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RADIATIVE TRANSFER IN LINEARLY ANISOTROPIC-SCATTERING, CONSERVATIVE AND NON-CONSERVATIVE SLABS WITH REFLECTIVE BOUNDARIES

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Abstract—The normal-mode-expansion technique is used to obtain a semi-analytical solution for the angular distribution of radiation at any optical distance within a linearly anisotropic-scattering, absorbing, emitting, non-isothermal, gray medium between two parallel reflecting boundaries. Both conservative and non-conservative cases are considered. The general problem is decomposed into simpler problems, and the net radiative heat flux is calculated to "bench mark" accuracy for these basic problems for several representative combinations of surface reflectivities and emissivities. By the superposition of these basic solutions, the net radiative heat flux can be determined for an absorbing, emitting, scattering slab with reflecting boundaries for the cases of uniform temperature and linearly varying fourth power of the temperature within the medium. Simple analytical expressions are presented for the intensity of radiation by utilizing first- and second-order approximations to the exact solution, and the accuracy of these approximations is evaluated for a variety of cases.

1. INTRODUCTION

THE PURPOSE of this paper is to develop a semi-analytical solution to a general radiative heat transfer problem in a linearly anisotropic-scattering, absorbing, emitting, non-isothermal gray medium confined between two parallel reflecting boundaries. The analysis is based on the singular-eigenfunction-expansion technique developed by Case [1] for treating one-dimensional neutron transport problems. This method has been applied only recently in the field of radiative heat transfer. One of the greater advantages of Case's method derives from the fact that analytical approximations obtained from the rigorous solution are simple yet accurate. One of the earlier applications of the normal-mode-expansion technique was made by McCormick and Mendelson [2] who discussed the slab-albedo problem; Siewert and McCormick [3] solved rigorously the problem of an absorbing, emitting, anisotropically scattering, semi-infinite medium with a linear source term and a free boundary. Ferziger and Simmons [4] solved the radiative transfer problem for a conservative medium between heated black boundaries, and later Simmons and Ferziger [5] extended this work to the two-group non-gray model. Özişik and Siewert [6] have solved a radiative transfer problem for an isotropically scattering, absorbing, and emitting slab with specularly reflecting boundaries, and recently Siewert and Özişik [7] reported a rigorous solution to a line formation problem based on the interlocked-doublet model. Heaslet and Warming [8, 9] considered radiative transfer in finite and semi-infinite non-conservative media. Bowden *et al.* [10] examined numerical methods of solving the one-dimensional transport equation with anisotropic scattering in slab geometry. Here we do not cite the many references on radiative heat transfer treated by classical techniques, but we note that the papers by Özişik and Siewert [6] and Heaslet and Warming [11] contain a more complete literature survey of work utilizing classical or numerical methods.

The outline of the paper is as follows: in Section 2 the problem of radiative heat transfer is formulated and the analysis basic to the normal-mode-expansion technique is given for the non-conservative case (i.e. $0 < c < 1$). In addition, various analytical approximations obtained from the rigorous solution are given, and the particular solutions of the transport equation for source terms represented as polynomials are presented. In Section 3 the conservative case ($c = 1$) is discussed in a similar manner. In Section 4 the physical quantities of interest, such as the incident radiation and the net radiative heat flux, are expressed explicitly in terms of the expansion coefficients, and in Section 5 the method of superposition of simpler problems to obtain solutions to more general problems is discussed. Section 6 is devoted to a presentation of highly accurate numerical results for the net radiative heat flux pertinent to the basic problems for a variety of boundary surface reflectivities. Solutions to the more general cases are thus readily available by superposition of these basic solutions. Also, the accuracy of analytical approximations for several representative cases is evaluated, and a discussion of the various methods used to establish confidence in our "exact" calculations is given.

2. THE NON-CONSERVATIVE CASE

General analysis

We consider the equation of transfer

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = (1 - c) I_b[T(\tau)] + \tfrac{1}{2} c \int_{-1}^{1} (1 + \beta \mu \mu') I(\tau, \mu') \, d\mu', \qquad (1)$$

where τ is the optical variable, and μ is the direction cosine (as measured from the positive τ axis) of the propagating radiation. The constant c is the ratio of the scattering coefficient to the extinction coefficient, $I_b[T(\tau)]$ is the prescribed frequency-integrated Planck function, and the constant β is the anisotropy factor.

The boundary surfaces 1 and 2 are positioned at $\tau = 0$ and $\tau = \tau_0$ respectively and are kept at uniform temperatures T_1 and T_2. The surfaces are diffuse emitters with emissivities ε_1 and ε_2, and the reflectivities are expressed as the sum of diffuse and specular reflectivity components: $\rho_i = \rho_i^d + \rho_i^s$, $i = 1$ or 2. The boundary conditions subject to which equation (1) is to be solved are written as

$$I(0, \mu) = \varepsilon_1 \frac{\sigma}{\pi} T_1^4 + \rho_1^s I(0, -\mu) + 2\rho_1^d \int_0^1 I(0, -\mu') \mu' \, d\mu', \quad \mu \in (0, 1), \qquad (2a)$$

and

$$I(\tau_0, -\mu) = \varepsilon_2 \frac{\sigma}{\pi} T_2^4 + \rho_2^s I(\tau_0, \mu) + 2\rho_2^d \int_0^1 I(\tau_0, \mu') \mu' \, d\mu', \quad \mu \in (0, 1), \qquad (2b)$$

where σ is the Stefan–Boltzmann constant. Clearly many special cases existing in the literature can be obtained from the considered problem.

For convenience in the analysis, we define

$$a_i \triangleq \varepsilon_i (\sigma/\pi) T_i^4, \quad b_i \triangleq \rho_i^s, \quad d_i \triangleq 2\rho_i^d, \quad i = 1 \text{ and } 2,$$

$$S(\tau) \triangleq (1 - c) I_b[T(\tau)], \qquad (3)$$

and thus consider the equations

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = S(\tau) + \tfrac{1}{2} c \int_{-1}^{1} (1 + \beta \mu \mu') I(\tau, \mu') \, d\mu', \qquad (4)$$

$$I(0, \mu) = a_1 + b_1 I(0, -\mu) + d_1 \int_0^1 I(0, -\mu') \mu' \, d\mu', \quad \mu \in (0, 1), \qquad (5a)$$

and

$$I(\tau_0, -\mu) = a_2 + b_2 I(\tau_0, \mu) + d_2 \int_0^1 I(\tau_0, \mu')\, \mu'\, d\mu', \quad \mu \in (0, 1). \tag{5b}$$

