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ON THE HALF-RANGE ORTHOGONALITY THEOREM APPROPRIATE TO THE SCATTERING OF POLARIZED LIGHT

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Abstract—A half-range orthogonality theorem relevant to the normal modes of a two-vector equation of transfer is proved. All appropriate normalization integrals are evaluated so that required expansion coefficients may be expressed concisely in terms of inner products.

1. INTRODUCTION

ONE OF the principal merits of the singular-eigenfunction-expansion technique, introduced initially by CASE[1] in regard to one-speed neutron transport theory, is the systematic and classical manner in which solutions to boundary-value problems in particle transport theory are constructed. In general, we first construct a sufficiently general set of solutions, denoted as normal modes, to the homogeneous equation of transfer. To a resulting superposition (with arbitrary expansion coefficients) we add a particular solution to account for any inhomogeneous source term, and we then constrain the complete solution to meet the boundary conditions of a specified problem. At this point, a completeness theorem (either full or half range) is proved, which in fact ensures that a sufficiently general set of normal modes has been obtained. In many cases,[2] the various completeness proofs are actually constructive, in that, in addition to proving an expansion theorem, we can construct analytical results for the desired expansion coefficients.

KUŠČER *et al.*[3] were the first to observe that the results of the half-range completeness theorem related to one-speed theory could be expressed quite naturally in terms of scalar products based on the proof of a half-range orthogonality theorem. Though the results so expressed were, of course, identical to those patiently deduced from the completeness theorem, the use of orthogonality relations provided a significant impetus to a more universal appeal of the singular-eigenfunction method.

We should like to demonstrate here the manner in which an orthogonality theorem in two-vector transport theory can be used to codify the results of half-range expansions in normal modes. Half-range orthogonality relations related to the generalized picket-fence model in radiative transfer, a special case of multi-group neutron transport theory, have been reported by SIEWERT and ZWEIFEL.[4] There, because of the rather special structure of the picket-fence model, SIEWERT and ZWEIFEL[4] obtained exact closed-form results and were able to express the half-range weight matrix solely in terms of scalar X- or H-functions.[5] Here the weight function is written in terms of an **H**-matrix for which no quadrature expression is available, though existence and uniqueness theorems[6] and rapidly convergent computational methods[7] are available.

We consider the two-vector equation of transfer

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}(\tau,\mu)+\mathbf{I}(\tau,\mu) = \frac{1}{2}\omega\mathbf{Q}(\mu)\int_{-1}^{1}\tilde{\mathbf{Q}}(\mu')\mathbf{I}(\tau,\mu')\,\mathrm{d}\mu', \tag{1}$$

where τ is the optical variable, μ is the direction cosine (as measured from the positive τ-axis) of the propagating radiation and $\omega \in [0, 1]$ is the single-scattering albedo. The superscript tilde is used to denote the transpose operation. Though the analysis here is essentially independent of the form of the **Q**-matrix, we are concerned principally with

$$\mathbf{Q}(\mu) = \frac{3(c+2)^{1/2}}{2(c+2)}\begin{vmatrix} c\mu^2+\frac{2}{3}(1-c) & (2c)^{1/2}(1-\mu^2) \\ \frac{1}{3}(c+2) & 0 \end{vmatrix}, \tag{2}$$

which is appropriate to studies of the scattering of polarized light.[8]

As previously reported,[8] seeking solutions to equation (1) of the form

$$\mathbf{I}_{\xi}(\tau,\mu) = \mathbf{F}(\xi,\mu)\,\mathrm{e}^{-\tau/\xi}, \tag{3}$$

we obtain

$$(1-\mu/\xi)\mathbf{F}(\xi,\mu) = \tfrac{1}{2}\omega\mathbf{Q}(\mu)\int_{-1}^{1}\tilde{\mathbf{Q}}(\mu')\mathbf{F}(\xi,\mu')\,\mathrm{d}\mu', \tag{4}$$

and we thus find the following set of normal modes: for the discrete spectrum,

$$\mathbf{F}(\pm\eta_0,\mu) = \frac{1}{2}\omega\eta_0\frac{1}{\eta_0\mp\mu}\mathbf{Q}(\mu)\mathbf{M}(\eta_0) \tag{5}$$

where

$$\mathbf{M}(\eta_0) = \int_{-1}^{1}\tilde{\mathbf{Q}}(\mu)\mathbf{F}(\pm\eta_0,\mu)\,\mathrm{d}\mu \tag{6}$$

is a null-vector of $\mathbf{\Lambda}(\eta_0)$. Here

$$\mathbf{\Lambda}(\eta_0)\mathbf{M}(\eta_0) = \mathbf{0}, \tag{7}$$

where $\pm\eta_0$ are the two zeros, in the complex plane cut from -1 to 1 along the real axis, of the dispersion function $\Lambda(z) = \det \mathbf{\Lambda}(z)$, with

$$\mathbf{\Lambda}(z) = \mathbf{I}+z\int_{-1}^{1}\mathbf{\Psi}(\mu)\frac{\mathrm{d}\mu}{\mu-z}, \tag{8}$$

