping in mind these assumptions, we integrate Eq. (1) over the frequency range Δ_n in which K_ν and S_ν have values K_n and S_n, respectively (this includes contributions from all $\Delta\nu_i$), to obtain

$$\mu \frac{\partial}{\partial z} \psi_n(z, \mu) + \rho(z)(K_n + S_n)\psi_n = \rho(z)K_nB_n[T(z)] + \rho(z)S_n \int \psi_n(z, \mu')f(\mathbf{\Omega}'\cdot\mathbf{\Omega})\, d\mathbf{\Omega}', \quad n = 1, 2, \cdots, N, \qquad (3)$$

where

$$\psi_n(z, \mu) \triangleq \int_{\Delta_n} d\nu\, \psi_\nu(z, \mu), \qquad (4)$$

and $B_n[T(z)]$ has a similar definition. The Schwarzs-

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[1] C. E. Siewert and P. F. Zweifel, Ann. Phys. (N. Y.) **36**, 61 (1966).

[2] We have assumed that the steps for K_ν and S_ν always occur at the same value of ν. Also when K_ν has the value K_n, S_ν has the value S_n.

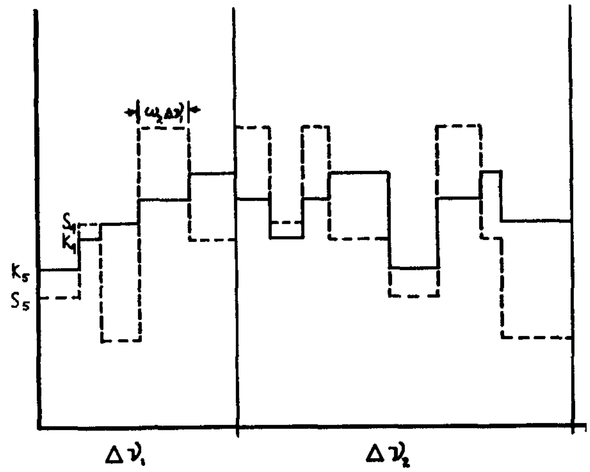

FIG. 1. Generalized picket-fence model, $N = 5$. We see that K_ν (solid lines) and S_ν (dashed lines) take on 5 different values in $\Delta\nu_i$ and that the frequency ranges over which K_ν and S_ν have constant values are the same. Further, the fractional width $w_n(i)$ of $\Delta\nu_i$ covered by K_ν and S_ν is the same for all i.

child condition,[1] which states local energy conservation, is

$$\int_0^\infty K_\nu B_\nu[T(z)]\,d\nu = \tfrac{1}{2}\int_0^\infty K_\nu\,d\nu \int_{-1}^{1} \psi_\nu(z,\mu')\,d\mu'. \tag{5a}$$

In the present model, it takes the form

$$\sum_{n=1}^{N} K_n B_n[T(z)] = \tfrac{1}{2}\sum_{n=1}^{N} K_n \int_{-1}^{1} \psi_n(z,\mu')\,d\mu'. \tag{5b}$$

Our uniformity assumptions (cf. Fig. 1) easily lead to the result

$$B_n[T(z)] = w_n \int_0^\infty d\nu\, B_\nu[T(z)] \tag{6a}$$

$$= (w_n\sigma/\pi)T^4(z), \tag{6b}$$

(σ is the Stefan–Boltzmann constant) so that Eq. (5b) can be written in the form

$$\frac{\sigma T^4(z)}{\pi}\sum_{n=1}^{N} K_n w_n = \tfrac{1}{2}\sum_{n=1}^{N} K_n \int_{-1}^{1} \psi_n(z,\mu')\,d\mu'. \tag{7}$$

We can now eliminate $B_n[T(z)]$ among Eqs. (3), (6), and (7), and obtain ($dx \triangleq \rho(z)K_N dz$; $K_N \le K_i$, $i < N$):

$$\mu\frac{\partial}{\partial x}\mathbf{\Psi}(x,\mu) + \mathbf{\Sigma}'\mathbf{\Psi}(x,\mu) = \mathbf{C}\int_{-1}^{1}\mathbf{\Psi}(x,\mu')\,d\mu' + \mathbf{C}'\int \mathbf{\Psi}(x,\mu')f(\mathbf{\Omega}'\cdot\mathbf{\Omega})\,d\mathbf{\Omega}', \tag{8}$$

where

$$(\mathbf{\Sigma}')_{ij} = (\sigma_i + S_i/K_N)\,\delta_{ij}, \tag{9a}$$

$$\sigma_i \triangleq K_i/K_N, \qquad (\sigma_N = 1), \tag{9b}$$

$$(\mathbf{C})_{ij} = \sigma_i\sigma_j w_i \Big/ 2\sum_{\alpha=1}^{N}\sigma_\alpha w_\alpha, \tag{10}$$

and

$$(\mathbf{C}')_{ij} = (S_i/K_N)\,\delta_{ij}. \tag{11}$$

Even for isotropic scattering [$f(\mathbf{\Omega}'\cdot\mathbf{\Omega}) = (4\pi)^{-1}$] we do not know how to obtain explicit solutions to this set of equations [det $(\mathbf{C}+\mathbf{C}') \neq 0$]. However, if the scattering term may be neglected, we obtain a transport equation of the form

$$\mu\frac{\partial}{\partial x}\mathbf{\Psi}(x,\mu) + \mathbf{\Sigma}\mathbf{\Psi}(x,\mu) = \mathbf{C}\int_{-1}^{1}\mathbf{\Psi}(x,\mu')\,d\mu', \tag{12}$$

where

$$(\mathbf{\Sigma})_{ij} = \sigma_i\,\delta_{ij}. \tag{13}$$

We note

$$\det \mathbf{C} = 0 \tag{14a}$$

and, in fact,

$$\det \mathbf{M} = 0, \tag{14b}$$

where $\mathbf{M}$ is any minor of any rank > 1 of $\mathbf{C}$. Equation (12) can be solved explicitly by an extension of the technique described in I. The procedure is described in the subsequent sections.

II. THE EIGENVALUES AND EIGENSOLUTIONS

As in I, translational invariance suggests we seek solutions of Eq. (12) in the form[3]

$$\mathbf{\Psi}(\eta, x, \mu) = e^{-x/\eta}\mathbf{F}(\eta,\mu). \tag{15}$$

Substituting this ansatz into Eq. (12), we obtain an equation for

$$\mathbf{F}(\eta,\mu),\ (\mathbf{\Sigma} - \mu/\eta\,\mathbf{E})\mathbf{F}(\eta,\mu) = \mathbf{C}\int_{-1}^{1}\mathbf{F}(\eta,\mu')\,d\mu', \tag{16}$$

where $\mathbf{E}$ is the unit matrix. First, we consider the continuum solutions, i.e., $\eta \in [-1, 1]$. In I it was necessary to divide this range into two subranges. Here, as we shall see, there are N such subranges

Region 1: $\eta \in [-1/\sigma_1, 1/\sigma_1]$; (17a)

Region 2: $\eta \in [-1/\sigma_2, -1/\sigma_1]$ and $[1/\sigma_1, 1/\sigma_2]$; (17b)

thus, in general,

Region i: $\eta \in [-1/\sigma_i, -1/\sigma_{i-1}]$ and $[1/\sigma_{i-1}, 1/\sigma_i]$, $i > 1$. (17c)

[3] K. M. Case, Ann. Phys. (N. Y.) 9, 1 (1960).