In equation (4), $S(\tau)$ represents a prescribed inhomogeneous source term, which later will be represented by a polynomial expansion in the optical variable. The desired solution can be written as a linear combination of normal modes, satisfying the homogeneous version of equation (4), and a particular solution:

$$I(\tau, \mu) = A(\eta_0)\, \Phi(\eta_0, \mu)\, e^{-\tau/\eta_0} + A(-\eta_0)\, \Phi(-\eta_0, \mu)\, e^{\tau/\eta_0} + \int_0^1 A(\eta)\, \Phi(\eta, \mu)\, e^{-\tau/\eta}\, d\eta + \int_0^1 A(-\eta)\, \Phi(-\eta, \mu)\, e^{\tau/\eta}\, d\eta + I_p(\tau, \mu), \tag{6}$$

where the normal modes, due to Mika [12] and collected by Case and Zweifel [13], can be written as

$$\Phi(\pm \eta_0, \mu) = \tfrac{1}{2} c \eta_0 \frac{1}{\eta_0 \mp \mu} R(\pm \eta_0\, \mu) \tag{7a}$$

and

$$\Phi(\eta, \mu) = \tfrac{1}{2} c \eta\, R(\eta\, \mu) \frac{P}{\eta - \mu} + \lambda(\eta)\delta(\eta - \mu), \tag{7b}$$

with

$$R(\pm \xi\, \mu) = 1 \pm \beta(1 - c)\xi\mu, \quad \xi = \eta_0 \text{ or } \eta \in (0, 1), \tag{7c}$$

and

$$\lambda(\eta) = R(\eta\eta)(1 - c\eta \tanh^{-1} \eta) - \beta(1 - c)^2 \eta^2; \tag{7d}$$

we consider those cases [12] for which there are only two discrete zeros, $\pm \eta_0$, of the dispersion function

$$\Lambda(z) = R(zz)\,[1 - cz \tanh^{-1}(1/z)] - \beta(1 - c)^2 z^2. \tag{7e}$$

Here P is a mnemonic symbol used to denote that all ensuing integrals are to be evaluated in the Cauchy principal-value sense, $\delta(x)$ denotes the Dirac delta function, and $I_p(\tau, \mu)$ is the particular solution to be determined for source terms of interest. To complete the analysis, the arbitrary expansion coefficients $A(\eta_0)$, $A(-\eta_0)$, $A(\eta)$ and $A(-\eta)$, $\eta \in (0, 1)$ must be determined by constraining equation (6) to meet the considered boundary conditions. We thus introduce equation (6) into equations (5) and rearrange the resulting expressions to find

$$f_1(\mu) + [b_1 A(\eta_0) - A(-\eta_0)]\, \Phi(-\eta_0, \mu) + \int_0^1 [b_1 A(\eta) - A(-\eta)]\, \Phi(-\eta, \mu) d\eta = [A(\eta_0) - b_1 A(-\eta_0)]\, \Phi(\eta_0, \mu) + \int_0^1 [A(\eta) - b_1 A(-\eta)]\, \Phi(\eta, \mu) d\eta, \quad \mu \in (0, 1), \tag{8}$$

and

$$f_2(\mu) + [b_2 A(-\eta_0) e^{\tau_0/\eta_0} - A(\eta_0) e^{-\tau_0/\eta_0}]\, \Phi(-\eta_0, \mu) + \int_0^1 [b_2 A(-\eta) e^{\tau_0/\eta} - A(\eta) e^{-\tau_0/\eta}]\, \Phi(-\eta, \mu)\, d\eta = [A(-\eta_0) e^{\tau_0/\eta_0} - b_2 A(\eta_0) e^{-\tau_0/\eta_0}]\, \Phi(\eta_0, \mu) + \int_0^1 [A(-\eta) e^{\tau_0/\eta} - b_2 A(\eta) e^{-\tau_0/\eta}]\, \Phi(\eta, \mu)\, d\eta, \quad \mu \in (0, 1), \tag{9}$$

where we have introduced the definitions

$$f_1(\mu) \triangleq a_1 + b_1 I_p(0, -\mu) - I_p(0, \mu) + d_1[A(\eta_0) J(-\eta_0) + A(-\eta_0) J(\eta_0) + \int_0^1 A(-\eta) J(\eta) \, d\eta + \int_0^1 A(\eta) J(-\eta) \, d\eta + J_p(0)], \tag{10a}$$

$$J(\pm\xi) \triangleq \int_0^1 \Phi(\pm\xi, \mu')\mu' \, d\mu', \quad \xi = \eta_0 \quad \text{or} \quad \eta \in (0, 1), \tag{10b}$$

$$J_p(0) \triangleq \int_0^1 I_p(0, -\mu')\mu' \, d\mu', \tag{10c}$$

$$f_2(\mu) \triangleq a_2 + b_2 I_p(\tau_0, \mu) - I_p(\tau_0, -\mu) + d_2[A(\eta_0) e^{-\tau_0/\eta_0} J(\eta_0) + A(-\eta_0) e^{\tau_0/\eta_0} J(-\eta_0) + \int_0^1 A(-\eta) e^{\tau_0/\eta} J(-\eta) \, d\eta + \int_0^1 A(\eta) e^{-\tau_0/\eta} J(\eta) \, d\eta + J_p(\tau_0)], \tag{11a}$$

and

$$J_p(\tau_0) \triangleq \int_0^1 I_p(\tau_0, \mu')\mu' \, d\mu'. \tag{11b}$$

Equations (8) and (9) are the defining constraints from which the unknown expansion coefficients are to be determined. To solve these equations we make use of the half-range completeness theorem proved initially by Mika [12] and the half-range bi-orthogonality relations developed by McCormick and Kuščer [14] and given by Case and Zweifel [13].

In order to isolate the discrete coefficients $[A(\eta_0) - b_1 A(-\eta_0)]$ on the right-hand side of equation (8), we multiply that equation by $\tilde{\Phi}(\eta_0, \mu) W(\mu)$ and integrate over μ from zero to unity. We then utilize the half-range bi-orthogonality relations and various normalization integrals to obtain

$$-(\tfrac{1}{2}c\eta_0)^2 X(\eta_0) R(\eta_0\eta_0) [A(\eta_0) - b_1 A(-\eta_0)] = F_1(\eta_0) + (\tfrac{1}{2}c\eta_0)^2 X(-\eta_0) \frac{R(\eta_0\eta_0) R(-\eta_0\bar{\eta})}{R(\eta_0\bar{\eta})} \times [b_1 A(\eta_0) - A(-\eta_0)] + (1/4)c^2\eta_0 \frac{R(\eta_0\eta_0)}{R(\eta_0\bar{\eta})} \times \int_0^1 \eta X(-\eta) R(-\eta\bar{\eta}) [b_1 A(\eta) - A(-\eta)] \, d\eta. \tag{12}$$

Similarly, to isolate the factor $[A(-\eta_0)e^{\tau_0/\eta_0} - b_2 A(\eta_0) e^{-\tau_0/\eta_0}]$ on the right-hand side of equation (9) we apply the same operation to find

$$(\tfrac{1}{2}c\eta_0)^2 X(\eta_0) R(\eta_0\eta_0) [b_2 A(\eta_0) e^{-\tau_0/\eta_0} - A(-\eta_0) e^{\tau_0/\eta_0}] = F_2(\eta_0) - (\tfrac{1}{2}c\eta_0)^2 X(-\eta_0) \times \frac{R(\eta_0\eta_0) R(-\eta_0\bar{\eta})}{R(\eta_0\bar{\eta})} [A(\eta_0) e^{-\tau_0/\eta_0} - b_2 A(-\eta_0) e^{\tau_0/\eta_0}] - \tfrac{1}{4}c^2\eta_0 \frac{R(\eta_0\eta_0)}{R(\eta_0\bar{\eta})} \times \int_0^1 \eta X(-\eta) R(-\eta\bar{\eta}) \times [A(\eta) e^{-\tau_0/\eta} - b_2 A(-\eta) e^{\tau_0/\eta}] \, d\eta. \tag{13}$$

To isolate the continuum coefficient $[A(\eta) - b_1 A(-\eta)]$ on the right-hand side of equation (8), we multiply that equation by $\tilde{\Phi}(\eta', \mu) W(\mu)$, $\eta' \in (0, 1)$, integrate over μ from zero to one and utilize the half-range bi-orthogonality relations and the normalization integrals to obtain (after interchanging η and η')

$$\frac{W(\eta)}{g(c, \eta)} [A(\eta) - b_1 A(-\eta)] = F_1(\eta) + c\eta_0 \eta X(-\eta_0) \Phi(-\eta_0, \eta) \times \frac{R(\eta_0\eta_0) R(-\eta\bar{\eta})}{R(\eta_0\bar{\eta}) R(-\eta\eta_0)} \times [b_1 A(\eta_0) - A(-\eta_0)] + \frac{c\eta}{2} \times \int_0^1 (\eta_0 + \eta') X(-\eta') \tilde{\Phi}(\eta', -\eta) \times [b_1 A(\eta') - A(-\eta')] \, d\eta', \quad \eta \in (0, 1). \tag{14}$$

