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ON EXISTENCE AND UNIQUENESS THEOREMS CONCERNING THE *H*-MATRIX OF RADIATIVE TRANSFER

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ABSTRACT

A solution to the system of singular-integral equations and the linear constraint which define mathematically the $\boldsymbol{H}$-matrix relevant to the scattering of polarized light is shown to exist and to be unique. A discussion of a class of canonical solutions to the matrix Riemann problem is given, and the $\boldsymbol{H}$-matrix is then expressed in terms of a convenient canonical solution. Cauchy's theorem is then used to develop the nonlinear integral equation convenient, when used with the linear constraint, for computing the $\boldsymbol{H}$-matrix, and the required existence and uniqueness theorem is proved. For the special case of conservative Rayleigh scattering, an explicit analytical result for the appropriate canonical matrix is given.

I. INTRODUCTION

We consider here the vector equation of transfer

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}(\tau, \mu) + \boldsymbol{I}(\tau, \mu) = \tfrac{1}{2}\omega \boldsymbol{Q}(\mu) \int_{-1}^{1} \boldsymbol{Q}^T(\mu')\boldsymbol{I}(\tau, \mu')d\mu' \tag{1}$$

relevant to the scattering of polarized light by a combination of Rayleigh and isotropic scattering. As discussed by Chandrasekhar (1950), the two-vector $\boldsymbol{I}(\tau, \mu)$ has elements $I_l(\tau, \mu)$ and $I_r(\tau, \mu)$, where τ is the optical variable and μ is the direction cosine of the propagating radiation (as measured from the positive τ-axis). In addition, the $\boldsymbol{Q}$-matrix introduced by Burniston and Siewert (1970) can be written as

$$\boldsymbol{Q}(\mu) = \frac{3(c+2)^{1/2}}{2(c+2)} \begin{bmatrix} c\mu^2 + \frac{2}{3}(1-c) & (2c)^{1/2}(1-\mu^2) \\ \frac{1}{3}(c+2) & 0 \end{bmatrix}, \tag{2}$$

where $c \in [0, 1]$ is a measure of the Rayleigh component of the scattering law; $\omega \in [0, 1]$ is the albedo for single scattering, and the superscript T denotes the transpose operation.

We note that a general solution to equation (1) may be written as

$$\begin{aligned} \boldsymbol{I}(\tau, \mu) = {} & A(\eta_0)\boldsymbol{\Phi}(\eta_0, \mu) \exp(-\tau/\eta_0) + A(-\eta_0)\boldsymbol{\Phi}(-\eta_0, \mu) \exp(\tau/\eta_0) \\ & + \int_{-1}^{1} [A_1(\eta)\boldsymbol{\Phi}_1(\eta, \mu) + A_2(\eta)\boldsymbol{\Phi}_2(\eta, \mu)] \exp(-\tau/\eta)d\eta\,, \end{aligned} \tag{3}$$

where $A(\pm\eta_0)$ and $A_\alpha(\eta)$, $\alpha = 1$ and 2, are the arbitrary expansion coefficients to be determined once the boundary conditions of a given problem are specified. The discrete solutions are given by (Siewert 1972)

$$\boldsymbol{\Phi}(\pm\eta_0, \mu) = \tfrac{3}{2}\omega\eta_0 \frac{1}{\eta_0 \mp \mu} \begin{bmatrix} c(1-\mu^2)\Lambda_2(\eta_0) + \omega_2(\eta_0) \\ \omega_2(\eta_0) \end{bmatrix}, \tag{4a}$$

and the two generalized solutions are

$$\boldsymbol{\Phi}_1(\eta, \mu) = \begin{bmatrix} \frac{3}{2}\omega c\eta(1-\eta^2)(1-\mu^2)Pv\left(\frac{1}{\eta-\mu}\right) + \omega_1(\eta)\delta(\eta-\mu) \\ -\omega_2(\eta)\delta(\eta-\mu) \end{bmatrix} \tag{4b}$$

and

$$\boldsymbol{\Phi}_2(\eta, \mu) = \begin{bmatrix} \frac{3}{2}\omega\eta(1-\eta^2)Pv\left(\frac{1}{\eta-\mu}\right) + \lambda_1(\eta)\delta(\eta-\mu) \\ \frac{3}{2}\omega\eta(1-\eta^2)Pv\left(\frac{1}{\eta-\mu}\right) + \lambda_2(\eta)\delta(\eta-\mu) \end{bmatrix}, \tag{4c}$$

where the distribution $Pv\left(\frac{1}{x}\right)$ denotes that ensuing integrals are to be evaluated in the Cauchy principal-value sense, and $\delta(x)$ is the Dirac delta distribution.

Here the discrete eigenvalues $\pm\eta_0$ are the zeros of the dispersion function

$$\Lambda(z) = \det \boldsymbol{\Lambda}(z), \tag{5}$$

where

$$\boldsymbol{\Lambda}(z) = \boldsymbol{I} + z\int_{-1}^{1} \boldsymbol{\Psi}(\mu)\frac{d\mu}{\mu - z}, \tag{6}$$

with $\boldsymbol{I}$ denoting the unit matrix and the characteristic function given by

$$\boldsymbol{\Psi}(\mu) = \tfrac{1}{2}\omega \boldsymbol{Q}^T(\mu)\boldsymbol{Q}(\mu). \tag{7}$$