The eigensolutions for the ith-region take the form

$$\mathbf{F}^{(i)}(\eta, \mu) = (\mathbf{P}_i + \mathbf{\Lambda}_i)\mathbf{C} \int_{-1}^{1} \mathbf{F}^{(i)}(\eta, \mu')\, d\mu', \qquad (18)$$

where

$$(\mathbf{P}_i)_{jk} = \eta P\left(\frac{1}{\sigma_j\eta - \mu}\right) \delta_{jk} \qquad (19a)$$

and

$$(\mathbf{\Lambda}_i)_{jk} = \lambda_j^{(i)}(\eta)\, \delta(\sigma_j\eta - \mu)\, \delta_{ij}. \qquad (19b)$$

Here the $\lambda_j^{(i)}(\eta)$ are unspecified functions that must be selected so that Eq. (18) is consistent. We note that for $j < i$ the symbol "P", denoting the Cauchy principal value in Eq. (19a), is superfluous because the denominator can never vanish. Similarly $\lambda_j^{(i)}(\eta)$ may be taken to be zero for $j < i$ since the argument of the delta function never vanishes. Thus, we see that $\mathbf{F}^{(i)}(\eta, \mu)$ contains $(N + 1 - i)$ unknown functions $\lambda_j^{(i)}(\eta)$. In addition, there are N unknown functions $A_j^{(i)}(\eta)$ in Eq. (16), defined by

$$\int_{-1}^{1} \mathbf{F}^{(i)}(\eta, \mu')\, d\mu' = \mathbf{A}^{(i)}(\eta) = \begin{bmatrix} a_1^{(i)}(\eta) \\ \vdots \\ \alpha_N^{(i)}(\eta) \end{bmatrix}. \qquad (20)$$

Thus the solutions $\mathbf{F}^{(i)}(\eta, \mu)$ are $(N + 1 - i)$-fold degenerate. There are $(N + 1 - i)$ linearly independent eigensolutions in region i, which are denoted by

$$\mathbf{F}_\alpha^{(i)}(\eta, \mu), \qquad \alpha = i, i + 1, \cdots, N. \qquad (21)$$

(For notational convenience we run α from i to N rather than from 1 to $N + i - 1$.)

It is a straightforward matter to obtain the explicit form of the $F_\alpha^{(i)}(\eta, \mu)$. We find $[\tau(x) \triangleq \tanh^{-1}(x)]$

$$\mathbf{F}_\alpha^{(i)}(\eta, \mu) = \begin{bmatrix} (\sigma_1\eta - \mu)^{-1}\eta C_{1\alpha} \\ \vdots \\ (\sigma_{i-1}\eta - \mu)^{-1}\eta C_{i-1,\alpha} \\ P(\sigma_i\eta - \mu)^{-1}\eta C_{i\alpha} - 2\eta C_{i\alpha}\tau(\sigma_i\eta)\delta(\sigma_i\eta - \mu) \\ \vdots \\ P(\sigma_\alpha\eta - \mu)^{-1}\eta C_{\alpha\alpha} + \left[1 - 2\eta C_{\alpha\alpha}\tau(\sigma_\alpha\eta) - 2\eta \sum_{\beta=1}^{i-1} C_{\beta\beta}\tau\left(\frac{1}{\sigma_\beta\eta}\right)\right]\delta(\sigma_\alpha\eta - \mu) \\ P(\sigma_{\alpha+1}\eta - \mu)^{-1}\eta C_{\alpha+1,\alpha} - 2\eta C_{\alpha+1,\alpha}\tau(\sigma_{\alpha+1}\eta - \mu) \\ \vdots \end{bmatrix}. \qquad (22)$$

Although the derivation of Eq. (22) is tedious, it is easy to verify that it is a solution. In doing so, the relation

$$C_{j\alpha}C_{\alpha k} = C_{jk}C_{\alpha\alpha} \qquad (23)$$

must be kept in mind.

Next, we consider the discrete spectrum, i.e., $\eta \notin [-1, 1]$. Thus from Eq. (16) we obtain

$$\mathbf{F}(\eta, \mu) = \mathbf{DC} \int_{-1}^{1} \mathbf{F}(\eta, \mu')\, d\mu', \qquad (24)$$

where

$$(\mathbf{D})_{ij} = (\sigma_i\eta - \mu)^{-1}\eta\, \delta_{ij}. \qquad (25)$$

The eigenvalues are obtained by integrating Eq. (24) and noting that nontrivial solutions exist only if

$$\Omega(z) \triangleq \det(\mathbf{E} - \boldsymbol{\tau}\mathbf{C}) = 0, \qquad (26)$$

where

$$(\boldsymbol{\tau})_{jk} = 2z\tau\left(\frac{1}{\sigma_j z}\right)\delta_{jk}. \qquad (27)$$

To evaluate Eq. (26), we write[4]

$$\det(\lambda\mathbf{E} - \boldsymbol{\tau}\mathbf{C}) = \lambda^N - \mathrm{Tr}(\boldsymbol{\tau}\mathbf{C})\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_N. \qquad (28)$$

Here, the coefficients of λ^{N-k}, $k = 2, 3, \cdots, N$ are defined as the sum of all the k by k minor determinants that can be formed using k of the diagonal elements of $\boldsymbol{\tau}\mathbf{C}$ (there are $N!/k!(N - k)!$ such determinants). One easily verifies that all such minor determinants of $\boldsymbol{\tau}\mathbf{C}$ vanish. Thus, setting $\lambda = 1$ in Eq. (28), we obtain

$$\Omega(z) = 1 - 2z \sum_{\beta=1}^{N} C_{\beta\beta}\tau\left(\frac{1}{\sigma_\beta z}\right). \qquad (29)$$

In Appendix A we show that $\Omega(z)$ has only two zeros which, from Eqs. (29) and (10), are $\eta_0 = \pm\infty$. Thus, the discrete eigenvalues are identical with those obtained in I. The discrete eigensolutions are similar; we find

[4] J. H. M. Wedderburn, *Lectures on Matrices* (American Mathematical Society, New York, 1934), Chap. II.

$$\Psi_+(x,\mu) = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} \triangleq \Phi_+ \tag{30a}$$

and

$$\Psi_-(x,\mu) = \begin{bmatrix} w_1(x-\mu/\sigma_1) \\ w_2(x-\mu/\sigma_2) \\ \vdots \\ w_N(x-\mu/\sigma_N) \end{bmatrix}. \tag{30b}$$

As in I, we choose to work mostly with certain linear combinations of the $\mathbf{F}_\alpha^{(i)}(\eta,\mu)$ which we call $\Phi_\alpha^{(i)}(\eta,\mu)$. These are defined by

$$\left\{1 - 2\eta \sum_{\beta=1}^{i-1} C_{\beta\beta}\tau\left(\frac{1}{\sigma_\beta\eta}\right)\right\}\Phi_\alpha^{(i)}(\eta,\mu) = \frac{1}{C_{1\alpha}}\mathbf{F}_\alpha^{(i)}(\eta,\mu) - \frac{1}{C_{1,\alpha+1}}\mathbf{F}_{\alpha+1}^{(i)}(\eta,\mu), \tag{31a}$$

where $\alpha = i, i+1, \cdots, N-1$. Thus

$$\Phi_\alpha^{(i)}(\eta,\mu) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ (C_{1\alpha})^{-1}\delta(\sigma_\alpha\eta-\mu) \\ (-C_{1,\alpha+1})^{-1}\delta(\sigma_{\alpha+1}\eta-\mu) \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad \alpha = i, \cdots, N-1. \tag{31b}$$

Also, taking

$$\Phi_N^{(i)}(\eta,\mu) = \mathbf{F}_N^{(i)}(\eta,\mu), \tag{32}$$

we see that, in each region, there are $(N-i)$ eigensolutions of the form (31b), containing only two nonzero elements, which involve only delta functions and one eigensolution of the form (32).