I denoting the unit matrix, and the characteristic matrix given by

$$\mathbf{\Psi}(\mu) = \tfrac{1}{2}\omega\tilde{\mathbf{Q}}(\mu)\mathbf{Q}(\mu). \tag{9}$$

For the continuum, $\eta \in (-1, 1)$, two linearly independent generalized solutions are available:[8]

$$\mathbf{F}_\alpha(\eta, \mu) = \frac{1}{2}\omega\left(\eta \frac{P}{\eta-\mu} + \lambda_\alpha^*(\eta)\delta(\eta-\mu)\right)\mathbf{Q}(\mu)\mathbf{M}_\alpha(\eta), \eta \in (-1, 1), \qquad \alpha = 1 \text{ or } 2, \tag{10}$$

where $\lambda_1^*(\eta)$ and $\lambda_2^*(\eta)$ are the two solutions of

$$\det[\boldsymbol{\lambda}(\eta) - \lambda^*(\eta)\boldsymbol{\Psi}(\eta)] = 0, \tag{11}$$

with

$$\boldsymbol{\lambda}(\eta) = \mathbf{I} + \eta \int_{-1}^{1} \boldsymbol{\Psi}(\mu)\frac{P}{\mu-\eta}\,\mathrm{d}\mu. \tag{12}$$

In addition, the normalization vectors $\mathbf{M}_\alpha(\eta)$ follow from

$$[\boldsymbol{\lambda}(\eta) - \lambda_\alpha^*(\eta)\boldsymbol{\Psi}(\eta)]\mathbf{M}_\alpha(\eta) = \mathbf{0}. \tag{13}$$

With this formalism established, we write our general solutions to equation (1) as

$$\mathbf{I}(\tau, \mu) = A(\eta_0)\mathbf{F}(\eta_0, \mu)\,\mathrm{e}^{-\tau/\eta_0} + A(-\eta_0)\mathbf{F}(-\eta_0, \mu)\,\mathrm{e}^{\tau/\eta_0} + \int_{-1}^{1} [A_1(\eta)\mathbf{F}_1(\eta, \mu) + A_2(\eta)\mathbf{F}_2(\eta, \mu)]\,\mathrm{e}^{-\tau/\eta}\,\mathrm{d}\eta. \tag{14}$$

2. HALF-RANGE ORTHOGONALITY

Relying on a paper by Bond and Siewert[9] for proofs of the full-range completeness and orthogonality theorems, we consider here a typical half-space problem where we seek a solution to a boundary-value constraint of the form

$$\mathbf{I}(\mu) = A(\eta_0)\mathbf{F}(\eta_0, \mu) + \int_{0}^{1} [A_1(\eta)\mathbf{F}_1(\eta, \mu) + A_2(\eta)\mathbf{F}_2(\eta, \mu)]\,\mathrm{d}\eta, \qquad \mu \in (0, 1). \tag{15}$$

Here $\mathbf{I}(\mu)$ is the expansion function and is considered to be given; it may contain a diverging (at infinity) solution of the homogeneous equation of transfer, as for the Milne problem, or a particular solution relevant to an inhomogeneous source term and/or a specified incident distribution.

Burniston and Siewert[8] have shown that equation (15) is a valid expansion for arbitrary Hölder vectors $\mathbf{I}(\mu)$; we shall now prove an orthogonality theorem that will allow the solution to equation (15) to be written in a systematic manner:

$$A(\eta_0) = \frac{1}{N(\eta_0)}[\mathbf{F}(\eta_0, \mu), \mathbf{I}(\mu)] \tag{16a}$$

and

$$A_\alpha(\eta) = \frac{1}{N_\alpha(\eta)}[\mathbf{F}_\alpha(\eta, \mu), \mathbf{I}(\mu)], \qquad \alpha = 1 \text{ and } 2, \tag{16b}$$

where $[\mathbf{X}, \mathbf{Y}]$ denotes an appropriate inner product and $N(\eta_0)$ and $N_\alpha(\eta)$ are the normalization factors.