The same operation when applied to equation (9) yields

$$\frac{W(\eta)}{g(c,\eta)}[A(-\eta)\,e^{\tau_0/\eta} - b_2A(\eta)\,e^{-\tau_0/\eta}]$$
$$= F_2(\eta) + c\eta_0\eta X(-\eta_0)\,\Phi(-\eta_0,\eta)$$
$$\times \frac{R(\eta_0\eta_0)\,R(-\eta\bar{\eta})}{R(\eta_0\bar{\eta})\,R(-\eta\eta_0)}$$
$$\times [b_2A(-\eta_0)\,e^{\tau_0/\eta_0} - A(\eta_0)\,e^{-\tau_0/\eta_0}]$$
$$+ \tfrac{1}{2}c\eta \int_0^1 (\eta_0+\eta')\,X(-\eta')\,\tilde{\Phi}(\eta',-\eta)$$
$$\times [b_2A(-\eta')e^{\tau_0/\eta'} - A(\eta')e^{-\tau_0/\eta'}]\,d\eta',$$
$$\eta \in (0,1). \qquad (15)$$

Here we have used the expressions [13]

$$F_\alpha(\xi) = \int_0^1 f_\alpha(\mu)\,\tilde{\Phi}(\xi,\mu)\,W(\mu)\,d\mu,$$
$$\alpha = 1 \text{ and } 2, \qquad (16a)$$

$$g(c,\eta) = [\lambda^2(\eta) + \tfrac{1}{4}c^2\eta^2\pi^2 R^2(\eta\eta)]^{-1}, \qquad (16b)$$

$$\tilde{\Phi}(\xi,\mu) = \Phi(\xi,\mu) + \tfrac{1}{2}c\beta\xi\frac{(1-c)(\eta_0-\bar{\eta})}{R(\eta_0\bar{\eta})}, \qquad (16c)$$

$$\bar{\eta} = \frac{\int_0^1 \mu'\gamma(\mu')\,d\mu'}{\int_0^1 \gamma(\mu')\,d\mu'} = \frac{\gamma^{(1)}}{\gamma^{(0)}}, \qquad (16d)$$

$$\gamma(\mu) = c\mu[2(1-c)(1-\tfrac{1}{3}c\beta)(\eta_0^2-\mu^2)X(-\mu)]^{-1}, \qquad (16e)$$

and

$$W(\mu) = (\eta_0-\mu)\,\gamma(\mu). \qquad (16f)$$

Also, $X(z)$ is Case's X-function for linearly anisotropic scattering. Equations (12)–(15) are the four basic equations to be considered; they are written more compactly in matrix notation:

$$\boldsymbol{M}\boldsymbol{A}(\eta_0) = \boldsymbol{G} + \int_0^1 \boldsymbol{B}(\eta')\,\boldsymbol{A}(\eta')\,d\eta', \qquad (17a)$$

and

$$\boldsymbol{M}(\eta)\,\boldsymbol{A}(\eta) = \boldsymbol{G}(\eta) + \boldsymbol{D}(\eta)\,\boldsymbol{A}(\eta_0)$$
$$+ \int_0^1 \boldsymbol{B}(\eta'\to\eta)\,\boldsymbol{A}(\eta')\,d\eta', \quad \eta\in(0,1), \qquad (17b)$$

where various quantities are defined as

$$\boldsymbol{M} = e^{-2\tau_0/\eta_0}\,\boldsymbol{U}(\eta_0) - \boldsymbol{M}(\eta_0) - \tilde{J}(\eta_0)\left(\frac{2}{c\eta_0}\right)^2$$
$$\times \frac{1}{X(\eta_0)\,R(\eta_0\,\eta_0)}\boldsymbol{V}(\eta_0), \qquad (18a)$$

$$\boldsymbol{M}(\xi) = \begin{vmatrix} 1 & -b_1 \\ -b_2\,e^{-\tau_0/\xi} & e^{\tau_0/\xi} \end{vmatrix},$$

$$\boldsymbol{U}(\xi) = \begin{vmatrix} b_1 & -1 \\ -e^{-\tau_0/\xi} & b_2\,e^{\tau_0/\xi} \end{vmatrix}, \qquad (18b)$$

$$\boldsymbol{V}(\xi) = \begin{vmatrix} d_1J(-\xi) & d_1J(\xi) \\ d_2J(\xi)e^{-\tau_0/\xi} & d_2J(-\xi)e^{-\tau_0/\xi} \end{vmatrix}, \qquad (18c)$$

$$\boldsymbol{D}(\eta) = K_1(\eta)\,\boldsymbol{U}(\eta) + \frac{\tilde{J}(\eta)\,g(c,\eta)}{W(\eta)}\boldsymbol{V}(\eta_0), \qquad (18d)$$

$$\boldsymbol{B}(\eta') = K_0(\eta')\,\boldsymbol{U}(\eta') + \tilde{J}(\eta_0)\left(\frac{2}{c\eta_0}\right)^2$$
$$\times \frac{1}{X(\eta_0)\,R(\eta_0\eta_0)}\boldsymbol{V}(\eta'),$$

and

$$\boldsymbol{B}(\eta'\to\eta) = K(\eta'\to\eta)\,\boldsymbol{U}(\eta') + \frac{\tilde{J}(\eta)\,g(c,\eta)}{W(\eta)}$$
$$\times \boldsymbol{V}(\eta'), \qquad (18e)$$

$$\boldsymbol{G} \triangleq \left(\frac{2}{c\eta_0}\right)^2 \frac{1}{X(\eta_0)R(\eta_0\eta_0)} \begin{vmatrix} \int_0^1 [a_1 + b_1I_p(0,-\mu) - I_p(0,\mu) + d_1J_p(0)]\,\tilde{\Phi}(\eta_0,\mu)\,W(\mu)\,d\mu \\ \int_0^1 [a_2 + b_2I_p(\tau_0,\mu) - I_p(\tau_0,-\mu) + d_2J_p(\tau_0)]\,\tilde{\Phi}(\eta_0,\mu)\,W(\mu)\,d\mu \end{vmatrix}, \qquad (18f)$$

$$G(\eta) \triangleq \frac{g(c,\eta)}{W(\eta)} \begin{vmatrix} \int_0^1 [a_1 + b_1 I_p(0,-\mu) - I_p(0,\mu) + d_1 J_p(0)]\,\tilde{\Phi}(\eta,\mu)\,W(\mu)\,d\mu \\ \\ \int_0^1 [a_2 + b_2 I_p(\tau_0,\mu) - I_p(\tau_0,-\mu) + d_2 J_p(\tau_0)]\,\tilde{\Phi}(\eta,\mu)\,W(\mu)\,d\mu \end{vmatrix}, \quad (18g)$$

$$K_0(\eta) \triangleq \frac{\eta X(-\eta)R(-\eta\bar{\eta})}{\eta_0 X(\eta_0)R(\eta_0\bar{\eta})}, \quad K_1(\eta) \triangleq \frac{c\eta_0\eta X(-\eta_0)\tilde{\Phi}(-\eta_0,\eta)g(c,\eta)R(\eta_0\eta_0)R(-\eta\bar{\eta})}{W(\eta)R(\eta_0\bar{\eta})R(-\eta\eta_0)}, \quad (18h)$$

$$K(\eta' \to \eta) \triangleq \tfrac{1}{2}c\eta\,\frac{(\eta_0 + \eta')X(-\eta')\tilde{\Phi}(\eta',-\eta)g(c,\eta)}{W(\eta)}, \quad (18i)$$

and

$$\tilde{J}(\xi) \triangleq \int_0^1 \tilde{\Phi}(\xi,\mu')W(\mu')\,d\mu'. \quad (18j)$$