In addition,

$$\boldsymbol{M}(\eta_0) = \int_{-1}^{1} \boldsymbol{Q}^T(\mu)\boldsymbol{\Phi}(\pm\eta_0, \mu)d\mu, \tag{8a}$$

or alternatively

$$\boldsymbol{M}(\eta_0) = 3(c+2)^{-1/2}\begin{bmatrix} 2\omega_2(\eta_0) \\ (2c)^{1/2}[\frac{1}{3}(c+2)\Lambda_2(\eta_0) + \omega_2(\eta_0)] \end{bmatrix}, \tag{8b}$$

is a null-vector of $\boldsymbol{\Lambda}(\eta_0)$:

$$\boldsymbol{\Lambda}(\eta_0)\boldsymbol{M}(\eta_0) = \boldsymbol{0}. \tag{9}$$

We note further that

$$\omega_1(\eta) = c(1-\eta^2)\lambda_1(\eta) + \omega_2(\eta), \tag{10}$$

$$\omega_2(\eta) = \tfrac{4}{3}(1-c) + 2c(1-\omega)\eta^2, \tag{11}$$

$$\lambda_\alpha(\eta) = (-1)^\alpha[1 - 3(1-\omega)\eta^2] + 3(1-\eta^2)[1 - \omega\eta\tanh^{-1}\eta], \quad \alpha = 1 \text{ or } 2, \tag{12}$$

and

$$\Lambda_\alpha(z) = (-1)^\alpha[1 - 3(1-\omega)z^2] + 3(1-z^2)\left(1 - \omega z\tanh^{-1}\frac{1}{z}\right), \quad \alpha = 1 \text{ or } 2. \tag{13}$$

The eigenvectors $\boldsymbol{\Phi}(\pm\eta_0, \mu)$, $\boldsymbol{\Phi}_1(\eta, \mu)$, and $\boldsymbol{\Phi}_2(\eta, \mu)$, $\eta \in (-1, 1)$, have been shown (Bond and Siewert 1971) to be a complete basis set for the expansion of Hölder vectors defined on the full range $\mu \in (-1, 1)$, and similarly $\boldsymbol{\Phi}(\eta_0, \mu)$, $\boldsymbol{\Phi}_1(\eta, \mu)$, and $\boldsymbol{\Phi}_2(\eta, \mu)$, $\eta \in (0, 1)$, are a complete basis set (Burniston and Siewert 1970) for Hölder vectors defined on the half-range $\mu \in (0, 1)$. For problems defined by full-range boundary conditions, the expansion coefficients $A(\pm\eta_0)$, $A_1(\eta)$, and $A_2(\eta)$, $\eta \in (-1, 1)$, can be expressed in terms of elementary functions, whereas for half-space applications the $\boldsymbol{H}$-matrix is required.

Siewert (1972) has proved the half-range orthogonality theorem and has shown that solutions to half-space problems may be expressed concisely in terms of the $\boldsymbol{H}$-matrix. We should like therefore to demonstrate the existence and uniqueness of the $\boldsymbol{H}$-matrix.

As previously reported (Schnatz and Siewert 1971), the half-range expansion theorem proved by Burniston and Siewert (1970) can be used to develop the equations

$$\boldsymbol{H}^T(\eta)\boldsymbol{\lambda}(\eta) = \boldsymbol{I} + \eta P \int_0^1 \boldsymbol{H}^T(\mu)\boldsymbol{\Psi}(\mu)\frac{d\mu}{\mu-\eta}, \qquad \eta \in (0, 1), \tag{14a}$$

and

$$\left[\boldsymbol{I} + \eta_0 \int_0^1 \boldsymbol{H}^T(\mu)\boldsymbol{\Psi}(\mu)\frac{d\mu}{\mu-\eta_0}\right]\boldsymbol{M}(\eta_0) = \boldsymbol{0}, \tag{14b}$$

which we will consider to define the $\boldsymbol{H}$-matrix, with the proviso that the required existence and uniqueness proofs will be established. Here

$$\boldsymbol{\lambda}(\eta) = \boldsymbol{I} + \eta P \int_{-1}^1 \boldsymbol{\Psi}(\mu)\frac{d\mu}{\mu-\eta}. \tag{15}$$

Equation (14a) clearly is a singular-integral equation for $\boldsymbol{H}(\eta)$, and equation (14b) is the required linear constraint.

After a general discussion of canonical matrices is given in § II, we prove in § III that a unique solution to equations (14) exists. In § IV we discuss the relationship between $\boldsymbol{H}(z)$ and a certain class of canonical matrices, and we illustrate the manner in which the half-range expansion theorem reveals the appropriate orthogonality result. In conclusion, we devote § V to the special case $\omega = c = 1$, for which equations (14) may be solved analytically to yield a closed-form result for the $\boldsymbol{H}$-matrix.

II. CANONICAL MATRICES

In this section we present some results concerning the so-called (Muskhelishvili 1953; Vekua 1967) canonical solutions to the matrix Riemann problem with the homogeneous boundary condition

$$\boldsymbol{\Phi}^+(\mu) = \boldsymbol{G}(\mu)\boldsymbol{\Phi}^-(\mu), \qquad \mu \in C, \tag{16}$$

where

$$\boldsymbol{G}(\mu) = \boldsymbol{\Lambda}^+(\mu)[\boldsymbol{\Lambda}^-(\mu)]^{-1}, \qquad \mu \in [0, 1], \tag{17a}$$

$$\boldsymbol{G}(\mu) = \boldsymbol{I}, \qquad \mu \in C_1, \tag{17b}$$

where C_1 is any arc, in the upper half-plane, which is added to the line segment [0, 1] such that the resulting C is a simple closed Liapunov contour. As usual, the superscripts + and − denote the limiting values taken along nontangential paths lying respectively inside and outside of C. We adopt here the standard terminology by which a canonical solution to equation (16) is meant to be any solution which is nonsingular, except perhaps at infinity. If, in addition, the columns of a canonical matrix have orders κ_1 and κ_2, respectively, at infinity, where the κ_α's denote the partial indices, then the canonical matrix is said to be of normal form at infinity. In the manner of Muskhelishvili (1953) we can compute the total index $\kappa = \kappa_1 + \kappa_2$ from equations (17); for the particular $\boldsymbol{\Lambda}$-matrix given by equation (6), we find $\kappa = 1$.