THEOREM: *The eigenfunctions* $\mathbf{F}(\eta_0, \mu)$, $\mathbf{F}_1(\eta, \mu)$ *and* $\mathbf{F}_2(\eta, \mu)$, $\eta \in (0, 1)$, *are orthogonal on the half range,* $\mu \in (0, 1)$, *to the related set* $\mathbf{G}(\eta_0, \mu)$, $\mathbf{G}_1(\eta, \mu)$ *and* $\mathbf{G}_2(\eta, \mu)$, $\eta \in (0, 1)$, *in the sense that*

$$\int_0^1 \tilde{\mathbf{G}}(\xi', \mu)\mathbf{F}(\xi, \mu)\mu \, d\mu = 0, \quad \xi \neq \xi'; \qquad \xi, \xi' = \eta_0 \text{ or} \in (0, 1). \tag{17}$$

Here

$$\mathbf{G}(\eta_0, \mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}^{-1}(\eta_0)\mathbf{Q}^{-1}(\mu)\mathbf{F}(\eta_0, \mu) \tag{18a}$$

and

$$\mathbf{G}_\alpha(\eta, \mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}^{-1}(\eta)\mathbf{Q}^{-1}(\mu)\mathbf{F}_\alpha(\eta, \mu), \qquad \alpha = 1 \text{ and } 2, \qquad \eta \in (0, 1). \tag{18b}$$

In addition, $\mathbf{H}(\mu)$ is the $\mathbf{H}$-matrix discussed by PAHOR[10] and SCHNATZ and SIEWERT.[11] We note that $\mathbf{H}(\mu)$ exists and is uniquely specified[6] by the singular-integral equation

$$\tilde{\mathbf{H}}(\mu)\boldsymbol{\lambda}(\mu) = \mathbf{I} + \mu P \int_0^1 \tilde{\mathbf{H}}(\mu')\boldsymbol{\Psi}(\mu')\frac{d\mu'}{\mu' - \mu}, \qquad \mu \in (0, 1), \tag{19a}$$

and the linear constraint,

$$\mathbf{H}^{-1}(-\eta_0)\mathbf{M}(\eta_0) = \mathbf{0}, \tag{19b}$$

where $\mathbf{H}(z)$ is defined in the complex plane by

$$\tilde{\mathbf{H}}(z)\boldsymbol{\Lambda}(z) = \mathbf{I} + z \int_0^1 \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\frac{d\mu}{\mu - z}. \tag{20}$$

Alternatively, the non-linear equation

$$\mathbf{H}(\mu) = \mathbf{I} + \mu\mathbf{H}(\mu) \int_0^1 \tilde{\mathbf{H}}(\mu')\boldsymbol{\Psi}(\mu')\frac{d\mu'}{\mu' + \mu}, \qquad \mu \in [0, 1], \tag{21a}$$

and the constraint

$$\mathbf{H}^{-1}(-\eta_0)\mathbf{M}(\eta_0) = \mathbf{0} \tag{21b}$$

uniquely specify $\mathbf{H}(\mu)$; a variation of these equations has proved to have merit for computational purposes.[7] We shall also require here the identity[10,11]

$$\tilde{\mathbf{H}}(z)\boldsymbol{\Lambda}(z)\mathbf{H}(-z) = \mathbf{I}, \tag{22}$$

an extension of CHANDRASEKHAR'S[12] scalar H-function expression to the case of matrices.

To prove the theorem, we first multiply equation (4) from the left by $\tilde{\mathbf{G}}(\xi', \mu)$. The transpose of equation (4) with $\xi \rightarrow \xi'$ is then multiplied from the right by

$\tilde{\mathbf{Q}}^{-1}(\mu)\tilde{\mathbf{H}}^{-1}(\xi')\tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{F}(\xi,\mu)$. We now integrate the two resulting equations over μ from zero to unity and subtract one from the other to obtain

$$\left(\frac{1}{\xi}-\frac{1}{\xi'}\right)\int_0^1 \tilde{\mathbf{G}}(\xi',\mu)\mathbf{F}(\xi,\mu)\mu\,\mathrm{d}\mu = \frac{1}{2}\omega[K_1(\xi',\xi)-K_2(\xi',\xi)], \tag{23}$$

where

$$K_1(\xi',\xi) = \tilde{\mathbf{M}}(\xi')\tilde{\mathbf{H}}^{-1}(\xi')\int_0^1 \tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{F}(\xi,\mu)\,\mathrm{d}\mu \tag{24a}$$

and

$$K_2(\xi',\xi) = \int_0^1 \tilde{\mathbf{F}}(\xi',\mu)\tilde{\mathbf{Q}}^{-1}(\mu)\tilde{\mathbf{H}}^{-1}(\xi')\tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{Q}(\mu)\,\mathrm{d}\mu\,\mathbf{M}(\xi). \tag{24b}$$

Here we consider $\xi, \xi' = \eta_0$ or $\in (0, 1)$ and note that

$$\mathbf{M}(\xi) = \int_{-1}^{1} \tilde{\mathbf{Q}}(\mu)\mathbf{F}(\xi,\mu)\,\mathrm{d}\mu. \tag{25}$$

To complete the proof, we clearly need only to show $K_1(\xi',\xi) = K_2(\xi',\xi)$ for all appropriate ξ and ξ'.

