Reprint Technical Note NUCLEAR SCIENCE and ENGINEERING: 45, 331-332 (1971)

Technical Notes

Matrix Reimann-Hilbert Problems Related to Neutron Transport Theory

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Received April 26, 1971

In recent years there has been considerable interest[1-4] in utilizing the method of singular-eigenfunction expansions, introduced by Case[5] for the one-speed neutron transport equation, in regard to the degenerate-kernel approximation of the energy-dependent transport equation, or the essentially equivalent multigroup transport equation.

Although the eigensolutions appropriate to these particular cases may, in fact, be complete, we would like to argue that the completeness proofs offered[1,3,4] are not entirely satisfactory. The completeness or expansion properties of the elementary solutions are crucial to the singular-eigenfunction-expansion technique and hence warrant the extreme care that, in general, must be exercised in establishing the required theorems. We note, for example, that Nicolaenko[6] has reported, albeit for the continuous energy model, an example for which the normal modes were, in fact, not complete.

The completeness proofs are normally based on the reduction of a system of singular-integral equations to the matrix version of the Reimann-Hilbert problem with an inhomogeneous boundary condition

$$\boldsymbol{M}^{+}(\mu) = \boldsymbol{G}(\mu)\boldsymbol{M}^{-}(\mu) + \boldsymbol{K}(\mu), \quad \mu \in L \ , \tag{1}$$

where L is an open arc and $\boldsymbol{G}(\mu)$ and $\boldsymbol{K}(\mu)$ are given. Here we seek a vector $\boldsymbol{M}(z)$ analytic in the complex plane cut along L and with prescribed behavior at infinity. Once $\boldsymbol{M}(z)$ is established, the desired expansion coefficients normally can be determined from the Plemelj formulas in the usual manner.[7]

We would first like to make a polemic remark concerning the fact that for full-range applications several authors[3,8] have used equations essentially equivalent to Eq. (1) and allowed $\boldsymbol{K}(\mu)$ to depend on the unknown expansion coefficients. Their ensuing analysis therefore yielded Fredholm equations rather than explicit results for the desired expansion coefficients. It is clear that no full-range completeness theorem can be considered definitive until the Fredholm equation resulting from such an approach is shown to be soluble. Two recent papers[9,10] have illustrated that this difficulty can be avoided.

Concerning the full-range theory, we note that the essential structure of the Reimann-Hilbert problem encountered is revealed by setting $\boldsymbol{G}(\mu) = \boldsymbol{I}$, with $\boldsymbol{I}$ denoting the unit matrix in Eq. (1). The solution to the resulting special case (sometimes referred to as the Sokhotski problem) can, of course, be written down immediately. We would therefore like to confine our attention to the considerably more interesting half-range[1,2,3] or two half-space applications[4] for which we require, in addition, a canonical solution $\boldsymbol{\Phi}_0(z)$ which satisfies the homogeneous boundary condition

$$\boldsymbol{\Phi}_0^{+}(\mu) = \boldsymbol{G}(\mu)\boldsymbol{\Phi}_0^{-}(\mu), \quad \mu \in L \ , \tag{2}$$

where typically the $\boldsymbol{G}$ matrix is of the form

$$\boldsymbol{G}(\mu) = [\boldsymbol{\Lambda}^{+}(\mu)]^{-1}\boldsymbol{\Lambda}^{-}(\mu) \ , \quad \mu \in L \ , \tag{3}$$

with $\boldsymbol{\Lambda}^{\pm}(\mu)$ denoting the limiting values of the dispersion matrix $\boldsymbol{\Lambda}(z)$.

The completeness proofs reported are all based on Muskhelishvili's[11] or Vekua's[12] theory of matrix Riemann-Hilbert problems. Muskhelishvili's and Vekua's analyses are confined to boundary value problems defined on *closed* contours, not open arcs, which are relevant to neutron transport theory. They require, in addition, the $\boldsymbol{G}$ matrix in Eq. (2) to be Hölder continuous on a closed contour C. All the authors mentioned[1,2,4] have converted the open-arc problem encountered to one based on a closed contour C simply by defining $\boldsymbol{G}(\mu) = \boldsymbol{I}$ off of the initial arc. Clearly

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[12] N. P. VEKUA, *Systems of Singular Integral Equations*, P. Noordhoff, Groningen, The Netherlands (1967).

this can be done; however, a matrix $G(\mu)$ with behavior[11,12] of class H^* or H^*_ϵ at the end points of an arc L will not, in general, satisfy the H condition when that arc is extended to become the closed contour C. Although Vekua[12] allows for discontinuous G matrices, his theory requires that the G matrix be Hölder on each arc of the contour C and, in addition, satisfy a Lipschitz condition near the end points of each arc. This clearly is not the case for the problems discussed by Leonard and Ferziger,[1] Kuščer,[2] Shultis,[3] and Jauho and Rajamäki.[4]

Thus, we believe it is clear that without modification, the standard works of Muskhelishvili[11] and Vekua[12] are not sufficient for the class of matrix Reimann-Hilbert problems encountered in neutron transport theory.[1-4] In fact, Muskhelishvili's Fredholm equation [Eq. (126.5), p. 386] discussed by Leonard and Ferziger[1] and Kuščer[2] is not necessarily even quasi-regular if $G(\mu)$ does not satisfy the Hölder condition on L.