The $\boldsymbol{G}$-matrix defined by equations (17) clearly is continuous on C, but fails to be Hölder continuous due to the behavior of $\boldsymbol{G}(\mu)$ at $\mu = 1$. Consequently the results of Mandžavidze and Hvedelidze (1958) rather than those of Muskhelishvili (1953) or Vekua (1967) are required to establish the existence of a canonical solution. In addition, for the particular $\boldsymbol{G}$-matrix we consider, we are able to establish other relevant results.

THEOREM I. *If $\boldsymbol{\Phi}(z)$ is a solution to the Riemann problem defined by equations (16) and (17), then so is*

$$\boldsymbol{\Phi}_*(z) = [\boldsymbol{\Phi}(z^*)]^*. \tag{18}$$

THEOREM II. *There exists a canonical matrix $\boldsymbol{\Phi}_1(z)$ of ordered normal form at infinity, for the Riemann problem defined by equations (16) and (17), such that*

$$\boldsymbol{\Phi}_{1*}(z) = \boldsymbol{\Phi}_1(z) . \tag{19}$$

The proof of Theorem I is an immediate consequence of the fact that $G^{-1}(\mu) = [G(\mu)]^*$. Theorem II, on the other hand, may be established by first writing

$$\boldsymbol{\Phi}_{1*}(z) = \boldsymbol{\Phi}_1(z)\boldsymbol{E}(z) , \tag{20}$$

where $\boldsymbol{E}(z)$ is a matrix of polynomials. Noting that a canonical matrix of ordered ($\kappa_2 \geq \kappa_1$) normal form at infinity satisfies

$$\boldsymbol{\Phi}_1(z)\begin{bmatrix} z^{\kappa_1} & 0 \\ 0 & z^{\kappa_2} \end{bmatrix} \to \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \text{as} \quad |z| \to \infty , \quad \alpha\delta \neq \beta\gamma , \tag{21}$$

we conclude that the proof of Theorem II can be established by considering only the two cases $\alpha = \delta = 1, \gamma = 0$ and $\alpha = \delta = 0, \beta = \gamma = 1$. If we now make use of equation (21) for either of these cases to study the form of equation (20) at infinity, we conclude that $\boldsymbol{E}(z) \equiv \boldsymbol{I}$, which proves Theorem II. Clearly $\boldsymbol{\Phi}_1(z)$ is real on the entire real axis complementary to (0, 1].

We have previously shown (Burniston and Siewert 1970) that the $\boldsymbol{\Lambda}$-matrix could be factored in the manner

$$\boldsymbol{\Lambda}(z) = \boldsymbol{\Phi}(z)\boldsymbol{P}(z)\boldsymbol{\Phi}^T(-z) , \tag{22}$$

where $\boldsymbol{\Phi}(z)$ is any canonical matrix, and $\boldsymbol{P}(z)$ is a matrix of polynomials determined by the particular choice of $\boldsymbol{\Phi}(z)$. Using the fact that $\kappa_1 + \kappa_2 = 1$ and restricting $\boldsymbol{\Phi}(z)$ to be of normal form at infinity enables us to prove that κ_1 and κ_2 must be nonnegative. Indeed, if we allow one of the partial indices to be negative then the right-hand side of equation (22) can yield the correct form at infinity [$\boldsymbol{\Lambda}(\infty)$ is bounded] only if one of the diagonal elements, depending on the ordering of the partial indices, of the matrix $\boldsymbol{P}(z)$ is identically zero. Since from equation (22) we note that

$$\boldsymbol{P}(0) = \boldsymbol{\Phi}^{-1}(0)[\boldsymbol{\Phi}^T(0)]^{-1} , \tag{23}$$

it is clear that by choosing $\boldsymbol{\Phi}(z) = \boldsymbol{\Phi}_1(z)$ and invoking Theorem II we can be sure that for this canonical matrix neither diagonal element of the resulting $\boldsymbol{P}_1(z)$ can be zero identically. Appealing therefore to the fact that the partial indices are invariants, we deduce that since κ_1 and κ_2 must be nonnegative and sum to unity, the only possibilities are zero and unity. Without loss of generality, we now select $\kappa_1 = 0$ and $\kappa_2 = 1$.

We may therefore establish the following result:

THEOREM III. *If $\boldsymbol{\Phi}(z)$ is a canonical matrix of ordered normal form at infinity, for the boundary value problem defined by equations (16) and (17), then so is*

$$\boldsymbol{\Phi}(z)\begin{bmatrix} l & 0 \\ r + sz & m \end{bmatrix}, \quad lm \neq 0 ,$$

where l, m, r, and s are constants.

We note from equation (22) that the matrix of polynomials must satisfy $\boldsymbol{P}(z) = \boldsymbol{P}^T(-z)$, while the quadratic det $\boldsymbol{P}(z)$ must have zeros at $z = \pm\eta_0$. Making use of this information and taking into account the behavior of equation (22) as $|z|$ tends to infinity, enables us to reduce $\boldsymbol{P}(z)$ to Smith normal form (Browne 1958) and subsequently to write, for a general canonical matrix of ordered normal form at infinity and with z det $\boldsymbol{\Phi}(z) \to -1$ as $|z| \to \infty$,

$$\boldsymbol{P}(z) = \boldsymbol{L}(z)\boldsymbol{L}^T(-z) , \tag{24}$$

where

$$L(z) = \begin{bmatrix} g^{-1} & 0 \\ g(h+fz) & g[\Lambda(\infty)]^{1/2}(\eta_0 - z) \end{bmatrix}, \tag{25}$$

with f, g, and h being constants.