Considering first $K_1(\xi',\xi)$, we note that equation (10) can be pre-multiplied by $\tilde{\mathbf{Q}}(\mu)$ to yield, after using equation (13),

$$\tilde{\mathbf{Q}}(\mu)\mathbf{F}(\eta,\mu) = \left(\eta\frac{P}{\eta-\mu}\mathbf{\Psi}(\mu)+\delta(\eta-\mu)\boldsymbol{\lambda}(\eta)\right)\mathbf{M}(\eta). \tag{26}$$

If we now substitute equation (26) into equation (24a) and make use of equation (19a), we find immediately

$$K_1(\xi',\eta) = \tilde{\mathbf{M}}(\xi')\tilde{\mathbf{H}}^{-1}(\xi')\mathbf{M}(\eta), \qquad \eta \in (0,1). \tag{27}$$

For $\xi = \eta_0$, we enter equation (5) into equation (24a) to find

$$K_1(\xi',\eta_0) = \tilde{\mathbf{M}}(\xi')\tilde{\mathbf{H}}^{-1}(\xi')\eta_0\int_0^1 \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu)\frac{\mathrm{d}\mu}{\eta_0-\mu}\mathbf{M}(\eta_0), \tag{28}$$

which can be simplified, after use is made of the linear constraint equation (19b), rewritten as

$$\eta_0\int_0^1 \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu)\frac{\mathrm{d}\mu}{\eta_0-\mu}\mathbf{M}(\eta_0) = \mathbf{M}(\eta_0), \tag{29}$$

to yield

$$K_1(\xi',\eta_0) = \tilde{\mathbf{M}}(\xi')\tilde{\mathbf{H}}^{-1}(\xi')\mathbf{M}(\eta_0). \tag{30}$$

In a similar manner, we use equation (5) in equation (24b) to find, after noting equation (20),

$$K_2(\eta_0, \xi) = \tilde{\mathbf{M}}(\eta_0)\tilde{\mathbf{H}}^{-1}(\eta_0)[\mathbf{I}-\tilde{\mathbf{H}}(\eta_0)\mathbf{\Lambda}(\eta_0)]\mathbf{M}(\xi), \tag{31}$$

which reduces to

$$K_2(\eta_0, \xi) = \tilde{\mathbf{M}}(\eta_0)\tilde{\mathbf{H}}^{-1}(\eta_0)\mathbf{M}(\xi). \tag{32}$$

In writing equation (32), we have made use of equation (7) and the fact that $\mathbf{\Lambda}(z) = \tilde{\mathbf{\Lambda}}(z)$. Finally we substitute equation (10) into equation (24b) and use equations (19a) and (13) to obtain

$$K_2(\eta, \xi) = \tilde{\mathbf{M}}(\eta)\tilde{\mathbf{H}}^{-1}(\eta)\mathbf{M}(\xi), \qquad \eta \in (0, 1). \tag{33}$$

Having shown that

$$K_\alpha(\xi', \xi) = \tilde{\mathbf{M}}(\xi')\tilde{\mathbf{H}}^{-1}(\xi')\mathbf{M}(\xi), \qquad \alpha = 1 \text{ or } 2; \qquad \xi, \xi' = \eta_0 \text{ or} \in (0, 1), \tag{34}$$

we write equation (23) as

$$\left(\frac{1}{\xi}-\frac{1}{\xi'}\right)\int_0^1 \tilde{\mathbf{G}}(\xi', \mu)\mathbf{F}(\xi, \mu)\mu \,\mathrm{d}\mu = 0, \qquad \xi, \xi' = \eta_0 \text{ or} \in (0, 1), \tag{35}$$

which proves the theorem.

3. NORMALIZATION INTEGRALS

The half-range orthogonality theorem has been established, and we would therefore like to evaluate the necessary normalization integrals, so that all expansion coefficients in equation (15) can be expressed concisely and explicitly in terms of integrals of the expansion function $\mathbf{I}(\mu)$. First, however, we should like to make several general observations.

The explicit form of the $\mathbf{Q}$-matrix given by equation (2) was utilized in the previous sections only to the extent that we considered the dispersion function $\Lambda(z)$ to have only two zeros $\pm\eta_0$ in the cut complex plane. Clearly, to include the possibility of more discrete solutions would require only a minor modification of the formalism established here. Further, since the given proof of half-range orthogonality is essentially independent of the order of equation (1), the general analysis reported may be considered applicable to a N-vector version of equation (1).