Aside from the simple form taken by the $\Phi_\alpha^{(i)}(\eta,\mu)$, we see that these solutions take the same form in different regions; thus, we can recombine the $\Phi_\alpha^{(i)}(\eta,\mu)$ in the following way. First, for $K < N$,

$$\Phi_K^{(i)}(\eta,\mu) = \Phi_K^{(j)}(\eta,\mu), \quad j = i, i+1, \cdots, K. \tag{33}$$

This suggests defining[5]

$$\Phi_K(\eta,\mu) = \begin{cases} \Phi_K^{(j)}(\eta,\mu), & j \le K < N, \quad (34a) \\ 0, & K < j. \quad (34b) \end{cases}$$

There are $(N-1)$ eigensolutions of this form; since there are N of the $\Phi_N^{(i)}(\eta,\mu)$, we are now dealing with only $(2N-1)$ different eigensolutions instead of $[\frac{1}{2}N(N+1)]$. If we, in fact, also define

$$\Phi_N(\eta,\mu) = \sum_{i=1}^{N} \Phi_N^{(i)}(\eta,\mu) \oplus_i(\eta), \tag{35}$$

where

$$\oplus_i(\eta) = \begin{cases} 1, & \eta \in \text{region } i, \\ 0, & \text{otherwise}, \end{cases} \tag{36}$$

we need only consider N eigensolutions, $\Omega_1(\eta,\mu)$, $\cdots$, $\Phi_N(\eta,\mu)$. To recapitulate, we have $(N-1)$ continuum eigensolutions of the forms

$$\Phi_1(\eta,\mu) = \begin{bmatrix} (C_{11})^{-1}\delta(\sigma_1\eta-\mu) \\ (-C_{12})^{-1}\delta(\sigma_2\eta-\mu) \\ 0 \\ 0 \\ \vdots \end{bmatrix} \oplus_1(\eta), \tag{37a}$$

$$\Phi_2(\eta,\mu) = \begin{bmatrix} 0 \\ (C_{12})^{-1}\delta(\sigma_2\eta-\mu) \\ (-C_{13})^{-1}\delta(\sigma_3\eta-\mu) \\ 0 \\ 0 \\ \vdots \end{bmatrix} \times \{\oplus_1(\eta) + \oplus_2(\eta)\}, \tag{37b}$$

or, in general,

$$\Phi_i(\eta,\mu) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ (C_{1i})^{-1}\delta(\sigma_i\eta-\mu) \\ (-C_{1,i+1})^{-1}\delta(\sigma_{i+1}\eta-\mu) \\ 0 \\ 0 \\ \vdots \end{bmatrix} \times \sum_{\alpha=1}^{i} \oplus_\alpha(\eta). \tag{37c}$$

[5] This method of attack was suggested by J. Mika (private communication).

In addition, we have one continuum eigensolution [Eq. (35)] as well as two discrete eigenmodes [Eqs. (30)].

III. COMPLETENESS

Theorem I: The functions $\mathbf{\Phi}_i(\eta, \mu)$, $i = 1 \cdots N$, $\eta \geq 0$ and $\mathbf{\Phi}_+$ are complete for arbitrary N-vector functions, $\mathbf{\Psi}(\mu)$, defined on the "half-range," $0 \leq \mu \leq 1$.

This theorem means that an N-component function $\mathbf{\Psi}(\mu)$ can be expanded in the form

$$\mathbf{\Psi}(\mu) = A_+\mathbf{\Phi}_+ + \sum_{\beta=1}^{N} \int_0^{1/\sigma_\beta} \alpha_\beta(\eta)\mathbf{\Phi}_\beta(\eta, \mu)\, d\eta, \qquad \mu \in [0, 1], \tag{38}$$

where A_+ and $\alpha_\beta(\eta)$ are "scalar" functions.

Proof: We proceed as in I, i.e., we attempt an expansion in terms of the continuum modes alone:

$$\mathbf{\Psi}(\mu) = \sum_{\beta=1}^{N-1} \int_0^{1/\sigma_\beta} \alpha_\beta(\eta)\mathbf{\Phi}_\beta(\eta, \mu)\, d\eta + \int_0^1 \alpha_N(\eta)\mathbf{\Phi}_N(\eta, \mu)\, d\eta. \tag{39}$$

Here, the last term has been split off from the sum because the first $(N - 1)$ terms have the simple form given in Eq. (37c). These integrals can all be carried out explicitly,

$$\int_0^{1/\sigma_\beta} \alpha_\beta(\eta)\mathbf{\Phi}_\beta(\eta, \mu)\, d\eta = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ (\sigma_\beta C_{1\beta})^{-1}\alpha_\beta(\mu/\sigma_\beta) \\ -(\sigma_{\beta+1}C_{1,\beta+1})^{-1}\alpha_\beta(\mu/\sigma_{\beta+1})S(\sigma_{\beta+1}/\sigma_\beta - \mu) \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \qquad \beta = 1, \cdots, N-1, \tag{40}$$

where $S(X)$ is the unit step function

$$S(X) = 1, \quad X \geq 0, \qquad = 0, \quad \text{otherwise}. \tag{41}$$

When these results are substituted into Eq. (39), we obtain

$$\begin{aligned}\psi_i(\mu) = {} & -(C_{1i}\sigma_i)^{-1}\alpha_{i-1}(\mu/\sigma_i)S(\sigma_i/\sigma_{i-1} - \mu) \\ & + (C_{1i}\sigma_i)^{-1}\alpha_i(\mu/\sigma_i) \\ & - 2\mu C_{iN}\alpha_N(\mu/\sigma_i)\tau(\mu)\sigma_i^{-2} \\ & + C_{iN}P \int_0^1 \frac{\alpha_N(\eta)\eta\, d\eta}{\sigma_i\eta - \mu}, \\ & i = 1, 2, \cdots, N-1, \end{aligned} \tag{42a}$$

and

$$\begin{aligned}\psi_N(\mu) = {} & -C_{1N}^{-1}\alpha_{N-1}(\mu)S(\sigma_{N-1}^{-1} - \mu) \\ & + C_{NN}P \int_0^1 \frac{\alpha_N(\eta)\eta\, d\eta}{\eta - \mu} \\ & + \alpha_N(\mu)[1 - 2\mu C_{NN}\tau(\mu)] \\ & - 2\mu \sum_{\beta=1}^{N-1} C_{\beta\beta}\tau\left(\frac{1}{\sigma_\beta\mu}\right)S(\mu - \sigma_\beta^{-1}). \end{aligned} \tag{42b}$$

Here $\psi_i(\mu)$ is the ith component of $\mathbf{\Psi}(\eta)$. [We recall in obtaining Eq. (42b) that $\sigma_N = 1$.]