We can now make use of $\mathbf{\Phi}_1(z)$ and write equation (22) as

$$\mathbf{\Lambda}(z) = \mathbf{\Phi}_1(z)\mathbf{P}_1(z)\mathbf{\Phi}_1^T(-z), \tag{26}$$

where $\mathbf{P}_1(z) = \mathbf{L}_1(z)\mathbf{L}_1^T(-z)$ and $\mathbf{L}_1(z)$ is of the form of equation (25). Now since $\mathbf{\Lambda}_*(z) = [\mathbf{\Lambda}(z^*)]^* = \mathbf{\Lambda}(z)$, we observe that $\mathbf{P}_*(z) = \mathbf{P}(z)$, and since $[\Lambda(\infty)]^{1/2}\eta_0$ is real and

$$\mathbf{P}_1(0) = \mathbf{\Phi}_1^{-1}(0)[\mathbf{\Phi}_1^T(0)]^{-1}, \tag{27}$$

we conclude that the constants used in the matrix $\mathbf{L}_1(z)$ must be real. We may therefore define

$$\mathbf{\Phi}_0(z) = \mathbf{\Phi}_1(z)\begin{bmatrix} g_1^{-1} & 0 \\ g_1(h_1+f_1z) & g_1[\Lambda(\infty)]^{1/2}\eta_0 \end{bmatrix}, \tag{28}$$

which by Theorem III is of ordered normal form at infinity, and thus write equation (26) as

$$\mathbf{\Lambda}(z) = \mathbf{\Phi}_0(z)\mathbf{D}(z)\mathbf{D}(-z)\mathbf{\Phi}_0^T(-z), \tag{29}$$

where $\mathbf{D}(z)$ is the diagonal matrix

$$\mathbf{D}(z) = \begin{bmatrix} 1 & 0 \\ 0 & \eta_0^{-1}(\eta_0 - z) \end{bmatrix}. \tag{30}$$

Clearly since $\mathbf{\Phi}_{0_*}(z) = \mathbf{\Phi}_0(z)$, our final choice of canonical solutions $\mathbf{\Phi}_0(z)$ is real on the real axis complementary to (0, 1]. In general,

$$\mathbf{\Phi}_0(z) \sim \begin{bmatrix} \alpha + \ldots & \dfrac{\beta}{z} + \ldots \\ \gamma + \ldots & \dfrac{\delta}{z} + \ldots \end{bmatrix}, \quad |z| \to \infty, \quad \alpha\delta \neq \beta\gamma, \tag{31}$$

where the constants α, β, γ, and δ are real.

Of course, we may also factor $\mathbf{\Lambda}(z)$ as

$$\mathbf{\Lambda}(z) = \mathbf{F}(z)\mathbf{F}^T(-z), \tag{32}$$

where

$$\mathbf{F}(z) = \mathbf{\Phi}_0(z)\mathbf{D}(z)\mathbf{\Phi}_0^{-1}(0), \tag{33}$$

and hence $\mathbf{F}(0) = \mathbf{I}$. It is apparent that except for the special case $\omega = 1$, which implies $\eta_0 = \infty$, $\mathbf{\Lambda}(z)$ cannot be factored into a product of two canonical matrices, since by definition such matrices are nonsingular.

Finally we prove that for all canonical matrices for which equations (32) and (33) are valid, $\mathbf{F}(z)$ is invariant. Indeed, let $\mathbf{\Theta}_1(z)$ and $\mathbf{\Theta}_2(z)$ be two such canonical matrices yielding $\mathbf{F}_1(z)$ and $\mathbf{F}_2(z)$, and consider

$$\mathbf{A}(z) = \mathbf{F}_1(z) - \mathbf{F}_2(z). \tag{34}$$

The matrix $\mathbf{A}(z)$ is clearly a solution of the Riemann problem defined by equation (16); and since $\mathbf{A}(0) = \mathbf{0}$, we can write

$$\mathbf{A}(z) = z\mathbf{\Phi}(z)\mathbf{K}(z) \tag{35}$$

where $\Phi(z)$ is any canonical matrix of ordered normal form at infinity and $K(z)$ is a matrix of polynomials. Considering equation (33) as $|z|$ tends to infinity, we find it a simple matter to show that

$$K(z) = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}, \tag{36}$$

where k_1 and k_2 are constants. On using the normalization condition, $\Lambda(\eta_0)M(\eta_0) = 0$, we observe that simultaneously we must have

$$D(\eta_0)\Phi^T(\eta_0)M(\eta_0) = 0 \tag{37a}$$

and

$$K^T(\eta_0)\Phi^T(\eta_0)M(\eta_0) = 0\,, \tag{37b}$$

which is impossible unless $k_1 = k_2 = 0$. It follows therefore that $F_1(z) = F_2(z)$.

III. EXISTENCE AND UNIQUENESS THEOREMS

Having established the basic formalism regarding matrix Riemann problems and canonical matrices, we should now like to investigate the equations defining the H-matrix.

THEOREM IV. *The equations*

$$H^T(\mu)\lambda(\mu) = I + \mu P\int_0^1 H^T(\eta)\Psi(\eta)\,\frac{d\eta}{\eta-\mu}\,, \qquad \mu \in (0, 1)\,, \tag{38a}$$

and

$$\left[I + \eta_0\int_0^1 H^T(\eta)\Psi(\eta)\,\frac{d\eta}{\eta-\eta_0}\right]M(\eta_0) = 0 \tag{38b}$$

possess a unique solution in the class of continuous functions $C[0, 1]$.