Though the representations of the two continuum solutions given by equation (10) are convenient for proving completeness and orthogonality theorems, we shall make use of more explicit forms for actual applications. In order to develop these explicit forms we consider only the $\mathbf{Q}$-matrix given by equation (2). We note that equation (11) is quadratic in $\lambda^*(\eta)$, and thus the two solutions $\lambda_1^*(\eta)$ and $\lambda_2^*(\eta)$ required in equation (10) will, in general, involve radicals. To avoid these radicals, we shall prefer eventually the linear combinations

$$\mathbf{\Phi}_\alpha(\eta, \mu) = A_{\alpha 1}(\eta)\mathbf{F}_1(\eta, \mu) + A_{\alpha 2}(\eta)\mathbf{F}_2(\eta, \mu), \qquad \alpha = 1 \text{ or } 2, \tag{36}$$

which yield the tractable solutions

$$\mathbf{\Phi}_1(\eta, \mu) = \begin{vmatrix} \frac{3}{2}\omega c\eta(1-\eta^2)(1-\mu^2)\frac{P}{\eta-\mu} + \omega_1(\eta)\delta(\eta-\mu) \\ -\omega_2(\eta)\delta(\eta-\mu) \end{vmatrix} \tag{37a}$$

and

$$\mathbf{\Phi}_2(\eta, \mu) = \begin{vmatrix} \frac{3}{2}\omega\eta(1-\eta^2)\frac{P}{\eta-\mu} + \lambda_1(\eta)\delta(\eta-\mu) \\ \frac{3}{2}\omega\eta(1-\eta^2)\frac{P}{\eta-\mu} + \lambda_2(\eta)\delta(\eta-\mu) \end{vmatrix}, \tag{37b}$$

where

$$\lambda_\alpha(\eta) = (-1)^\alpha[1-3(1-\omega)\eta^2] + 3(1-\eta^2)\lambda_0(\eta), \qquad \alpha = 1 \text{ or } 2, \tag{38a}$$

$$\lambda_0(\eta) = 1 - \omega\eta \tanh^{-1}\eta, \tag{38b}$$

$$\omega_1(\eta) = c(1-\eta^2)\lambda_1(\eta) + \omega_2(\eta) \tag{39a}$$

and

$$\omega_2(\eta) = \tfrac{4}{3}(1-c) + 2c(1-\omega)\eta^2. \tag{39b}$$

We note that the full-range orthogonality theorem appropriate to the considered eigenvectors $\mathbf{F}(\xi, \mu)$ has been proved by Bond and Siewert;[9] however, since the full-range normalization integrals previously evaluated were expressed in terms of the general $\mathbf{F}_\alpha(\eta, \mu)$, rather than the more explicit $\mathbf{\Phi}_\alpha(\eta, \mu)$, we shall summarize both cases, full and half range. For the full range, we write[9]

$$\int_{-1}^{1} \tilde{\mathbf{F}}(\xi', \mu)\mathbf{F}(\xi, \mu)\mu\, d\mu = 0, \qquad \xi \neq \xi'; \qquad \xi, \xi' = \pm\eta_0 \text{ or } \in (-1, 1), \tag{40a}$$

whereas for the half range

$$\int_{0}^{1} \tilde{\mathbf{G}}(\xi', \mu)\mathbf{F}(\xi, \mu)\mu\, d\mu = 0, \qquad \xi \neq \xi'; \qquad \xi, \xi' = \eta_0 \text{ or } \in (0, 1). \tag{40b}$$

Here $\mathbf{F}(\xi, \mu)$ is a solution of equation (4), and

$$\mathbf{G}(\xi, \mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}^{-1}(\xi)\mathbf{Q}^{-1}(\mu)\mathbf{F}(\xi, \mu). \tag{41}$$

Considering now the continuum normalization, we find

$$\int_{-1}^{1} \tilde{\mathbf{F}}_\alpha(\eta', \mu)\mathbf{F}_\beta(\eta, \mu)\mu\, d\mu = S_\alpha(\eta)\delta(\eta-\eta')\delta_{\alpha\beta}; \qquad \eta, \eta' \in (-1, 1), \tag{42a}$$

and

$$\int_{0}^{1} \tilde{\mathbf{G}}_\alpha(\eta', \mu)\mathbf{F}_\beta(\eta, \mu)\mu\, d\mu = S_\alpha(\eta)\delta(\eta-\eta')\delta_{\alpha\beta}; \qquad \eta, \eta' \in (0, 1). \tag{42b}$$

Here

$$S_\alpha(\eta) = \tfrac{1}{2}\omega\eta\tilde{\mathbf{M}}_\alpha(\eta)\mathbf{\Lambda}^+(\eta)\mathbf{\Psi}^{-1}(\eta)\mathbf{\Lambda}^-(\eta)\mathbf{M}_\alpha(\eta), \tag{43a}$$

or, alternatively,

$$S_\alpha(\eta) = \tfrac{1}{2}\omega\eta([\lambda_\alpha^*(\eta)]^2 + \pi^2\eta^2)\tilde{\mathbf{M}}_\alpha(\eta)\mathbf{\Psi}(\eta)\mathbf{M}_\alpha(\eta). \tag{43b}$$