We note from Eqs. (42) that the unknowns $\alpha_i(\eta)$, $i = 1, \cdots, N-1$, can be eliminated successively starting with $i = 1(\alpha_0(\eta) = 0)$. In this way, Eqs. (42) can be converted into a singular integral equation for $\alpha_N(\eta)$. To carry this out, we make the change of variable in Eq. (42a),

$$\mu/\sigma_i \rightarrow \mu \tag{43}$$

and multiply the equation by σ_i^2. Then we add all N equations [i.e., including (42b)] to obtain the simple result

$$\begin{aligned}\sum_{i=1}^{N} \sigma_i^2\psi_i(\sigma_i\mu)S(1/\sigma_i - \mu) = {} & \sum_{i=1}^{N} C_{ii}S(1/\sigma_i - \mu)P \int_0^1 \frac{\alpha_N(\eta)\eta\, d\eta}{\eta - \mu} + \alpha_N(\mu) \\ & \times \left\{1 - 2\mu \sum_{i=1}^{N} C_{ii}[\tau(1/\sigma_i\mu)\ \delta(\mu - 1/\sigma_i) + \tau(\sigma_i\mu)S(1/\sigma_i - \mu)]\right\}. \end{aligned} \tag{44}$$

The various step functions were introduced by the variable change, since we must require the argument of $\psi_i(\eta)$, for example, to be less than or equal to unity; thus, under (43),

$$\psi_i(\mu) \rightarrow \psi_i(\sigma_i\mu)\{\oplus_1(\mu) + \cdots + \oplus_i(\mu)\}. \tag{45}$$

We note

$$\sum_{\alpha=1}^{i} \oplus_\alpha(\mu) = S\left(\frac{1}{\sigma_i} - \mu\right). \tag{46}$$

Equation (44) is now in canonical form, since, we note that, from the definition of $\Omega(z)$ in Eq. (29),

$$\Omega^{\pm}(\mu) = 1 - 2\mu \sum_{\beta=1}^{N} C_{\beta\beta}\tau(1/\sigma_\beta\mu)S(\mu - 1/\sigma_\beta) - 2\mu \sum_{\beta=1}^{N} C_{\beta\beta}[\tau(\sigma_\beta\mu) \mp \tfrac{1}{2}\pi i]S\left(\frac{1}{\sigma_\beta} - \mu\right). \quad (47)$$

[$\Omega(z)$ is analytic in the complex z plane cut from -1 to $+1$ along the real line; $\Omega^{\pm}$ represents the boundary values above and below the branch cut.] Thus, Eq. (44) can be written as

$$\sum_{i=1}^{N} \sigma_i^2\psi_i(\sigma_i\mu)S\left(\frac{1}{\sigma_i} - \mu\right) = \frac{\Omega^+(\mu) - \Omega^-(\mu)}{2\pi i\mu} P \int_0^1 (\eta - \mu)^{-1}\alpha_N(\eta)\, d\eta + \tfrac{1}{2}(\Omega^+(\mu) - \Omega^-(\mu))\alpha_N(\mu). \quad (48)$$

This equation is in standard form, (c.f, I) and its solution is well known in terms of the X-function, which is analytic in the complex plane cut from 0 to 1 along the real line.

$$X(z) = (1 - z)^{-1} \times \exp\{\pi^{-1}\int_0^1 \arg\Omega^+(\mu)(\mu - z)^{-1}\, d\mu\}. \quad (49)$$

Once $\alpha_N(\eta)$ is known, the other $\alpha_i(\eta)$ may all be found from Eqs. (42). Also, the discrete mode is introduced, as in I, by the condition at infinity on the auxiliary function

$$N(z) = (2\pi i)^{-1}\int_0^1 (\eta - z)^{-1}\alpha_N(\eta)\eta\, d\eta. \quad (50)$$

Since the details are identical to those in I, we go no further than to note that the completeness theorem is hereby proved. The coefficients in the expansion, Eq. (38), can be found in principal from the above solution; but it is simpler to use the orthogonality relations derived in the next section.

IV. ORTHOGONALITY

Theorem II: The continuum functions $\mathbf{\Phi}(\eta, \mu)$, $\eta \in [0, 1]$ and the discrete mode $\mathbf{\Phi}_+$ are orthogonal to the adjoint eigensolutions $\mathbf{\Phi}^\dagger(\eta, \mu)$, $\eta \in [0, 1]$ and $\mathbf{\Phi}_+^\dagger$ on the range $0 \le \mu \le 1$ with weight function $\mathbf{W}(\mu)$, where

$$[\mathbf{W}(\mu)]_{ij} = \sigma_i\gamma(\mu/\sigma_i)\delta_{ij}, \quad (51)$$

and $\gamma(\mu)$ is defined as

$$\gamma(\mu) = \mu X^+(\mu)/\Omega^+(\mu). \quad (52)$$

The proof of this theorem is a generalization of the one given in I for $N = 2$. There, we prove orthogonality of the $\mathbf{F}_\alpha^i(\eta, \mu)$ rather than the $\mathbf{\Phi}_K(\eta, \mu)$. However, this makes no difference, since the theorem states that[6] (the over tilde denotes transpose)

$$\int_0^1 \tilde{\mathbf{\Phi}}^\dagger(\eta', \mu)\mathbf{W}(\mu)\mathbf{\Phi}(\eta, \mu)\, d\mu = 0, \qquad \eta \neq \eta', \quad (53a)$$

and we *prove* that

$$\int_0^1 \tilde{\mathbf{F}}^\dagger(\eta', \mu)\mathbf{W}(\mu)\mathbf{F}(\eta, \mu)\, d\mu = 0, \qquad \eta = \eta'. \quad (53b)$$

Clearly, (53a) and (53b) imply each other. $\mathbf{F}^\dagger(\eta, \mu)$, obeying the adjoint equation given below, may be obtained from $\mathbf{F}(\eta, \mu)$ under the interchange $C_{ij} \Leftrightarrow C_{ji}$; $\mathbf{\Phi}^\dagger(\eta, \mu)$ and $\mathbf{\Phi}(\eta, \mu)$ are related in the same way. As in I, we note that care must be taken when this interchange is made. For example, $\mathbf{\Phi}_+$ is given by Eq. (30a). However, this form is a reduction of the form obtained if Eq. (16) with $\eta_0 = \pm\infty$ is solved directly; this is to say that C_{ij} no longer appear. One finds easily, however,

$$\mathbf{\Phi}_+^\dagger = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \quad (54)$$

Equations (22) for $\mathbf{F}_\alpha^i(\eta, \mu)$,and Eqs. (31b) and (32) for $\mathbf{\Phi}_\alpha(\eta, \mu)$ are in the form so that the interchange $C_{ij} \Leftrightarrow C_{ji}$ gives the adjoint solutions.