To prove the theorem we make use of the equivalence of the above singular-integral equations to a certain matrix Riemann problem. We therefore, in the manner of Muskhelishvili (1953), introduce the sectionally analytic matrix

$$N(z) = \frac{1}{2\pi i}\int_0^1 H^T(\eta)\Psi(\eta)\,\frac{d\eta}{\eta - z}\,, \tag{39}$$

which vanishes at least as fast as $1/z$ for $|z|$ tending to infinity. Invoking the Plemelj formulae, we note that the boundary values of $N(z)$ satisfy

$$\pi i[N^+(\mu) + N^-(\mu)] = P\int_0^1 H^T(\eta)\Psi(\eta)\,\frac{d\eta}{\eta-\mu} \tag{40a}$$

and

$$N^+(\mu) - N^-(\mu) = H^T(\mu)\Psi(\mu)\,. \tag{40b}$$

We observe that the boundary values of the dispersion matrix are similarly related,

$$\tfrac{1}{2}[\Lambda^+(\mu) + \Lambda^-(\mu)] = \lambda(\mu) \tag{41a}$$

and

$$\Lambda^+(\mu) - \Lambda^-(\mu) = 2\pi i\mu\Psi(\mu)\,, \tag{41b}$$

and thus equation (38a) can be cast in the form

$$[N^+(\mu)]^T = G_1(\mu)[N^-(\mu)]^T + \Psi(\mu)[\Lambda^-(\mu)]^{-1}\,, \qquad \mu \in (0, 1)\,, \tag{42}$$

where

$$G_1(\mu) = \Psi(\mu)[\Lambda^-(\mu)]^{-1}\Lambda^+(\mu)\Psi^{-1}(\mu)\,. \tag{43}$$

Since

$$\Lambda^+(\mu)\Psi^{-1}(\mu)\Lambda^-(\mu) = \Lambda^-(\mu)\Psi^{-1}(\mu)\Lambda^+(\mu)\,, \tag{44}$$

we can simplify equation (43) to yield

$$G_1(\mu) = G(\mu) = \Lambda^+(\mu)[\Lambda^-(\mu)]^{-1} . \tag{45}$$

As discussed in the previous section, the analysis of Mandžavidze and Hvedelidze (1958) is sufficient to ensure that a canonical solution $\Phi_0(z)$, of ordered normal form at infinity, to the homogeneous problem

$$\Phi_0^+(\mu) = G(\mu)\Phi_0^-(\mu) \tag{46}$$

exists, and therefore equation (42) may be reduced to the following Sokhotski problem:

$$[\Phi_0^+(\mu)]^{-1}[N^+(\mu)]^T - [\Phi_0^-(\mu)]^{-1}[N^-(\mu)]^T = [\Phi_0^+(\mu)]^{-1}\Psi(\mu)[\Lambda^-(\mu)]^{-1} . \tag{47}$$

Equation (47) may be solved immediately to yield

$$N^T(z) = \frac{1}{2\pi i}\,\Phi_0(z)\left[\int_0^1 K(\eta)\,\frac{d\eta}{\eta - z} + P(z)\right] , \tag{48}$$

where $P(z)$ is a matrix of polynomials and

$$K(\eta) = [\Phi_0^+(\eta)]^{-1}\Psi(\eta)[\Lambda^-(\eta)]^{-1} . \tag{49}$$

Since $\Phi_0(z)$ has normal form at infinity, with partial indices $\kappa_1 = 0$ and $\kappa_2 = 1$, we use the fact that

$$\Phi_0(z) \sim \begin{bmatrix} \alpha + \ldots & \beta/z + \ldots \\ \gamma + \ldots & \delta/z + \ldots \end{bmatrix} , \qquad |z| \to \infty , \tag{50}$$

where $\Delta = \alpha\delta - \beta\gamma \neq 0$, to prove that $N(z)$ will vanish at least as fast as $1/z$ for $|z|$ tending to infinity only if $P(z)$ is a constant matrix of the form

$$P = \begin{bmatrix} 0 & 0 \\ P_{21} & P_{22} \end{bmatrix} . \tag{51}$$

Cauchy's integral theorem may now be applied to represent $\Phi_0^{-1}(z)$ as

$$\Phi_0^{-1}(z) = \Phi_{0\infty}^{-1}(z) - \Phi_{0\infty}^{-1}(0) + \Phi_0^{-1}(0) - z\int_0^1 K(\eta)\,\frac{d\eta}{\eta - z} , \tag{52}$$

where $\Phi_{0\infty}^{-1}(z)$ denotes the principal part of $\Phi_0^{-1}(z)$ at infinity. Equation (48) may thus be used to evaluate the integral term in equation (52); we obtain then the concise expression

$$N^T(z) = \frac{1}{2\pi i}\,\frac{1}{z}\,[\Phi_0(z)B(z) - I] , \tag{53}$$

where $B(z)$ is the linear matrix

$$B(z) = zP + \Phi_{0\infty}^{-1}(z) - \Phi_{0\infty}^{-1}(0) + \Phi_0^{-1}(0) . \tag{54}$$