The Kronecker $\delta_{\alpha\beta}$ appearing in each of equations (42) should be noted since it illustrates that the general forms given by equation (10) are orthogonal even for $\eta = \eta'$. The explicit solutions $\mathbf{\Phi}_1(\eta, \mu)$ and $\mathbf{\Phi}_2(\eta, \mu)$ given by equations (37), though more concise than the $\mathbf{F}_\alpha(\eta, \mu)$, are not orthogonal for $\eta = \eta'$; however, we can use a Schmidt-type procedure to construct suitable adjoint vectors.

The four elementary integrals

$$\int_{-1}^{1} \tilde{\mathbf{\Phi}}_\alpha(\eta', \mu)\mathbf{\Phi}_\beta(\eta, \mu)\mu \, \mathrm{d}\mu = \eta M_{\alpha\beta}(\eta)\delta(\eta-\eta') \tag{44}$$

may be evaluated to yield

$$M_{11}(\eta) = c^2(1-\eta^2)^2[\lambda_1^2(\eta) + \tfrac{9}{4}\omega^2\eta^2(1-\eta^2)^2\pi^2] + 2\omega_2(\eta)[c(1-\eta^2)\lambda_1(\eta) + \omega_2(\eta)], \tag{45a}$$

$$M_{12}(\eta) = M_{21}(\eta) = c(1-\eta^2)[\lambda_1^2(\eta) + \tfrac{9}{4}\omega^2\eta^2(1-\eta^2)^2\pi^2] - 2\omega_2(\eta)[1-3(1-\omega)\eta^2] \tag{45b}$$

and

$$M_{22}(\eta) = 2[1-3(1-\omega)\eta^2]^2 + 18(1-\eta^2)^2[\lambda_0^2(\eta) + \tfrac{1}{4}\omega^2\eta^2\pi^2]. \tag{45c}$$

If we now introduce the full-range adjoint vectors

$$\mathbf{X}_1(\eta, \mu) = M_{22}(\eta)\mathbf{\Phi}_1(\eta, \mu) - M_{12}(\eta)\mathbf{\Phi}_2(\eta, \mu) \tag{46a}$$

and

$$\mathbf{X}_2(\eta, \mu) = M_{11}(\eta)\mathbf{\Phi}_2(\eta, \mu) - M_{21}(\eta)\mathbf{\Phi}_1(\eta, \mu), \tag{46b}$$

and hence the half-range adjoint vectors

$$\mathbf{\Theta}_1(\eta, \mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}^{-1}(\eta)\mathbf{Q}^{-1}(\mu)\mathbf{X}_1(\eta, \mu) \tag{47a}$$

and

$$\mathbf{\Theta}_2(\eta, \mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}^{-1}(\eta)\mathbf{Q}^{-1}(\mu)\mathbf{X}_2(\eta, \mu), \tag{47b}$$

the final results for the continuum can be summarized as

$$\int_{-1}^{1} \tilde{\mathbf{X}}_\alpha(\eta', \mu)\mathbf{\Phi}_\beta(\eta, \mu)\mu \, \mathrm{d}\mu = N(\eta)\delta(\eta-\eta')\delta_{\alpha\beta}; \qquad \eta, \eta' \in (-1, 1), \tag{48}$$

and

$$\int_{0}^{1} \tilde{\mathbf{\Theta}}_\alpha(\eta', \mu)\mathbf{\Phi}_\beta(\eta, \mu)\mu \, \mathrm{d}\mu = N(\eta)\delta(\eta-\eta')\delta_{\alpha\beta}; \qquad \eta, \eta' \in (0, 1). \tag{49}$$

Here

$$N(\eta) = 64\eta(1-\eta^2)^2\Lambda^+(\eta)\Lambda^-(\eta), \tag{50}$$

with $\Lambda^{+}(\eta)$ and $\Lambda^{-}(\eta)$ denoting the limiting values of the dispersion function $\Lambda(z)$ as the branch cut $[-1, 1]$ is approached from above $(+)$ and below $(-)$. More explicitly, equation (8) can be integrated [for $\mathbf{Q}(\mu)$ given by equation (2)] to yield

$$\Lambda(z) = \tfrac{1}{8}c\Lambda_1(z)\Lambda_2(z)+[(1-c)+\tfrac{3}{2}c(1-\omega)z^2]\Lambda_0(z), \tag{51}$$

where

$$\Lambda_\alpha(z) = (-1)^\alpha[1-3(1-\omega)z^2]+3(1-z^2)\Lambda_0(z) \tag{52}$$

and

$$\Lambda_0(z) = 1+\frac{1}{2}\omega z\int_{-1}^{1}\frac{\mathrm{d}\mu}{\mu-z}. \tag{53}$$