Proof: We turn now to the proof of Theorem II, i.e., Eq. (53b). We begin, as in I, with the equation obeyed by $\mathbf{F}(\eta, \mu)$ and the adjoint equation obeyed by $\mathbf{F}^\dagger(\eta', \mu)$. [To simplify notation, we allow $\mathbf{F}(\eta, \mu)$ and $\mathbf{F}^\dagger(\eta', \mu)$ to denote either discrete or continuum modes.] Thus,

$$\left(\frac{\mathbf{\Sigma}}{\mu} - \frac{\mathbf{C}}{\mu}\int_{-1}^1 d\mu'\right)\mathbf{F}(\eta, \mu) = \frac{1}{\eta}\mathbf{F}(\eta, \mu) \quad (55a)$$

and

$$\left(\frac{\mathbf{\Sigma}}{\mu} - \frac{\tilde{\mathbf{C}}}{\mu}\int_{-1}^1 d\mu'\right)\mathbf{F}^+(\eta', \mu) = \frac{1}{\eta}\mathbf{F}^\dagger(\eta', \mu). \quad (55b)$$

The method of proof is to multiply Eq. (55a) from the left by $\tilde{\mathbf{F}}^+(\eta', \mu)\mathbf{W}(\mu)$ and to multiply the transpose of Eq. (55b) from the right by $\mathbf{W}(\mu)\mathbf{F}(\eta, \mu)$, integrate both over μ from 0 to 1 and subtract. The right-hand side of the resulting equation becomes simply

$$\left(\frac{1}{\eta'} - \frac{1}{\eta}\right)\int_0^1 \tilde{\mathbf{F}}^\dagger(\eta', \mu)\mathbf{W}(\mu)\mathbf{F}(\eta, \mu)\, d\mu,$$

[6] The function $\mathbf{\Psi}_-$ is not included in the orthogonal set because it is not a solution of Eqs. (55) [it satisfies Eq. (12)].

and thus we wish to prove that the left-hand side vanishes if $\eta \neq \eta'$. By following a procedure identical to that outlined in I the proof can be reduced to showing that a quantity J_2 is a constant, where

$$J_2 = X(0) - \omega(\eta)\eta^{-1}\gamma(\eta) + \sum_{\alpha=1}^{N} C_{\alpha\alpha}\lambda^{\alpha}(\eta)\eta^{-1}\gamma(\eta), \quad (56)$$

with

$$\omega(\eta) = \tfrac{1}{2}[\Omega^{+}(\eta) - \Omega^{-}(\eta)]. \quad (57)$$

The proof of the theorem hinges on proving that J_2 is independent of η and symmetric in i, j. We showed this in I by direct substitution of the $\lambda^{\alpha}(\eta)$ (see note added in proof). Here, we prove it in general. We should clarify what is meant by the symbol $\lambda^{\alpha}(\eta)$. We note from Eq. (16) that any of the components $f_i(\eta, \mu)$ of $\mathbf{F}(\eta, \mu)$ can be written in the form

$$f_{\alpha}(\eta, \mu) = \{P\eta(\sigma_{\alpha}\eta - \mu)^{-1} + \lambda^{\alpha}(\eta)\delta(\sigma_{\alpha}\eta - \mu)\} \times \sum_{\beta=1}^{N} C_{\alpha\beta}a_{\beta}(\eta), \qquad \sigma_{\alpha}\eta \leq 1, \quad (58a)$$

or

$$f_i(\eta, \mu) = \eta(\sigma_i\eta - \mu)^{-1} \times \sum_{\beta=1}^{N} C_{i\beta}a_{\beta}(\eta), \qquad \sigma_i\eta \geq 1, \quad (58b)$$

where

$$a_{\beta}(\eta) = \int_{-1}^{1} f_{\beta}(\eta, \mu')\, d\mu'. \quad (59)$$

[Thus the $\lambda^{\alpha}(\eta)$ are the unknown coefficients in Eq. (19b) which one must find for explicit evaluation of the eigensolutions; thus they should be denoted by $\lambda_{ik}^{\alpha}(\eta)$ in order to distinguish the region and the degeneracy. However, we do not have to use their explicit form. The symbol $\lambda^{\alpha}(\eta)$ thus denotes *any* of the $\lambda_{ik}^{\alpha}(\eta)$.] Multiply Eq. (58a) by $C_{N\alpha}$ and sum over all α for which $\sigma_{\alpha}n \leq 1$; then multiply Eq. (58b) by C_{Nj} and sum over all j for which $\sigma_j n > 1$. We integrate the resulting two equations over μ from -1 to 1 and add them to obtain

$$\sum_{\alpha}{}^{s} C_{N\alpha}a_{\alpha}(\eta) + \sum_{i}{}^{ns} C_{Ni}a_i(\eta) = \sum_{\alpha}{}^{s} C_{N\alpha}[2\eta\tau(\sigma_{\alpha}\eta) + \lambda^{\alpha}(\eta)] \times \sum_{\beta=1}^{N} C_{\alpha\beta}a_{\beta}(\eta) + \sum_{i}{}^{ns} C_{Ni}2\eta\tau\left(\frac{1}{\sigma_i\eta}\right) \sum_{\beta=1}^{N} C_{i\beta}a_{\beta}(\eta). \quad (60)$$

Here the superscripts s and ns indicate that the sums are to be taken only over the singular [Eq. (58a)] and the nonsingular [Eq. (58b)] sets respectively. Using $C_{\alpha\beta}C_{N\alpha} = C_{\alpha\alpha}C_{N\beta}$ in the right-hand side of Eq. (60), the sums separate and a factor $\sum_{\beta=1}^{N} C_{N\beta}A_{\beta}(\eta)$ can therefore be canceled to give

$$1 - 2\eta \sum_{\alpha}{}^{s} C_{\alpha\alpha}\tau(\sigma_{\alpha}\eta) - \sum_{i}{}^{ns} 2\eta C_{ii}\tau\left(\frac{1}{\sigma_i\eta}\right) = \sum_{\alpha}{}^{s} C_{\alpha\alpha}\lambda^{\alpha}(\eta). \quad (61)$$

The left-hand side of Eq. (61) is exactly $\omega(\eta)$, thus

$$\sum_{\alpha}{}^{s} C_{\alpha\alpha}\lambda^{\alpha}(\eta) = \omega(\eta). \quad (62)$$

Substituting this result into Eq. (56) [remembering that $\lambda^{\alpha}(\eta)$ appears only when $f_{\alpha}(\eta, \mu)$ is singular] we find

$$J_2 \equiv 1, \quad (63)$$

the theorem is thus proved.

V. NORMALIZATION

The results of the previous two sections can be used to expand functions $\mathbf{\Psi}(\mu)$ for $\mu \in [0, 1]$ and to obtain the expansion coefficients if

(i) the normalization integrals are known, and
(ii) the degenerate eigenfunctions are orthogonalized.