Further z_0 is the Milne-problem extrapolated end-point [13], and the unknowns have been written as

$$\boldsymbol{A}(\xi) \triangleq \begin{vmatrix} A(\xi) \\ \\ A(-\xi) \end{vmatrix}, \quad \xi = \eta_0 \text{ or } \eta \in (0,1). \quad (18k)$$

Up to this point our analysis has been mathematically rigorous; however, it is highly unlikely that analytical solutions to equations (17) exist. Although these equations are formidable analytically, they certainly pose no problem for existing computing facilities. Thus if highly accurate "bench mark" solutions are sought, an iterative numerical procedure can be used to solve these equations to any desired degree of accuracy. The degree of precision with which we can complete the desired solution will be measured by how accurately we determine $\boldsymbol{A}(\eta_0)$ and $\boldsymbol{A}(\eta)$ from equations (17).

Analytical approximations can also be obtained from equations (17) to yield solutions of sufficient accuracy. Ferziger and Simmons [4] obtained two different approximate solutions to a related problem; they showed that the lowest-order solution was better than classical diffusion theory, whereas the second-order solution was highly accurate.

Approximations

In the present analysis, the lowest-order (or first-order) solution is obtained by neglecting the continuum coefficients entirely; the discrete solutions are thus readily available from equation (17);

$$\boldsymbol{A}^{(1)}(\eta) = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{A}^{(1)}(\eta_0) = \boldsymbol{M}^{-1}\boldsymbol{G}, \quad (19)$$

where superscripts are used to denote the order of approximation.

The second-order continuum coefficients $\boldsymbol{A}^{(2)}(\eta)$ are found by neglecting the contribution from the kernel $\boldsymbol{B}(\eta' \to \eta)$ in equation (17b) and by using $\boldsymbol{A}^{(1)}(\eta_0)$ in that equation. Finally $\boldsymbol{A}^{(2)}(\eta)$ is substituted into equation (17a) to yield $\boldsymbol{A}^{(2)}(\eta_0)$. Thus

$$\boldsymbol{A}^{(2)}(\eta) = \boldsymbol{M}^{-1}(\eta)\,[\boldsymbol{G}(\eta) + \boldsymbol{D}(\eta)\boldsymbol{A}^{(1)}(\eta_0)] \quad (20)$$

and

$$\boldsymbol{A}^{(2)}(\eta_0) = \boldsymbol{M}^{-1}\,[\boldsymbol{G} + \int_0^1 \boldsymbol{B}(\eta')\boldsymbol{A}^{(2)}(\eta')\,d\eta']. \quad (21)$$

Particular solutions

In order to evaluate the vectors $\boldsymbol{G}$ and $\boldsymbol{G}(\eta)$, the particular solution $I_p(\tau, \mu)$ for the source term of interest is needed. Several particular solutions of the equation of radiative transfer have been reported by Özişik and Siewert [15]. Here we list the particular solutions for source terms represented by polynomials in the optical variable τ. Considering a source term of the type

$$S^{(i)}(\tau) = \tau^i, \qquad i = 0, 1, 2, \ldots, \tag{22}$$

we find the first three solutions to be

$$I_p^{(0)}(\tau, \mu) = (1 - c)^{-1}, \tag{23a}$$

$$I_p^{(1)}(\tau, \mu) = (1 - c)^{-1}\left(1 - \frac{c\beta}{3}\right)^{-1} \times \left[\tau\left(1 - \frac{c\beta}{3}\right) - \mu\right], \tag{23b}$$

and

$$I_p^{(2)}(\tau, \mu) = (1 - c)^{-1}\left(1 - \frac{c\beta}{3}\right)^{-1} \times\left[\frac{2c}{3(1 - c)} - 2\tau\mu + \tau^2\left(1 - \frac{c\beta}{3}\right) + 2\mu^2\right]. \tag{23c}$$

To determine solutions for higher-order polynomials is a straightforward but tedious task. Of course, solutions corresponding to sources of the form $S(\tau) = \sum \alpha_i S^{(i)}(\tau)$ are obtained by superposition. Once the particular solution is known, the evaluation of the G-vectors follows in a simple manner [6].

Relations between X- *and* H-*functions*

In the foregoing analysis our results are given in terms of Case's X-function. However, it may be desirable to express these results in terms of Chandrasekhar's [16] well known H-function. For linearly anisotropic scattering, Case's X-function is related to Chandrasekhar's H-function by [14]

$$\frac{1}{X(-z)} = \left[(1 - c)\left(1 - \frac{c\beta}{3}\right)\right]^{\frac{1}{2}}(\eta_0 + z)H(z), \tag{24}$$

where the H-function is a solution of

$$\frac{1}{H(z)} = \left[1 - 2\int_0^1 \psi(\mu)\,\mathrm{d}\mu\right]^{\frac{1}{2}} + \int_0^1 \mu\psi(\mu)\,H(\mu)\frac{\mathrm{d}\mu}{\mu + z}, \tag{25a}$$

with

$$\psi(\mu) = (c/2)\,[1 + \beta(1 - c)\,\mu^2]. \tag{25b}$$

To solve equation (25a) iteratively for $H(\mu)$ certainly poses no problem for modern computing facilities; however, reasonably accurate predictions of $H(\mu)$ follow from Shure and Natelson's [17] concise approximation to the X-function.

3. THE CONSERVATIVE CASE

General analysis

Though the foregoing analysis can be interpreted in the limit $c \to 1$ and thus $\eta_0 \to \infty$, we prefer to develop the special forms required since in some cases the limiting values are tedious to obtain. This conservative case clearly corresponds to a purely scattering gray medium or to a gray medium in radiative equilibrium. We consider the equation of transfer

$$\mu\frac{\partial}{\partial\tau} I(\tau,\mu) + I(\tau,\mu) = \tfrac{1}{2}\int_{-1}^{1}(1 + \beta\mu\mu') \times I(\tau,\mu')\mathrm{d}\mu' \tag{26}$$

subject to the boundary conditions given by equations (5).