We note that

$$\Lambda_0^{\pm}(\eta) = 1-\omega\eta\tanh^{-1}\eta\pm\tfrac{1}{2}\pi i\omega\eta, \qquad \eta\in(-1,1). \tag{54}$$

For the discrete spectrum, $\xi = \pm\eta_0$, we prefer to normalize the general expressions given by equation (5) to obtain the solutions

$$\mathbf{\Phi}(\pm\eta_0,\mu) = \frac{3}{2}\omega\eta_0\frac{1}{\eta_0\mp\mu}\begin{vmatrix}c(1-\mu^2)\Lambda_2(\eta_0)+\omega_2(\eta_0)\\ \omega_2(\eta_0)\end{vmatrix}. \tag{55}$$

The discrete normalization integrals may thus be evaluated to yield

$$\int_{-1}^{1}\tilde{\mathbf{X}}(\pm\eta_0,\mu)\mathbf{\Phi}(\pm\eta_0,\mu)\mu\,\mathrm{d}\mu = \pm N(\eta_0) \tag{56}$$

and

$$\int_{0}^{1}\tilde{\mathbf{\Theta}}(\eta_0,\mu)\mathbf{\Phi}(\eta_0,\mu)\mu\,\mathrm{d}\mu = N(\eta_0), \tag{57}$$

where

$$N(\eta_0) = 12\omega\eta_0^2[c(1-\eta_0^2)\Lambda_2(\eta_0)+\omega_2(\eta_0)]\frac{\mathrm{d}}{\mathrm{d}z}\Lambda(z)\bigg|_{z=\eta_0}. \tag{58}$$

In addition, the discrete adjoint vectors used here are

$$\mathbf{X}(\pm\eta_0,\mu) = \mathbf{\Phi}(\pm\eta_0,\mu) \tag{59}$$

and

$$\mathbf{\Theta}(\eta_0,\mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}^{-1}(\eta_0)\mathbf{Q}^{-1}(\mu)\mathbf{X}(\eta_0,\mu). \tag{60}$$

Introducing the normalization vectors

$$\mathbf{U}(\eta_0) = \int_{-1}^{1}\tilde{\mathbf{Q}}(\mu)\mathbf{\Phi}(\pm\eta_0,\mu)\,\mathrm{d}\mu, \tag{61a}$$

and

$$\mathbf{U}_\alpha(\eta) = \int_{-1}^{1} \tilde{\mathbf{Q}}(\mu)\mathbf{\Phi}_\alpha(\eta, \mu)\,\mathrm{d}\mu, \qquad \alpha = 1 \text{ or } 2, \tag{61b}$$

and making use of the explicit forms given by equations (37) and (55), we find

$$\mathbf{U}(\eta_0) = 3(c+2)^{-1/2}\begin{vmatrix} 2\omega_2(\eta_0) \\ (2c)^{1/2}[\frac{1}{3}(c+2)\Lambda_2(\eta_0)+\omega_2(\eta_0)] \end{vmatrix}, \tag{62a}$$

$$\mathbf{U}_1(\eta) = 3(c+2)^{-1/2}\begin{vmatrix} 0 \\ \frac{1}{3}(c+2)(2c)^{1/2} \end{vmatrix}(1-\eta^2) \tag{62b}$$

and

$$\mathbf{U}_2(\eta) = 3(c+2)^{-1/2}\begin{vmatrix} 2 \\ (2c)^{1/2} \end{vmatrix}(1-\eta^2). \tag{62c}$$

The following inner-product integrals may thus be expressed as

$$\int_0^1 \tilde{\mathbf{\Theta}}(\eta_0, \mu)\mathbf{\Phi}(-\eta_0, \mu)\mu\,\mathrm{d}\mu = \tfrac{1}{4}\omega\eta_0\tilde{\mathbf{U}}(\eta_0)\tilde{\mathbf{H}}^{-1}(\eta_0)\mathbf{H}^{-1}(\eta_0)\mathbf{U}(\eta_0), \tag{63a}$$

$$\int_0^1 \tilde{\mathbf{\Theta}}_\alpha(\eta', \mu)\mathbf{\Phi}(-\eta_0, \mu)\mu\,\mathrm{d}\mu = \frac{1}{2}\omega\eta'\eta_0\frac{1}{\eta'+\eta_0}\mathbf{V}_\alpha(\eta')\tilde{\mathbf{H}}^{-1}(\eta')\mathbf{H}^{-1}(\eta_0)\mathbf{U}(\eta_0), \qquad \eta' \in (0, 1), \tag{63b}$$