We note that the H-matrix determined from equation (53) as prescribed by equation (40b) will satisfy the singular-integral equation for $H(\mu)$; however, the H-matrix so determined clearly is not unique since the matrix P embodied in equation (53) is expressed in terms of the arbitrary constants P_{21} and P_{22}. We proceed therefore to show that the linear constraint equation (38b) determines these two constants. Equation (38b) may be written as

$$[I + 2\pi i\eta_0 N(\eta_0)]M(\eta_0) = \mathbf{0} , \tag{55}$$

and we can then invoke equation (53) to find

$$B^T(\eta_0)\Phi_0{}^T(\eta_0)M(\eta_0) = \mathbf{0} . \tag{56}$$

We note that equation (54) yields

$$B^T(z) = z\begin{bmatrix} 0 & P_{21} - \dfrac{\gamma}{\Delta} \\ 0 & P_{22} + \dfrac{\alpha}{\Delta} \end{bmatrix} + [\Phi_0{}^T(0)]^{-1} , \tag{57}$$

after use has been made of equation (50). Recalling equations (30) and (37a), we note that equation (56) is equivalent to

$$B^T(\eta_0)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{0} , \tag{58}$$

which can be used to write equation (57) as

$$B(z) = D(z)\Phi_0{}^{-1}(0) , \tag{59}$$

or alternatively

$$B(z) = D(z)\Phi_0{}^T(0) . \tag{60}$$

We can now enter equation (60) into equation (53) to establish the completely specified N-matrix:

$$N^T(z) = \frac{1}{2\pi i}\frac{1}{z}[\Phi_0(z)D(z)\Phi_0{}^T(0) - I] , \tag{61}$$

which, because of equation (29), may be written as

$$N^T(z) = \frac{1}{2\pi i}\frac{1}{z}[\Lambda(z)[\Phi_0{}^T(-z)]^{-1}D^{-1}(-z)\Phi_0{}^T(0) - I] . \tag{62}$$

The boundary values of equation (62) may now be evaluated to yield

$$[N^+(\mu)]^T - [N^-(\mu)]^T = \frac{1}{2\pi i}\frac{1}{\mu}[\Lambda^+(\mu) - \Lambda^-(\mu)][\Phi_0{}^T(-\mu)]^{-1}D^{-1}(-\mu)\Phi_0{}^T(0) , \tag{63}$$

and thus by making use of equations (40b) and (41b) we obtain our final result for $H(\mu)$:

$$H(\mu) = [\Phi_0{}^T(-\mu)]^{-1}D^{-1}(-\mu)\Phi_0{}^T(0) , \qquad \mu \in [0, 1] . \tag{64}$$

It therefore follows that $H(\mu)$ exists, and from the comments at the end of § II that it is unique. Furthermore, because of the particular canonical matrix $\Phi_0(z)$, $H(\mu)$ is obviously *real* since η_0 is *real* for all $\omega \in [0, 1)$.

Since we have now shown that $H(\mu)$ exists and is unique, we should like to illustrate that Cauchy's integral theorem may be used to derive the nonlinear H-equation useful for computational purposes. The canonical matrix $\Phi_0(z)$ is sectionally analytic and bounded at infinity; it therefore follows that

$$\Phi_0(z) = \Phi_0(\infty) + \frac{1}{2\pi i}\int_0^1 [\Phi_0{}^+(\mu) - \Phi_0{}^-(\mu)]\frac{d\mu}{\mu - z} , \tag{65}$$

which, after use of equations (46), (41b), and (29), may be simplified to

$$\Phi_0{}^T(z) = \Phi_0{}^T(0) + z\int_0^1 D^{-1}(\mu)D^{-1}(-\mu)\Phi_0{}^{-1}(-\mu)\Psi(\mu)\frac{d\mu}{\mu - z} . \tag{66}$$

If we now set $z = -\mu$ in equation (66) and consider only $\mu \in [0, 1]$, we can enter equation (64) into equation (66) to find, after invoking equations (39), (61) and the linear constraint equation (38b), equations suitable for numerical computation of the $\boldsymbol{H}$-matrix (Kriese and Siewert 1971):

$$\boldsymbol{H}(\mu) = \boldsymbol{I} + \mu \boldsymbol{H}(\mu) \int_0^1 \boldsymbol{H}^T(\mu') \boldsymbol{\Psi}(\mu') \frac{d\mu'}{\mu' + \mu}, \quad \mu \in [0, 1], \quad (67)$$

and

$$\left[\boldsymbol{I} + \eta_0 \int_0^1 \boldsymbol{H}^T(\mu') \boldsymbol{\Psi}(\mu') \frac{d\mu'}{\mu' - \eta_0}\right] \boldsymbol{M}(\eta_0) = \boldsymbol{0}. \quad (68)$$

We note that equation (64) provides a convenient way in which to extend the definition of $\boldsymbol{H}(\mu)$ to the complex plane:

$$\boldsymbol{H}(z) = [\boldsymbol{\Phi}_0^T(-z)]^{-1} \boldsymbol{D}^{-1}(-z) \boldsymbol{\Phi}_0^T(0); \quad (69)$$

or alternatively, in view of equation (66) we may write

$$\boldsymbol{H}(z) = \boldsymbol{I} + z \boldsymbol{H}(z) \int_0^1 \boldsymbol{H}^T(\mu') \boldsymbol{\Psi}(\mu') \frac{d\mu'}{\mu' + z}. \quad (70)$$

We conclude therefore from equation (69) that except for a pole at $z = -\eta_0$, $\boldsymbol{H}(z)$ is analytic in the entire complex plane cut from -1 to 0 along the real axis. With $\boldsymbol{H}(z)$ so defined, equations (29) and (69) yield the useful identity

$$\boldsymbol{H}^T(z) \boldsymbol{\Lambda}(z) \boldsymbol{H}(-z) = \boldsymbol{I}. \quad (71)$$

We have established proof of the existence of a unique solution to equations (38) and subsequently developed equation (67) specifically to be used, with equation (68), for computational purposes. It thus follows that we require proof of

THEOREM V. *The equations*

$$\boldsymbol{H}(\mu) = \boldsymbol{I} + \mu \boldsymbol{H}(\mu) \int_0^1 \boldsymbol{H}^T(\mu') \boldsymbol{\Psi}(\mu') \frac{d\mu'}{\mu' + \mu}, \quad \mu \in [0, 1], \quad (72a)$$

and

$$\left[\boldsymbol{I} + \eta_0 \int_0^1 \boldsymbol{H}^T(\mu') \boldsymbol{\Psi}(\mu') \frac{d\mu'}{\mu' - \eta_0}\right] \boldsymbol{M}(\eta_0) = \boldsymbol{0} \quad (72b)$$

possess a unique solution in the class of continuous functions $C[0, 1]$.