Since the normal modes of equation (26) are established [13], the desired solution can be written as

$$I(\tau,\mu) = A_+ + A_-\left(\tau - \frac{3\mu}{3 - \beta}\right) + \int_0^1 A(\eta)\phi(\eta,\mu)\mathrm{e}^{-\tau/\eta}\,\mathrm{d}\eta + \int_0^1 A(-\eta)\phi(-\eta,\mu) \times \mathrm{e}^{\tau/\eta}\,\mathrm{d}\eta, \tag{27}$$

where A_+, A_-, $A(\eta)$ and $A(-\eta)$ are the expansion coefficients to be determined from the boundary conditions. The continuum normal mode $\phi(\eta,\mu)$ takes the form

$$\phi(\eta,\mu) = \frac{\eta}{2}\frac{P}{\eta-\mu} + (1 - \eta\tanh^{-1}\eta)\,\delta(\eta-\mu). \tag{28}$$

We follow a procedure similar to that used in Section 2 to determine the unknown expansion coefficients. The equations obtained from the substitution of the solution given by equation (27) into the boundary conditions given by equations (5) are multiplied successively by $\gamma(\mu)$ and $\phi(\eta',\mu)\gamma(\mu)$, $\eta'\in(0,1)$, and integrated over μ from zero to unity. Various normalization integrals and orthogonality relations are then invoked to yield the following equations for the determination of the expansion coefficients A_+, A_-, $A(\eta)$ and $A(-\eta)$.

$$\boldsymbol{N}\boldsymbol{A} = \boldsymbol{S} + \int_0^1 \boldsymbol{P}(\eta')\boldsymbol{A}(\eta')\mathrm{d}\eta', \tag{29a}$$

$$\boldsymbol{N}(\eta)\,\boldsymbol{A}(\eta) = \boldsymbol{F}(\eta)\boldsymbol{A} + \int_0^1 \boldsymbol{P}(\eta'\to\eta)\boldsymbol{A}(\eta')\mathrm{d}\eta', \tag{29b}$$

where the various matrices are defined as

$$\boldsymbol{N} \triangleq \begin{vmatrix} 1 - b_1 - d_1/2 & \frac{3}{3-\beta}[-\gamma^{(1)}(1+b_1) - d_1/3] \\ 1 - b_2 - d_2/2 & \frac{3}{3-\beta}\left[\gamma^{(1)}(1+b_2) + \frac{d_2}{3}\right] + \tau_0\left[1 - b_2 - \frac{d_2}{2}\right] \end{vmatrix}, \tag{30a}$$

$$\boldsymbol{N}(\eta) \triangleq \begin{vmatrix} 1 & -b_1 \\ -b_2\mathrm{e}^{-\tau_0/\eta} & \mathrm{e}^{\tau_0/\eta} \end{vmatrix}, \quad \boldsymbol{S} \triangleq \begin{vmatrix} a_1 \\ a_2 \end{vmatrix}, \tag{30b}$$

$$\boldsymbol{F}(\eta) \triangleq \begin{vmatrix} 0 & -\frac{3}{3-\beta}(1+b_1)\frac{\eta g(1,\eta)}{2\gamma(\eta)} \\ 0 & \frac{3}{3-\beta}(1+b_2)\frac{\eta g(1,\eta)}{2\gamma(\eta)} \end{vmatrix}, \tag{30c}$$

$$\boldsymbol{P}(\eta') \triangleq \begin{vmatrix} d_1J(-\eta') + b_1\frac{\eta'X(-\eta')}{2} & d_1J(\eta') - \frac{\eta'X(-\eta')}{2} \\ \left[d_2J(\eta') - \frac{\eta'X(-\eta')}{2}\right]\mathrm{e}^{-\tau_0/\eta'} & \left[d_2J(-\eta') + b_2\frac{\eta'X(-\eta')}{2}\right]\mathrm{e}^{\tau_0/\eta'} \end{vmatrix}, \tag{30d}$$

$$\boldsymbol{P}(\eta'\to\eta) \triangleq \begin{vmatrix} b_1K(\eta'\to\eta) & -K(\eta'\to\eta) \\ -K(\eta'\to\eta)\,\mathrm{e}^{-\tau_0/\eta'} & b_2K(\eta'\to\eta)\,\mathrm{e}^{\tau_0/\eta'} \end{vmatrix}, \tag{30e}$$

$$\boldsymbol{A} \triangleq \begin{vmatrix} A_+ \\ \\ A_- \end{vmatrix} \text{ and } \boldsymbol{A}(\eta) \triangleq \begin{vmatrix} A(\eta) \\ \\ A(-\eta) \end{vmatrix}. \quad (30f)$$

Here we have also defined

$$\gamma(\eta) \triangleq \frac{3}{2}\frac{\eta}{X(-\eta)}\frac{1}{1-\beta/3}, \quad (31a)$$

$$\gamma^{(1)} \triangleq \int_0^1 \mu\gamma(\mu)\mathrm{d}\mu = 0{\cdot}71044609\ldots, \quad (31b)$$

$$K(\eta' \to \eta) \triangleq \frac{\eta\eta' g(1,\eta)X(-\eta')}{4(\eta+\eta')\gamma(\eta)}, \quad (31c)$$

$$J(\pm\xi) \triangleq \int_0^1 \phi(\pm\xi,\mu)\mu\mathrm{d}\mu, \quad \xi \in (0,1), \quad (31d)$$

and

$$\frac{1}{g(1,\eta)} = (1 - \eta\tanh^{-1}\eta)^2 + \left(\frac{\pi\eta}{2}\right)^2. \quad (31e)$$

Case's $X(-\eta)$ function can be related to Chandrasekhar's $H(\eta)$-function by utilizing equation (24) and noting that as $c \to 1$, $\eta_0 \to \infty$. We find

$$X(-\eta) = \sqrt{3}\frac{1}{H(\eta)}. \quad (31f)$$

Approximations

The lowest-order and second-order approximations to equations (29) are found in a manner identical to that used in Section 2; thus

$$\boldsymbol{A}^{(1)}(\eta) = \boldsymbol{0} \text{ and } \boldsymbol{A}^{(1)} = \boldsymbol{N}^{-1}\boldsymbol{S}, \quad (32a)$$

$$\boldsymbol{A}^{(2)}(\eta) = \boldsymbol{N}^{-1}(\eta)\boldsymbol{F}(\eta)\boldsymbol{A}^{(1)}, \quad (32b)$$

and

$$\boldsymbol{A}^{(2)} = \boldsymbol{N}^{-1}[\boldsymbol{S} + \int_0^1 \boldsymbol{P}(\eta')\boldsymbol{A}^{(2)}(\eta')\mathrm{d}\eta']. \quad (32c)$$

4. THE INCIDENT RADIATION AND THE NET RADIATIVE HEAT FLUX

It is apparent from the analysis given in the previous sections that the determination of the expansion coefficients is the most basic step for the solution of the considered problem. Once these expansion coefficients have been evaluated from the relations given in Sections 2 and 3, the intensity of radiation $I(\tau,\mu)$ everywhere in the medium is immediately available through equations (6) and (27) for the non-conservative and conservative cases respectively. Other physical quantities of interest, such as the incident radiation $E(\tau)$ and the net radiative heat flux $q(\tau)$ are evaluated from the definitions

$$E(\tau) = 2\pi \int_{-1}^{1} I(\tau,\mu)\mathrm{d}\mu \quad (33)$$

and

$$q(\tau) = 2\pi \int_{-1}^{1} I(\tau,\mu)\mu\mathrm{d}\mu. \quad (34)$$

For the non-conservative case, $0 < c < 1$, we find

$$E(\tau) = 2\pi[A(\eta_0)\mathrm{e}^{-\tau/\eta_0} + A(-\eta_0)\mathrm{e}^{\tau/\eta_0} + \int_0^1 A(\eta)\mathrm{e}^{-\tau/\eta}\mathrm{d}\eta + \int_0^1 A(-\eta)\mathrm{e}^{\tau/\eta}\mathrm{d}\eta + \int_{-1}^{1} I_p(\tau,\mu)\,\mathrm{d}\mu], \quad c \neq 1 \quad (35)$$

and

$$q(\tau) = 2\pi(1-c)\,[A(\eta_0)\eta_0\mathrm{e}^{-\tau/\eta_0} - A(-\eta_0)\eta_0\mathrm{e}^{\tau/\eta_0} + \int_0^1 A(\eta)\eta\mathrm{e}^{-\tau/\eta}\mathrm{d}\eta - \int_0^1 A(-\eta)\eta\mathrm{e}^{\tau/\eta}\mathrm{d}\eta + \frac{1}{1-c}\int_{-1}^{1} I_p(\tau,\mu)\mu\mathrm{d}\mu]. \quad (36)$$