$$\int_0^1 \tilde{\mathbf{\Theta}}(\eta_0, \mu)\mathbf{\Phi}_\beta(-\eta, \mu)\mu\,\mathrm{d}\mu = \frac{1}{2}\omega\eta_0\eta\frac{1}{\eta_0+\eta}\tilde{\mathbf{U}}(\eta_0)\tilde{\mathbf{H}}^{-1}(\eta_0)\mathbf{H}^{-1}(\eta)\mathbf{U}_\beta(\eta), \qquad \eta \in (0, 1), \tag{64a}$$

and

$$\int_0^1 \tilde{\mathbf{\Theta}}_\alpha(\eta', \mu)\mathbf{\Phi}_\beta(-\eta, \mu)\mu\,\mathrm{d}\mu = \frac{1}{2}\omega\eta'\eta\frac{1}{\eta'+\eta}\tilde{\mathbf{V}}_\alpha(\eta')\tilde{\mathbf{H}}^{-1}(\eta')\mathbf{H}^{-1}(\eta)\mathbf{U}_\beta(\eta), \qquad \eta, \eta' \in (0, 1), \tag{64b}$$

where

$$\mathbf{V}_1(\eta) = M_{22}(\eta)\mathbf{U}_1(\eta) - M_{12}(\eta)\mathbf{U}_2(\eta) \tag{65a}$$

and

$$\mathbf{V}_2(\eta) = M_{11}(\eta)\mathbf{U}_2(\eta) - M_{21}(\eta)\mathbf{U}_1(\eta). \tag{65b}$$

Having proved the half-range orthogonality theorem and evaluated all required normalization integrals, we may express the solution to expansions of the form

$$\mathbf{I}(\mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu) + \int_0^1 [A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]\,\mathrm{d}\eta, \qquad \mu \in (0, 1), \tag{66}$$

in a concise manner:

$$A(\eta_0) = \frac{1}{N(\eta_0)} \int_0^1 \tilde{\mathbf{\Theta}}(\eta_0, \mu)\mathbf{I}(\mu)\mu \, d\mu \tag{67a}$$

and

$$A_\alpha(\eta) = \frac{1}{N(\eta)} \int_0^1 \tilde{\mathbf{\Theta}}_\alpha(\eta, \mu)\mathbf{I}(\mu)\mu \, d\mu. \tag{67b}$$

Finally, in order to illustrate the half-range formalism established, we should like to solve a typical half-space problem—the Milne problem. We seek a diverging (at infinity) solution of equation (1) such that

$$\text{(i)} \lim_{\tau \to \infty} \mathbf{I}(\tau, \mu)\, e^{-\tau} = \mathbf{0}$$

$$\text{(ii)} \ \mathbf{I}(0, \mu) = \mathbf{0}, \qquad \mu \in (0, 1).$$

The solution may be written as

$$\mathbf{I}(\tau, \mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu)\, e^{-\tau/\eta_0} + \mathbf{\Phi}(-\eta_0, \mu)\, e^{\tau/\eta_0} + \int_0^1 [A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]\, e^{-\tau/\eta}\, d\eta, \ \tau \in [0, \infty), \mu \in [-1, 1], \tag{68}$$

where the expansion coefficients are to be determined from the free-surface boundary condition, that is from

$$-\mathbf{\Phi}(-\eta_0, \mu) = A(\eta_0)\mathbf{\Phi}(\eta_0, \mu) + \int_0^1 [A_1(\eta)\mathbf{\Phi}_1(\eta, \mu) + A_2(\eta)\mathbf{\Phi}_2(\eta, \mu)]\, d\eta, \qquad \mu \in (0, 1). \tag{69}$$

If we now multiply equation (69) by $\mu\tilde{\mathbf{\Theta}}(\eta_0, \mu)$, integrate over μ and invoke equations (35), (57) and (63a), we find

$$A(\eta_0) = -\frac{1}{N(\eta_0)} \frac{1}{4} \omega \eta_0 \tilde{\mathbf{U}}(\eta_0)\tilde{\mathbf{H}}^{-1}(\eta_0)\mathbf{H}^{-1}(\eta_0)\mathbf{U}(\eta_0). \tag{70a}$$

In a similar manner, we multiply equation (69) by $\mu\tilde{\mathbf{\Theta}}_\alpha(\eta', \mu)$, integrate over μ and use equations (35), (49) and (63b) to obtain

$$A_\alpha(\eta) = -\frac{1}{N(\eta)} \frac{1}{2} \omega \eta \eta_0 \frac{1}{\eta + \eta_0} \tilde{\mathbf{V}}_\alpha(\eta)\tilde{\mathbf{H}}^{-1}(\eta)\mathbf{H}^{-1}(\eta_0)\mathbf{U}(\eta_0). \tag{70b}$$

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