We might mention that a more recent paper by Mandžavidze and Hvedelidze[13] has extended much of the Muskhelishvili-Vekua theory to include the case of $G(\mu)$ being simply continuous, as opposed to Hölder continuous. Though many of the important concepts, such as partial indices and canonical solutions, discussed by Muskhelishvili and Vekua do carry over to the more general case, the theory of Mandžavidze and Hvedelidze[13] is constructed within the framework of L_p functions, and thus the existence of solutions that satisfy the Reimann-Hilbert boundary condition almost everywhere is established. There is consequently no mention of quasi-regular Fredholm equations. We note that Muskhelishvili's quasiregular equation has been solved numerically for two scalar cases for which the appropriate $G(\mu)$ does actually satisfy the Hölder condition.[14] A similar procedure has not been proven feasible, or even correct, for the simplest of all neutron transport problems—the one-speed case.

More importantly, we should like to point out that the completeness proofs offered by Leonard and Ferziger,[1] Shultis,[3] and Jauho and Rajamäki[4] are subject to further criticism. By failing to establish the nature (in regard to algebraic sign) of the partial indices, they have in essence, assumed to be true that which they actually propose to prove.

To clarify this point, consider the two-group model with isotropic scattering. Here we seek a solution to Eq. (1) that vanishes at least as fast as $1/z$ as $|z|$ tends to infinity. In addition, the vector $K(\mu)$ in Eq. (1) is expressed in terms of the function to be expanded and the nondiverging discrete normal modes, and the dispersion matrix required in Eq. (3) to specify the G matrix is

$$\Lambda(z) = I + z\int_{-1}^{1} \psi(\mu)\,\frac{d\mu}{\mu - z}\ , \tag{4}$$

where the characteristic matrix is

$$\psi(\mu) = \begin{bmatrix} \theta(\mu) & 0 \\ 0 & 1 \end{bmatrix} C\ , \tag{5}$$

with C being the constant, nonsingular transfer matrix. Further,

$$\theta(\mu) = 1\ , \quad \mu \in \left[-\frac{1}{\sigma}, \frac{1}{\sigma}\right]; \quad \theta(\mu) = 0\ , \quad \mu \notin \left[-\frac{1}{\sigma}, \frac{1}{\sigma}\right]\ , \tag{6}$$

and the constant $\sigma > 1$ denotes the ratio of the total cross sections in the two-group model.

The solution to Eq. (1) can be written as

$$M(z) = \frac{1}{2\pi i} X(z)\left\{\int_L [X^+(\mu)]^{-1} K(\mu)\,\frac{d\mu}{\mu - z} + P(z)\right\}\ , \tag{7}$$

where $P(z)$ is a vector with polynomial elements and $X(z)$ is a canonical solution to the homogeneous boundary value problem

$$X^+(\mu) = G(\mu)X^-(\mu)\ , \quad \mu \in C\ . \tag{8}$$

The theory of Mandžavidze and Hvedelidze[13] ensures that a canonical matrix $X(z)$ of normal form at infinity exists, and we can thus write

$$X(z) \sim \begin{bmatrix} az^{-\kappa_1} + \ldots & bz^{-\kappa_2} + \ldots \\ cz^{-\kappa_1} + \ldots & dz^{-\kappa_2} + \ldots \end{bmatrix}\ , \quad \text{as} \quad |z| \to \infty\ , \tag{9}$$

where κ_1 and κ_2 are the partial (or component) indices and $ad \neq bc$. Note that in Eq. (9) we have written explicitly only the leading terms. The total index here is $\kappa = \kappa_1 + \kappa_2 = -N$, where $2N$ is the total number of discrete solutions available.

To see that the vector $M(z)$ will vanish at infinity only if $\kappa_1 \leq 0$ and $\kappa_2 \leq 0$, suppose, for example, that $\kappa_1 = 1$. It follows that the first column of $X(z)$ will vanish at infinity, but the second column will consequently diverge as z^{N+1}. Thus, for $M(z)$ to vanish at infinity one must impose $N + 1$ conditions on the vector $K(\mu)$ which has only N degrees of freedom corresponding to the N discrete solutions used for half-range expansions. It is clear, therefore, that the required completeness proof can be established in this manner if and only if all partial indices are less than or equal to zero. [We note that some authors[15] have the condition reversed due simply to an alternative formulation of Eq. (1) and the appropriate G matrix.]

It is quite clear that one can make no definitive statement about the behavior at infinity of the X matrix until a proof of the algebraic signs of the partial indices is constructed. Thus, since the behavior at infinity of the X matrix is crucial to the half-range or two half-range completeness proofs[1,3,4] *any* proof of the considered completeness theorems lacking a proof to establish the nature of the partial indices is unsatisfactory. We note that Kuščer[2] and Burniston and Siewert[15] have reported proofs concerning the nature of the partial indices for two rather special dispersion matrices. An extension to non-self-adjoint kernels has been proposed,[16] but the proof is still inconclusive because of certain details arising from the fact that both the direct and adjoint problems must be considered simultaneously. Incidentally, we might mention that were the usual theory of Muskhelishvili[11] or Vekua[12] applicable, the partial indices would have some relation to the solubility of a certain Fredholm equation. Unfortunately, this concept evidently does not carry over to the L_p theory of Mandžavidze and Hvedelidze.[13]

ACKNOWLEDGMENT

This work was supported in part by National Science Foundation Grant GK-11935 and Joint Services Advisory Group Grant AFOSR-69-1779.

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