For the conservative case $c = 1$ we find

$$E(\tau) = 2\pi[2A_+ + 2A_-\tau + \int_0^1 A(\eta)\mathrm{e}^{-\tau/\eta}\mathrm{d}\eta + \int_0^1 A(-\eta)\mathrm{e}^{\tau/\eta}\mathrm{d}\eta], \quad c = 1, \quad (37)$$

and

$$q(\tau) = -\frac{4\pi}{3-\beta}A_-, \quad c = 1. \quad (38)$$

5. SUPERPOSITION OF ELEMENTARY SOLUTIONS

The most general problem considered in the basic analysis of Sections 2 and 3 contains many parameters and thus clearly includes many special cases. If some discussion of the numerical calculations is to be reported in a reasonable space, it is desirable to present the results in the form of dimensionless functions independent of several parameters, such that the solution to the general problem can be constructed by the superposition of these elementary functions. Fortunately, the linearity of the governing equations permits the construction of the solution to the general problem by the superposition of elementary solutions independent of the temperature. We present below the principle of superposition for the cases of a constant inhomogeneous source term and a linearly varying inhomogeneous source term.

(i) When $S(\tau)$ is a constant it can be shown that the net radiative heat flux is given by

$$q(\tau) = \sigma[T_0^4 Q_0(\tau) + T_1^4 Q_1(\tau) - T_2^4 Q_1^*(\tau_0 - \tau)]; \tag{39}$$

T_1 and T_2 are the temperatures at the boundary surfaces $\tau = 0$ and $\tau = \tau_0$ respectively, and T_0 is the uniform temperature in the medium. The dimensionless functions $Q_i(\tau)$ have been defined as

$$Q_i(\tau) = 2\pi \int_1^1 \psi_i(\tau,\mu)\mu \mathrm{d}\mu, \quad i = 0 \text{ or } 1, \tag{40}$$

where the functions $\psi_i(\tau,\mu)$ satisfy the following system

$$B\psi_i(\tau,\mu) = \frac{1-c}{\pi}\delta_{0i}, \quad i = 0 \text{ or } 1, \tag{41}$$

$$\psi_i(0,\mu) = \frac{1}{\pi}\varepsilon_1\delta_{1i} + b_1\psi_i(0,-\mu) + d_1\int_0^1 \psi_i(0,-\mu')\mu'\mathrm{d}\mu', \quad \mu \in (0,1), \tag{42a}$$

and

$$\psi_i(\tau_0,-\mu) = b_2\psi_i(\tau_0,\mu) + d_2\int_0^1 \psi_i(\tau_0,\mu')\mu'\mathrm{d}\mu', \quad \mu \in (0,1), \tag{42b}$$

where we have defined the operator B as

$$Bf(\tau,\mu) \triangleq \mu\frac{\partial}{\partial\tau}f(\tau,\mu) + f(\tau,\mu) - \frac{c}{2} \times \int_{-1}^1 f(\tau,\mu')[1 + \beta\mu\mu']\mathrm{d}\mu'. \tag{43}$$

The function $Q_1^*(\tau_0 - \tau)$ is obtained from the solution of equations defining $Q_1(\tau)$ by interchanging the radiative properties at the boundary surfaces 1 and 2.

For conservative media with opaque boundaries equation (39) simplifies to

$$q(\tau) = \sigma[T_1^4 - T_2^4]\,Q_1 \text{ for } c = 1, \tag{44}$$

where the constant Q_1 is a solution of the system of equations defining $Q_1(\tau)$ with, however, $c = 1$.

(ii). When the fourth power of the temperature of the medium varies linearly from the surface temperature T_1^4 at $\tau = 0$ to the surface temperature T_1^4 at $\tau = \tau_0$, it can be shown that the net radiative heat flux is given by

$$q(\tau) = \sigma[T_1^4\bar{Q}_1(\tau) - T_2^4\bar{Q}_1^*(\tau_0 - \tau)]. \tag{45}$$

The dimensionless function $\bar{Q}_1(\tau)$ has been defined as

$$\bar{Q}_1(\tau) = 2\pi \int_{-1}^1 \bar{\psi}(\tau,\mu)\mu\mathrm{d}\mu, \tag{46}$$

where $\bar{\psi}(\tau,\mu)$ satisfies the following problem

$$B\bar{\psi}(\tau,\mu) = \frac{1-c}{\pi}\left(1 - \frac{\tau}{\tau_0}\right), \tag{47}$$

$$\bar{\psi}(0,\mu) = \frac{1}{\pi}\varepsilon_1 + b_1\bar{\psi}(0,-\mu) + d_1\int_0^1 \bar{\psi}(0,-\mu') \times \mu'\mathrm{d}\mu', \quad \mu \in (0,1), \tag{48a}$$

and

$$\bar{\psi}(\tau_0,-\mu) = b_2\bar{\psi}(\tau_0,\mu) + d_2\int_0^1 \bar{\psi}(\tau_0,\mu')\mu'\mathrm{d}\mu', \quad \mu \in (0,1) \tag{48b}$$

where B is defined by equation (43).

The function $\bar{Q}_1^*(\tau_0 - \tau)$ is obtained from the solution of the system defining $\bar{Q}_1(\tau)$ by interchanging the radiative properties at the boundary surfaces 1 and 2.

6. DISCUSSION OF RESULTS

It is to be emphasized that the most general calculation to be made is that of the determination of the expansion coefficients $A(\eta_0)$ and $A(\eta)$ $\eta \in (0,1)$. Clearly, once these coefficients are established, the intensity of radiation, the heat flux and the incident radiation at any point in the medium are immediately available.

The integral equations corresponding to the linear and constant source problems discussed in Section 5 have been solved for the expansion coefficients by an iterative process with the integral terms being evaluated by a 41-point improved Gaussian quadrature scheme [18]. Starting values for this process were obtained from the approximations previously discussed, and the iteration process was terminated when successive values of the coefficients agreed to at least ten significant figures. The calculations were performed on the IBM 360/75 computer in double-precision arithmetic.

Since we consider our calculations to be highly accurate, it is appropriate to report here several checks made on the accuracy of our results. A check could be made by testing how accurately the computed expansion coefficients satisfy the boundary conditions, but since this would involve the evaluation of principal-value integrals which might introduce errors itself, we preferred to consider a check on the moments of the boundary conditions. We multiply equation (5) by μ^{α} and integrate over μ from zero to unity. In comparing the two sides of the resulting expression, for $\alpha = 0, 1, 2. \ldots 9$, we found agreement to the order of 10^{-5}.

Additional confidence in the calculations reported here is established by noting that doubling the order of the quadrature scheme (from 41 to 81) did not change the results to the accuracy presented.

Another check on the accuracy was made by calculating with the present analysis the radiative heat flux for the special case of $c = 0$, for which simple analytical solutions can be obtained, and by comparing the numerical and analytical results. The agreement was excellent.

The computer program prepared for the present analysis is capable of calculating the expansion coefficients, the intensity of radiation, the incident radiation and the net radiative heat flux anywhere in the medium. For most engineering applications the net radiative heat flux at the boundary surfaces is of primary interest. For this reason, and for brevity in the presentation of results, we have concentrated most of our attention on the net radiative heat flux at the boundaries. We present tabulations of the boundary values of the net radiative heat flux for several cases of isotropic scattering and diffusely reflecting boundaries, and report investigations of the effects of linearly anisotropic scattering and specular reflection.