As in I, we introduce a new set of functions $\mathbf{x}_K(\eta, \mu)$, constructed so as to be orthogonal to all of the $\mathbf{\Phi}_K(\eta, \mu)$. We define our scalar product as

$$(\mathbf{U}, \mathbf{V}) \triangleq \int_0^1 \tilde{\mathbf{U}}^{\dagger}(\eta', \mu)\mathbf{W}(\mu)\mathbf{V}(\eta, \mu)\, d\mu. \quad (64)$$

Abbreviating

$$(\mathbf{\Phi}_i, \mathbf{\Phi}_j) = (i, j)\delta(\eta - \eta'), \quad (65)$$

we easily calculate

$$(i, j) = -\gamma(\eta)[(C_{1,i+1}C_{i+1,1})^{-1}\delta_{i+1}^{j} + (C_{1i}C_{i1})^{-1}\delta_{i-1}^{j} - [(C_{1i}C_{i1})^{-1} + (C_{1,i+1}C_{i+1,1})^{-1}]\delta_i^j\}, \quad i, j < N. \quad (66)$$

Also, keeping Theorem II in mind, we have

$$(\mathbf{\Phi}_i, \mathbf{\Phi}_N) = \sum_{\alpha=1}^{i} (\mathbf{\Phi}_i, \mathbf{\Phi}_N^{(\alpha)}). \quad (67)$$

The product $(\mathbf{\Phi}_i, \mathbf{\Phi}_N^{(\alpha)})$ is also easily calculated, and is seen to be independent of α. Thus the sum in Eq. (67) merely introduces a factor i, and we find

$$(i, N) = i\frac{C_{iN}}{C_{i1}} 2\eta\gamma(\eta)\{\tau(\sigma_{i+1}\eta) - \tau(\sigma_i\eta)\}, \qquad i < N - 1, \quad (68a)$$

$$(N, i) = (i, N)\frac{C_{Ni}}{C_{1i}}\frac{C_{i1}}{C_{iN}}, \quad i < N-1, \quad (68b)$$

$$= w_N/w_i(i, N), \quad i < N-1. \quad (68c)$$

Also,

$$(N-1, N) = -\gamma(\eta)\left[(N-1)\frac{C_{N-1,N}}{C_{N-1,1}}2\eta\tau(\sigma_{N-1}\eta) + \frac{1}{C_{N1}}\sum_{\alpha=1}^{N-1}\left(1 - 2\eta\sum_{\beta=1}^{\alpha-1}C_{\beta\beta}\tau(1/\sigma_\beta\eta) - 2\eta C_{NN}T(\eta)\right)\right]. \quad (69)$$

$(N, N-1)$ may be found from $(N-1, N)$ under the interchange $C_{ij} \to C_{ji}$. Finally we need

$$(\mathbf{\Phi}_N, \mathbf{\Phi}_N) = \sum_{i=1}^{N}(\mathbf{\Phi}_N^{(i)}, \mathbf{\Phi}_N^{(i)}), \quad (70)$$

where $(\mathbf{\Phi}_N^{(i)}, \mathbf{\Phi}_N^{(i)})$ may be calculated from the explicit form, Eq. (22), by making use of the X-function identities of Appendix B. We quote the result,

$$(N, N) = \sum_{i=1}^{N}\left\{\sum_{\alpha=1}^{N}C_{NN}4\eta^2C_{\alpha\alpha}\tau^2(\sigma_\alpha\eta) + \left[1 - 2\eta\sum_{\beta=1}^{i-1}C_{\beta\beta}\tau(1/\sigma_\beta\eta) - 2\eta C_{NN}\tau(\eta)\right]^2 + \sum_{\alpha=i}^{N}C_{NN}C_{\alpha\alpha}\pi^2\eta^2\right\}\gamma(\eta). \quad (71)$$

Equations (66), (68), (69), and (71) give all the necessary normalization integrals for the construction of the $\mathbf{x}_K$. We write, in general,

$$\mathbf{x}_i(\eta, \mu) = \sum_{\alpha=1}^{N}\mathrm{N}_\alpha^{(i)}\mathbf{\Phi}_\alpha(\eta, \mu), \quad (72)$$

where $\mathrm{N}_\alpha^{(i)}$ are to be choosen such that

$$(\mathbf{x}_i, \mathbf{\Phi}_j) = \delta_{ij}\,\delta(\eta - \eta'). \quad (73)$$

The $\mathrm{N}_\alpha^{(i)}$ are readily found from Eqs. (72) and (73). We take the scalar product of Eq. (72) from the right successively with the $\mathbf{\Phi}_\beta(\eta, \mu)$, $\beta = 1, \cdots N$. This yields the following equations for the $\mathrm{N}_\alpha^{(i)}$:

$$\begin{aligned}
0 &= \mathrm{N}_1^{(i)}(1,1) + \mathrm{N}_2^{(i)}(2,1) + \mathrm{N}_3^{(i)}(3,1) + 0 + \cdots + 0 + \mathrm{N}_N^{(i)}(N,1),\\
0 &= \mathrm{N}_1^{(i)}(1,2) + \mathrm{N}_2^{(i)}(2,2) + \mathrm{N}_3^{(i)}(3,2) + 0 + \cdots + 0 + \mathrm{N}_N^{(i)}(N,2),\\
0 &= 0 + \mathrm{N}_2^{(i)}(2,3) + \mathrm{N}_3^{(i)}(3,3) + \mathrm{N}_4^{(i)}(4,3) + 0 + \cdots + 0 + \mathrm{N}_N^{(i)}(N,3),\\
&\vdots\\
1 &= 0 + \cdots \mathrm{N}_{i-1}^{(i)}(i-1,i) + \mathrm{N}_i^{(i)}(i,i) + \mathrm{N}_{i+1}^{(i)}(i+1,i) + 0 + \cdots \mathrm{N}_N^{(i)}(N,i),\\
&\vdots\\
0 &= \mathrm{N}_1^{(i)}(1,N) + \mathrm{N}_2^{(i)}(2,N) + \cdots \mathrm{N}_N^{(i)}(N,N).
\end{aligned} \quad (74)$$

We see that the first and the $(N-1)$th equations have only three nonvanishing coefficients, all the rest have four such coefficients except the Nth, which has N. The set of equations (74) is easily solved for the $\mathrm{N}_\alpha^{(i)}$.