Since we have shown that equations (38) possess a unique solution, we need simply show that any solution of equation (72a) is also a solution of equation (38a). We first write equation (72a) as

$$\boldsymbol{H}(\mu)\left[\boldsymbol{I} - \mu \int_0^1 \boldsymbol{H}^T(\mu') \boldsymbol{\Psi}(\mu') \frac{d\mu'}{\mu' + \mu}\right] = \boldsymbol{I}, \quad \mu \in [0, 1], \quad (73a)$$

or, alternatively,

$$\left[\boldsymbol{I} - \mu \int_0^1 \boldsymbol{H}^T(\mu') \boldsymbol{\Psi}(\mu') \frac{d\mu'}{\mu' + \mu}\right] \boldsymbol{H}(\mu) = \boldsymbol{I}, \quad \mu \in [0, 1]. \quad (73b)$$

If we now multiply the *transpose* of equation (73b) from the right by

$$\boldsymbol{I} + \mu P \int_0^1 \boldsymbol{H}^T(\mu') \boldsymbol{\Psi}(\mu') \frac{d\mu'}{\mu' - \mu}$$

and invoke equations (72a) and (15), then we find

$$\boldsymbol{H}^T(\mu)\boldsymbol{\lambda}(\mu) = \boldsymbol{I} + \mu P \int_0^1 \boldsymbol{H}^T(\mu')\boldsymbol{\Psi}(\mu')\frac{d\mu'}{\mu'-\mu}, \quad \mu \in (0, 1), \tag{74}$$

which proves Theorem V.

IV. THE RELATIONSHIP BETWEEN THE $\boldsymbol{X}$- AND $\boldsymbol{H}$-MATRICES

Because of the interest in the singular-eigenfunction-expansion technique (Case 1960), we should like to illustrate how the results of the half-range expansion theorem proved by Burniston and Siewert (1970) may be expressed quite concisely in terms of the $\boldsymbol{H}$-matrix. In the course of proving that

$$\boldsymbol{I}(\mu) = A(\eta_0)\boldsymbol{\Phi}(\eta_0, \mu) + \int_0^1 [A_1(\eta)\boldsymbol{\Phi}_1(\eta, \mu) + A_2(\eta)\boldsymbol{\Phi}_2(\eta, \mu)]d\eta, \quad \mu \in (0, 1), \tag{75}$$

is a valid expansion for arbitrary two-component Hölder vectors $\boldsymbol{I}(\mu)$, we deduced that the expansion coefficient $A(\eta_0)$ was specified by

$$\begin{bmatrix}0\\1\end{bmatrix}^T \int_0^1 \boldsymbol{\Gamma}(\mu)[\boldsymbol{I}(\mu) - A(\eta_0)\boldsymbol{\Phi}(\eta_0, \mu)]d\mu = 0, \tag{76}$$

where

$$\boldsymbol{\Gamma}(\mu) = \mu\boldsymbol{X}^+(\mu)[\boldsymbol{\Lambda}^+(\mu)]^{-1}\boldsymbol{Q}^T(\mu), \tag{77}$$

with $\boldsymbol{X}(z)$ being expressed in terms of the canonical matrix $\boldsymbol{\Phi}_0(z)$ as

$$\boldsymbol{X}(z) = \boldsymbol{\Phi}_0{}^T(z). \tag{78}$$

The $\boldsymbol{X}$-matrix here is analogous to Case's (1960) scalar X-function appropriate to problems in one-speed neutron-transport theory. From equation (69) we note immediately the relationship between $\boldsymbol{X}(z)$ and $\boldsymbol{H}(z)$:

$$\boldsymbol{H}(z) = \boldsymbol{X}^{-1}(-z)\boldsymbol{D}^{-1}(-z)\boldsymbol{X}(0); \tag{79}$$

from which we deduce

$$\boldsymbol{H}(z) = \boldsymbol{X}^{-1}(-z)\boldsymbol{D}^{-1}(-z)\boldsymbol{D}(-\eta_0)\boldsymbol{X}(-\eta_0)\boldsymbol{H}(\eta_0). \tag{80}$$

We now wish to express equation (77) in terms of the $\boldsymbol{H}$-matrix. Noting that

$$\boldsymbol{\Lambda}(z) = \boldsymbol{X}^T(z)\boldsymbol{D}(z)\boldsymbol{D}(-z)\boldsymbol{X}(-z), \tag{81}$$

we find that

$$\boldsymbol{\Gamma}(\mu) = \mu\boldsymbol{D}^{-1}(\mu)\boldsymbol{D}^{-1}(-\eta_0)[\boldsymbol{X}^T(-\eta_0)]^{-1}[\boldsymbol{H}^T(\eta_0)]^{-1}\boldsymbol{H}^T(\mu)\boldsymbol{Q}^T(\mu). \tag{82}$$