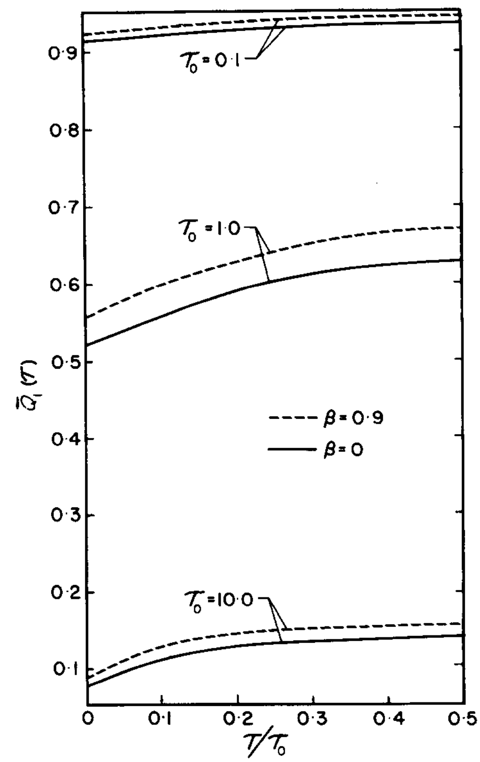

FIG. 1. The effects of linearly anisotropic scattering on the function $\bar{Q}_1(\tau)$ for c = 0·5.

We list in Table 1 the numerical values of the functions $Q_0(0)$, $Q_1(0)$ and $Q_1^*(\tau_0)$ for three optical thicknesses and for several combinations of diffuse surface reflectivities and emissivities for the case of isotropic scattering. The net radiative heat flux at the surface $\tau = 0$ can be

Table 1. The heat flux functions $Q_0(0)$, $Q_1(0)$ and $Q_1{}^*(\tau_0)$ for non-conservative media at constant temperature

Boundary at $\tau = 0$		Boundary at $\tau = \tau_0$		$\tau_0 = 0{\cdot}1$		$\tau_0 = 1{\cdot}0$		$\tau_0 = 10{\cdot}0$	
ε_1	$\rho_1{}^d$	ε_2	$\rho_2{}^d$	$c = 0$	$c = 0{\cdot}5$	$c = 0$	$c = 0{\cdot}5$	$c = 0$	$c = 0{\cdot}5$
				$-Q_0(0)$					
1·0	0·0	0·0	1·0	0·3068	0·1736	0·9519	0·7572	1·0000	0·8535
1·0	0·0	0·5	0·5	0·2371	0·1316	0·8662	0·6510	1·0000	0·8534
1·0	0·0	1·0	0·0	0·1674	0·0911	0·7806	0·5591	1·0000	0·8534
				$Q_1(0)$					
1·0	0·0	0·0	1.0	0·3068	0·1736	0·9519	0·7572	1·0000	0·8535
1·0	0·0	0·5	0·5	0·6534	0·5753	0·9759	0·8154	1·0000	0·8535
1·0	0·0	1·0	0·0	1·0000	0·9616	1·0000	0·8658	1·0000	0·8535
				$Q_1^*(\tau_0)$					
1·0	0·0	0·0	1·0	0·0	0·0	0·0	0·0	0·0	0·0
1·0	0·0	0·5	0·5	0·4163	0·4437	0·1097	0·1644	0·0	0·0
1·0	0·0	1·0	0·0	0·8326	0·8704	0·2194	0·3067	0·0	0·0

Table 2. The heat flux constant Q_1 for conservative media

				Q_1 for $c = 1$		
Boundary at $\tau = 0$		Boundary at $\tau = \tau_0$		$\tau_0 = 0{\cdot}1$	$\tau_0 = 1{\cdot}0$	$\tau_0 = 10{\cdot}0$
ε_1	$\rho_1{}^d$	ε_2	$\rho_2{}^d$			
1·0	0·0	0·0	1·0	1·0	0·0	0·0
1·0	0·0	0·5	0·5	0·4780	0·3562	0·10454
1·0	0·0	1·0	0·0	0·9157	0·5534	0·11675

Table 3. The heat flux functions $\bar{Q}_1(0)$ and $\bar{Q}_1{}^*(\tau_0)$ for non-conservative media with a linear fourth-power of temperature

Boundary at $\tau = 0$		Boundary at $\tau = \tau_0$		$\tau_0 = 0{\cdot}1$		$\tau_0 = 1{\cdot}0$		$\tau_0 = 10{\cdot}0$	
ε_1	$\rho_1{}^d$	ε_2	$\rho_2{}^d$	$c = 0$	$c = 0{\cdot}5$	$c = 0$	$c = 0{\cdot}5$	$c = 0$	$c = 0{\cdot}5$
				$\bar{Q}_1(0)$					
1·0	0·0	0·0	1·0	0·1527	0·0866	0·3860	0·3351	0·0667	0·0758
1·0	0·0	0·5	0·5	0·5323	0·5083	0·4403	0·4337	0·0667	0·0758
1·0	0·0	1·0	0·0	0·9119	0·9138	0·4945	0·5190	0·0667	0·0758
				$\bar{Q}_1^*(\tau_0)$					
1·0	0·0	0·0	1·0	0·0	0·0	0·0	0·0	0·0	0·0
1·0	0·0	0·5	0·5	0·4560	0·4658	0·2473	0·2781	0·0334	0·0409
1·0	0·0	1·0	0·0	0·9119	0·9138	0·4945	0·5190	0·0667	0·0758

Table 4. The heat flux function $\bar{Q}_1(0)$ for non-conservative media with a purely specularly reflecting boundary at $\tau = \tau_0$

Boundary at $\tau = 0$		Boundary at $\tau = \tau_0$		$\bar{Q}_1(0)$		
				$\tau_0 = 0{\cdot}1$	$\tau_0 = 1{\cdot}0$	$\tau_0 = 10{\cdot}0$
ε_1	ρ_1^s	ε_2	ρ_2^s	$c = 0{\cdot}5$	$c = 0{\cdot}5$	$c = 0{\cdot}5$
1·0	0·0	0·0	1·0	0·0845	0·3263	0·0758
1·0	0·0	0·5	0·5	0·5066	0·4285	0·0758

evaluated from equation (39) by obtaining the values of $Q_0(0)$, $Q_1^*(0)$ and $Q_1^*(\tau_0)$ from Table 1.

Table 2 gives the values of the heat flux constant Q_1 for use in equation (44) for the special case $c = 1$.

Table 3 gives the numerical values of the functions $\bar{Q}_1(0)$ and $\bar{Q}_1^*(\tau_0)$ for use in equation (45) for diffuse surface reflectivities and emissivities.

In Tables 1 and 3 we have included, for the sake of completeness, results for the special case of $c = 0$. This case, of course, can be evaluated analytically.

To investigate the effects of specular reflection on the radiative heat flux, we have evaluated the dimensionless heat-flux function $\bar{Q}_1(0)$ by assuming that the boundary surface at $\tau = \tau_0$ is a purely specular reflector. The results of these calculations are presented in Table 4. A comparison of the results given in Tables 3 and 4 shows that the heat fluxes for specularly reflecting and diffuse reflecting cases differ only slightly.