We note that we have set $(\mathbf{x}_K, \mathbf{\Phi}_K) = \delta(\eta - \eta')$. Also $(\mathbf{x}_K, \mathbf{\Phi}_+) = 0$. The discrete (asymptotic) coefficient is found from the relation

$$(\mathbf{\Phi}_+, \mathbf{\Phi}_+) = \sum_{i=1}^{N}\int_0^1 w_i\sigma_i\gamma(\mu/\sigma_i)\,d\mu. \quad (75)$$

Changing variables, this can be written

$$(\mathbf{\Phi}_+, \mathbf{\Phi}_+) = \sum_{i=1}^{N}\int_0^1 w_i\sigma_i^2\int_0^{1/\sigma_i}\gamma(\mu)\,d\mu,$$

$$= 2\sum_{K=1}^{N}w_K\sigma_K\sum_{i=1}^{N}C_{ii}\int_0^{1/\sigma_i}\gamma(\mu)\,d\mu. \quad (76)$$

This integral can be evaluated from Eq. (B1) (in Appendix B) in the limit $z \to \infty$,

$$-\lim_{z\to\infty} Z\,X(z) = \sum_{i=1}^{N}C_{ii}\int_0^{1/\sigma_i}\gamma(\mu)\,d\mu. \quad (77)$$

But from Eq. (49), we see that the limit is -1. Thus

$$(\mathbf{\Phi}_+, \mathbf{\Phi}_+) = 2\sum_{i=1}^{N}\sigma_iw_i. \quad (78)$$

In applying this result to obtain expansion coefficients, one might have an expansion of the form

$$\mathbf{\Psi}(\mu) = A_+\mathbf{\Phi}_+ + \sum_{i=1}^{N}\int_0^{1/\sigma_i}\alpha_i(\eta)\mathbf{\Phi}_i(\eta, \mu)\,d\eta, \quad \mu \in [0, 1]. \quad (79)$$

Then, from Eq. (78), we find

$$A_+ = (\mathbf{\Phi}_+, \mathbf{\Psi}(\mu))\left(2\sum_{i=1}^{N}\sigma_iw_i\right)^{-1}, \quad (80)$$

while

$$\alpha_i(\eta) = (\mathbf{x}_i, \mathbf{\Psi}(\mu)). \quad (81)$$

VI. THE MILNE PROBLEM

We seek the angular density, $\Psi_M(x, \mu)$, in the source-free half-space under the boundary conditions:

(a) $\Psi_M(x, \mu = 0, \mu \geq 0$ (zero re-entrant radiation),
(b) $\Psi_M(x, \mu) \sim \Psi_-(x, \mu)$ (for large x).

The second condition specifies that $\Psi_M(x, \mu)$ diverges no more rapidly than the slowest diverging mode $\Psi_-(x, \mu)$.

The solution can be constructed from the normal modes of the transport equation. Condition (b) requires that no $\Psi(\eta, x, \mu)$ be included for $\eta \in [-1, 0]$. Thus, we write

$$\Psi_M(x, \mu) = A_-\Psi_-(x, \mu) + A_+\Phi_+ + \sum_{i=1}^{N} \int_0^{1/\sigma_i} \alpha_i(\eta) e^{-x/\eta} \Phi_i(\eta, \mu)\, d\eta. \quad (82)$$

The coefficient A_- we leave arbitrary (it depends upon the normalization). The other coefficients are obtained from condition (a). Setting $x = 0$ in Eq. (82), we have

$$-A_-\Psi_-(0, \mu) = A_+\Phi_+ + \sum_{i=1}^{N} \int_0^{1/\sigma_i} \alpha_i(\eta) \Phi_i(\eta, \mu)\, d\eta, \qquad \mu \in [0, 1]. \quad (83)$$

Thus, the coefficients are just the half-range expansion coefficients for the function

$$-A_-\Psi_-(0, \mu) = A_- \begin{bmatrix} w_1/\sigma_1 \\ w_2/\sigma_2 \\ \vdots \\ w_N \end{bmatrix} \mu. \quad (84)$$

They are found immediately from the orthogonality relations once the $\mathbf{X}_i(\eta, \mu)$ are constructed. For the asymptotic solution (i.e., the part of Ψ_M involving Ψ_- and Φ_+), we have

$$\frac{A_+}{A_-} = \frac{-\int_0^1 \tilde{\Phi}_+^\dagger \mathbf{W}(\mu) \Psi_-(0, \mu)\, d\mu}{2 \sum_{i=1}^{N} \sigma_i w_i}, \quad (85)$$

where the normalization integral, Eq. (78), is used. Expanding Eq. (85), we obtain

$$\frac{A_+}{A_-} = \frac{\sum_{i=1}^{N} w_i \int_0^1 \gamma(\mu/\sigma_i) \mu\, d\mu}{2 \sum_{i=1}^{N} \sigma_i w_i}. \quad (86)$$

Changing variables and noting Eq. (10), (86) becomes

$$\frac{A_+}{A_-} = \sum_{i=1}^{N} C_{ii} \int_0^{1/\sigma_i} \gamma(\mu) \mu\, d\mu. \quad (87)$$

This expression can be put in terms of the X-function by use of Identity IV, Appendix B. We find

$$\frac{A_+}{A_-} = \frac{3}{2} \sum_{i=1}^{N} C_{ii} \int_0^{1/\sigma_i} \frac{\mu^2}{X(-\mu)}\, d\mu. \quad (88)$$

The continuum expansion coefficients can be found in just the same manner. [However, since, in general, one must solve the set of equations (74) and then use Eq. (72) to construct the $\mathbf{x}_i$'s we merely formally indicate the solution.]

$$\alpha_i(\eta)/A_- = -[\mathbf{x}_i, \Psi_-(0, \mu)]. \quad (89)$$

The customary normalization[1] is to set

$$-2\pi \int_{-1}^{1} \mu\, d\mu \int_0^\infty d\nu\, \psi_\nu(x, \mu) = \sigma T_e^4, \quad (90)$$

where T_e is the "effective temperature" and σ is the Stefan–Boltzmann constant. Equation (90) can be written as

$$\frac{-\sigma T_e^4}{2\pi} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^\sim \int_{-1}^{1} \mu \Psi_M(x, \mu) \quad (91a)$$

or

$$\frac{-\sigma T_e^4}{2\pi} = \int_{-1}^{1} \mu \tilde{\Phi}_+^\dagger \Psi_M(x, \mu)\, d\mu. \quad (91b)$$

Evaluating Eq. (91b), we find

$$A_- = 3\sigma T_e^4 \left(4\pi \sum_{i=1}^{N} \frac{w_i}{\sigma_i}\right)^{-1}. \quad (92)$$

The expansion coefficients are now found (in principle) to solve the problem. We have

$$\Psi_M(x, \mu) = \left[3\sigma T_e^4 \left(4\pi \sum_{i=1}^{N} \frac{w_i}{\sigma_i}\right)^{-1}\right] \left[\Psi_-(x, \mu) + \frac{A_+}{A_-} \times \Phi_+ + \sum_{i=1}^{N} \int_0^{1/\sigma_i} \frac{\alpha_i(\eta)}{A_-} e^{-x/\eta} \Phi_i(\eta, \mu)\, d\eta\right]. \quad (93)$$

The energy density,

$$\mathfrak{F}(x) \triangleq 2\pi \int_{-1}^{1} d\mu \int_0^\infty d\nu\, \psi_\nu(x, \mu) \quad (94)$$

is given by

$$\mathfrak{F}(x) = 2\pi \int_{-1}^{1} d\mu\, \tilde{\Phi}_+^\dagger \Psi_M(x, \mu). \quad (95)$$

The extrapolated endpoint is defined in terms of the quantity

$$\mathfrak{F}_{\text{asymp}}(x) = 2\pi \int_{-1}^{1} d\mu\, \tilde{\mathbf{\Phi}}_+^{\dagger} \mathbf{\Psi}_{\text{M asymp}}(x, \mu). \quad (96)$$

This, in our model, becomes

$$\mathfrak{F}_{\text{asymp}}(x) = \left[3\sigma T_e^4 \left(\sum_{i=1}^{N} \frac{w_i}{\sigma_i}\right)^{-1}\right] \times \left[\sum_{i=1}^{N} w_i \left(x + \frac{A_+}{A_-}\right)\right]. \quad (97)$$

Thus the asymptotic energy density extrapolates to zero at $x = -x_0$, where

$$x_0 = A_+/A_- \quad (98)$$

which is already given by Eq. (88).