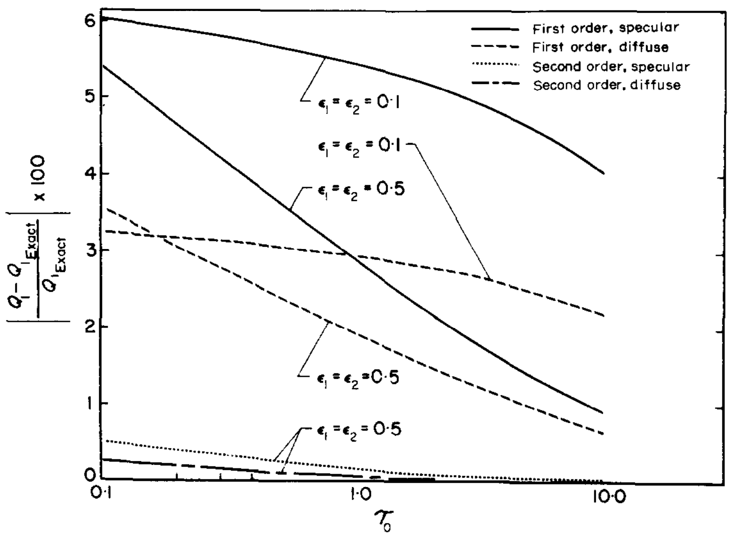

FIG. 2. The accuracy of the first- and second-order approximations to predict the heat-flux constant Q_1 for opaque boundaries with $c = 1$.

Figure 1 shows the effects of linearly anisotropic scattering on the radiative heat-flux function $\bar{Q}_1(\tau)$ as a function of the optical distance in the slab. The net radiative heat flux is slightly higher for linearly anisotropic scattering with $\beta = 0{\cdot}9$ than for isotropic scattering. The difference in heat fluxes becomes less for very large and very small values of the optical thickness τ_0.

The accuracy of the analytical approximations [i.e. equations (19)–(21) and (32)] to predict the net radiative heat flux is investigated by comparing the approximate results with the "exact" solutions obtained from the iterated coefficients.

Figure 2 shows the accuracy of the first-order and the second-order approximations to predict the net heat-flux constant Q_1 for a conservative medium (i.e. $c = 1$) as a function of the optical thickness τ_0 for both specularly and diffusely reflecting opaque boundaries. The accuracy of the second-order approximation appears to be very good even for optical thicknesses as small as $\tau_0 = 0{\cdot}1$. The accuracy of the approximations is better for purely diffuse reflection than for purely specular reflection.

Figure 3 shows the accuracy of the second-order approximation to predict the heat flux functions $Q_0(0)$ and $Q_1(0)$ for a non-conservative

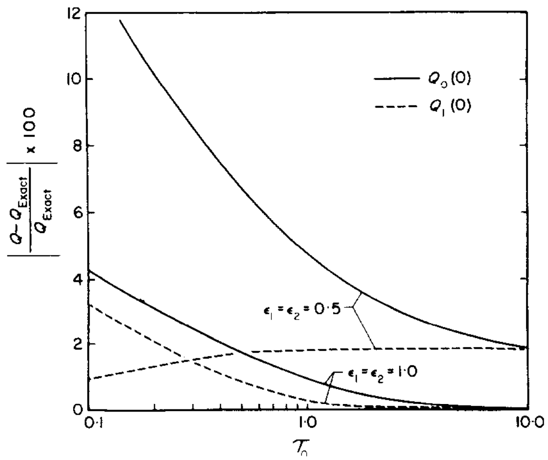

FIG. 3. The accuracy of the second-order approximation to predict the function $Q_0(0)$ and $Q_1(0)$ for diffuse, opaque boundaries with $c = 0{\cdot}5$.

medium for $c = 0{\cdot}5$ with diffusely reflecting and diffusely emitting opaque boundaries. The accuracy of the second-order approximation for a non-conservative medium with $c = 0{\cdot}5$ is not as good as that for the conservative media shown in Fig. 2.

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TRANSFERT PAR RAYONNEMENT AVEC DISPERSION LINÉAIRE ANISOTROPE DANS DES LAMES CONSERVATIVES OU NON CONSERVATIVES AVEC DES FRONTIÈRES RAYONNANTES

Résumé—On utilise la technique du développement à mode normal pour obtenir une solution semi-analytique pour une distribution angulaire de rayonnement à une distance optique quelconque dans un milieu entre deux frontières parallèles réfléchissantes. Ce milieu est dispersif linéairement et anisotropiquement, absorbant émissif non isotherme et gris. On considère les cas conservatifs ou non conservatifs. Le problème général est décomposé en problèmes plus simples et le flux net de chaleur rayonné est calculé pour plusieurs combinaisons de réflectivités et d'émissivités superficielles. En superposant ces solutions fondamentales on peut déterminer le flux net thermique rayonné pour une lame absorbante, émettrice, dispersante avec des frontières réfléchissantes dans le cas d'une température uniforme et de la puissance quatrième variant linéairement dans le milieu. Des expressions analytiques simples de l'intensité de rayonnement sont présentées en utilisant des approximations au premier et au second ordre de la solution exacte et la précision de ces approximations est évaluée pour une variété de cas.

TRANSPORT DURCH STRAHLUNG ZWISCHEN LINEAR ANISOTROP STREUENDEN KONSERVATIVEN UND NICHTKONSERVATIVEN PLATTEN MIT REFLEKTIERENDEN GRENZEN.

Zusammenfassung—Die Normal-Expansions-Technik wird benützt, um halbanalytische Lösungen für die Winkelverteilung der Strahlung in jeder optischen Entfernung in einem linear anisotrop streuenden, absorbierenden, emittierenden, nicht isothermen, grauen Medium zwischen zwei parallelen, reflektierenden Grenzen zu erhalten. Es werden sowohl "konservative" als auch "nicht-konservative" Fälle betrachtet. Das allgemeine Problem wird auf einfachere Probleme zurückgeführt, und der Netto-Wärmestrom wird bei diesen Grundproblemen für mehrere repräsentative Kombinationen aus Oberflächen-Reflektions- und Emissionsvermögen mit ausreichender Genauigkeit berechnet. Durch Überlagerung dieser Elementarlösungen kann der Netto-Wärmestrom für eine absorbierende, emittierende[i] streuende Platte mit reflektierenden Grenzen für die Fälle einer einheitlichen Temperatur im Medium berechnet werden. Es werden einfache analytische Ausdrücke für die Strahlungsintensität angegeben, indem die exakten Lösungen 1. und 2. Ordnung approximiert werden. Die Genauigkeit dieser Näherungen wird für mehrere Fälle untersucht.

РАДИАЦИОННЫЙ ТЕПЛООБМЕН В ЛИНЕЙНО АНИЗОТРОПНЫХ-РАССЕИВАЮЩИХ КОНСЕРВАТИВНЫХ И НЕКОНСЕРВАТИВНЫХ ПЛАСТИНАХ С ОТРАЖАЮЩИМИ ГРАНИЦАМИ

Аннотация—Используется метод разложения по нормальным колебаниям для того, чтобы получить полуаналитическое решение для углового распределения радиации для любой оптической длины в линейно анизотропной рассеивающей, поглощающей, излучающей, неизотермической серой среде между двумя параллельными отражающими границами. Рассматриваются случаи подвижной и неподвижной границ. Общая задача разбивается на более простые, а результирующий тепловой поток рассчитывается со стандартной точностью для этих задач с различными комбинациями отражательной и излучательной способности поверхности. Путем суперпозиции этих основных решений можно рассчитать результирующий тепловой поток для поглощающей, излучающей и рассеивающей пластины с отражающими границами для случая однородной температуры и линеаризованной четвертой степени температуры в среде. Представлены простые аналитические выражения для интенсивности радиации, применяя приближения первого и второго порядка к точному решению. Оценена точность этих приближений для целого ряда случаев.