The temperature distribution in this model is given by

$$\frac{\sigma T^4(x)}{\pi} = \left(2 \sum_{i=1}^{N} \sigma_i w_i\right)^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ 1 \end{bmatrix}^{\sim} \times \int_{-1}^{1} \mathbf{\Psi}_{\text{M}}(x, \mu)\, d\mu. \quad (99)$$

The asymptotic temperature distribution is easily obtained as

$$\frac{T_{\text{asymp}}^4(x)}{T_e^4} = \left[\frac{3}{4}\left(\sum_{i=1}^{N} \frac{w_i}{\sigma_i}\right)^{-1}\right][x + x_0]. \quad (100)$$

Just as in I, the law of darkening [for the integrated quantities, $\mathbf{\Psi}_{\text{M}}(0, \mu)$, $\mu < 0$] can be obtained. The fact that we restrict μ to be negative enables us to determine $\mathbf{\Psi}_{\text{M}}$ (for $x = 0$) explicitly without actually knowing any of the continuum expansion coefficients except $\alpha_N(\eta)$. The coefficient $\alpha_N(\eta)$ is expressed in terms of the N-function which then permits the evaluation of integrals involved. The procedure follows exactly as in I. We simply state the result,

$$\mathbf{\Psi}_{\text{M}}(0, -\mu) = \left[3\sigma T_e^4 \left(4\pi \sum_{i=1}^{N} \frac{w_i}{\sigma_i}\right)^{-1}\right] \begin{bmatrix} \dfrac{w_1}{X(-\mu/\sigma_1)} \\ \dfrac{w_2}{X(-\mu/\sigma_2)} \\ \vdots \\ \dfrac{w_N}{X(-\mu)} \end{bmatrix}, \qquad \mu \in [0, 1]. \quad (101)$$

It is clear how other half-range problems could be solved. For example, consider the albedo problem. Here we have a source-free half-space with incident distribution

$$\mathbf{\Psi}_{\text{inc}}(\mu) = \begin{bmatrix} \delta(\mu - \mu_1) \\ \delta(\mu - \mu_2) \\ \vdots \\ \delta(\mu - \mu_N) \end{bmatrix}, \qquad \mu_i, \mu \geq 0. \quad (102)$$

Here, the solution must not diverge at infinity, so we set

$$\mathbf{\Psi}_a(x, \mu) = A_+ \mathbf{\Phi}_+ + \sum_{i=1}^{N} \int_0^{1/\sigma_i} \alpha_i(\eta) e^{-x/\eta} \mathbf{\Phi}_i(\eta, \mu)\, d\eta. \quad (103)$$

Since

$$\mathbf{\Psi}_a(0, \mu) = \mathbf{\Psi}_{\text{inc}}(\mu), \qquad \mu \geq 0, \quad (104)$$

the expansion coefficients are found as integrals of the adjoint functions times delta functions. As in the Milne problem, the determination of the solution is quite trivial once the set of x-functions has been constructed.

The construction of the half-space Green's function requires a special technique, this is discussed in I. The procedure here for the case of general N follows in exactly the same manner.

Note added in proof: In I we "proved" $J_2 = 1$. Actually, $J_2 = X(0) \neq 1$. However, since $X(0) =$ const, symmetric in i, j (cf. Identity II, Appendix B), the proof is still valid.

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APPENDIX A. THE NUMBER OF DISCRETE EIGENVALUES

The discrete eigenvalues, $\eta_{i\pm}$, are defined as the zeros of the dispersion function $\Omega(z)$, Eq. (29). Here we verify that there are only two zeros for any N.

Since $\Omega(z)$ is analytic in the cut plane and vanishes at infinity, the number of zeros is $(2\pi)^{-1}$ times the change in the argument of $\Omega(z)$ as a contour encircling the cut is traversed.[7] Because $\Omega^+(\mu) = [\Omega^-(\mu)]^*$, and $\Omega(z) = \Omega(-z)$, the change in argument is four times the change in going from $0 + i\epsilon$ to $1 + i\epsilon$. Call this change $\Delta_+(0, 1)$. We have

$$\Delta_+(0, 1) = \sum_{i=0}^{N-1} \Delta_+\left(\frac{1}{\sigma_i}, \frac{1}{\sigma_{i+1}}\right), \tag{A1}$$

where we define $1/\sigma_0 = 0$.

From Eq. (47), we can write for the boundary value in region i

$$\Omega_i^{\pm}(\mu) = 1 - 2\mu \sum_{\beta=1}^{i-1} C_{\beta\beta} T\left(\frac{1}{\sigma_\beta \mu}\right) - 2\mu \sum_{\beta=\lambda}^{N} C_{\beta\beta}\left\{T(\sigma_\beta \mu) \pm \frac{\pi i}{2}\right\}. \tag{A2}$$

From (A2) it is easily verified that

$$\Delta_+(0, 1/\sigma_1) = \pi, \tag{A3}$$

$$\Delta_+\left(\frac{1}{\sigma_i}, \frac{1}{\sigma_{i+1}}\right) = 0, \qquad i = 1, 2, \cdots N - 1. \tag{A4}$$

Thus

$$\Delta_+(0, 1) = \pi, \tag{A5}$$

and the total change (for the encircled cut) is 4π. Thus $\Omega(z)$ has two zeros.

[7] R. V. Churchill, *Complex Variables and Applications* (McGraw-Hill Book Company, Inc., New York, 1960), Chap. 12.

APPENDIX B. X-FUNCTION IDENTITIES

The derivations of the X-function identities are trivial generalizations of the corresponding derivations in I (Appendix A), so we present them without proof.

Identity I:

$$X(z) = \sum_{i=1}^{N} C_{ii} \int_0^{1/\sigma_i} \frac{\gamma(\mu)\, d\mu}{\mu - z}. \tag{B1}$$

Identity II:

$$X(z)X(-z) = \left[\frac{3}{2}\left(\sum_{i=1}^{N} \frac{C_{ii}}{\sigma_1^3}\right)^{-1}\right]\Omega(z). \tag{B2}$$

By combining Identities I and II we get a nonlinear nonsingular integral equation for the numerical evaluation of $X(z)$. Thus we find

Identity III:

$$X(z) = \left[\frac{3}{2}\left(\sum_{i=1}^{N} \frac{C_{ii}}{\sigma_i^3}\right)^{-1}\right] \times \sum_{\alpha=1}^{N} C_{\alpha\alpha} \int_0^{1/\sigma_\alpha} \frac{\mu}{X(-\mu)} \frac{d\mu}{\mu - z}. \tag{B3}$$

Furthermore, Identity IV is the trivial result obtained by taking boundary values of Eq. (B2).

$$\gamma(\mu) = \left[\frac{3}{2}\left(\sum_{i=1}^{N} \frac{C_{ii}}{\sigma_i^3}\right)^{-1}\right] \frac{\mu}{X(-\mu)}. \tag{B4}$$