It is an elementary task to show that

$$(\eta_0 - \mu)\begin{bmatrix}0\\1\end{bmatrix}^T \boldsymbol{D}^{-1}(\mu)\boldsymbol{D}^{-1}(-\eta_0)[\boldsymbol{X}^T(-\eta_0)]^{-1} \propto \boldsymbol{M}^T(\eta_0),$$

so that equation (76) is equivalent to

$$\boldsymbol{M}^T(\eta_0)[\boldsymbol{H}^T(\eta_0)]^{-1}\int_0^1 \boldsymbol{H}^T(\mu)\boldsymbol{Q}^T(\mu)[\boldsymbol{I}(\mu) - A(\eta_0)\boldsymbol{\Phi}(\eta_0, \mu)]\mu\frac{d\mu}{\eta_0 - \mu} = 0. \tag{83}$$

If we now make use of the half-range adjoint vector introduced by Siewert (1972),

$$\boldsymbol{\Theta}(\eta_0, \mu) = \boldsymbol{Q}(\mu)\boldsymbol{H}(\mu)\boldsymbol{H}^{-1}(\eta_0)\boldsymbol{Q}^{-1}(\mu)\boldsymbol{\Phi}(\eta_0, \mu)\ , \tag{84}$$

we can express the solution to equation (83) concisely in terms of inner products:

$$A(\eta_0) = \int_0^1 \boldsymbol{\Theta}^T(\eta_0, \mu)\boldsymbol{I}(\mu)\mu d\mu \Big/ \int_0^1 \boldsymbol{\Theta}^T(\eta_0, \mu)\boldsymbol{\Phi}(\eta_0, \mu)\mu d\mu\ . \tag{85}$$

We note that the continuum coefficients $A_1(\eta)$ and $A_2(\eta)$ in equation (75) can also be expressed in terms of inner products (Siewert 1972).

V. THE CONSERVATIVE RAYLEIGH-SCATTERING LIMIT

Although we have established the existence of a unique solution $\boldsymbol{H}(\mu)$ to equations (38), explicit analytical results analogous to those applicable to scalar problems have not been constructed. On the other hand, essentially all of the identities normally associated with H-functions are available and a convenient method for computing the $\boldsymbol{H}$-matrix has been reported (Kriese and Siewert 1971). There is, however, one interesting case, viz., $\omega = c = 1$, for which an exact analytical result for the $\boldsymbol{H}$-matrix can be obtained. Though the solution for this case of conservative Rayleigh scattering has been developed by Chandrasekhar (1950), we wish to discuss briefly the manner in which Chandrasekhar's solution can be deduced from our general analysis.

We observe from equation (46) that if $\boldsymbol{G}(\mu)$ is either upper or lower triangular, the resulting matrix Riemann-Hilbert problem reduces to two solvable scalar problems. We note however that, apart from the two simple cases $\omega = 0$, $c \in [0, 1]$ and $c = 0$, $\omega \in [0, 1]$, the $\boldsymbol{G}$-matrix given by equation (45) is triangular only for the case $\omega = c = 1$. To construct the desired canonical matrix for this limiting case, we prefer to make use of the fact that here the $\boldsymbol{\Lambda}$-matrix can be factored as

$$\boldsymbol{\Lambda}(z) = [3(1 - z^2)]^{-1}\boldsymbol{Q}^T(z)\boldsymbol{\Omega}(z)\boldsymbol{Q}(z)\ , \tag{86}$$

where $\boldsymbol{\Omega}(z)$ is the diagonal matrix (Mullikin 1969)

$$\boldsymbol{\Omega}(z) = \begin{bmatrix} \Lambda_1(z) & 0 \\ 0 & \Lambda_2(z) \end{bmatrix}, \tag{87}$$

with

$$\Lambda_\alpha(z) = (-1)^\alpha + 3(1 - z^2)\left[1 + \tfrac{1}{2}z \int_{-1}^{1} \frac{d\mu}{\mu - z}\right], \qquad \alpha = 1 \text{ and } 2\ . \tag{88}$$

We seek therefore a canonical solution $\boldsymbol{\Phi}_0(z)$ to the boundary-value problem

$$[\boldsymbol{Q}^T(\mu)]^{-1}\boldsymbol{\Phi}^+(\mu) = \boldsymbol{G}_0(\mu)[\boldsymbol{Q}^T(\mu)]^{-1}\boldsymbol{\Phi}^-(\mu)\ , \tag{89}$$

where

$$\boldsymbol{G}_0(\mu) = \begin{bmatrix} \dfrac{\Lambda_1^+(\mu)}{\Lambda_1^-(\mu)} & 0 \\ 0 & \dfrac{\Lambda_2^+(\mu)}{\Lambda_2^-(\mu)} \end{bmatrix}. \tag{90}$$

If we now make use of the functions

$$F_1(z) = \frac{1}{1 - z} \exp\left[\frac{1}{\pi} \int_0^1 \operatorname{Arg} \Lambda_1^+(\mu) \frac{d\mu}{\mu - z}\right] \tag{91a}$$

and

$$F_2(z) = \exp\left[\frac{1}{\pi} \int_0^1 \operatorname{Arg} \Lambda_2^+(\mu) \frac{d\mu}{\mu - z}\right] \tag{91b}$$

introduced by Siewert and Fraley (1967), we conclude that there is a canonical matrix of the form

$$\boldsymbol{\Phi}(z) = Q^T(z)\begin{bmatrix} F_1(z) & 0 \\ 0 & F_2(z) \end{bmatrix} R(z) , \tag{92}$$

where the elements of $R(z)$ are rational functions (with singularities only at $z = \pm 1$).

Since det $\boldsymbol{\Phi}(z)$ must be nonvanishing in the finite plane and since $F_1(z)$ and $F_2(z)$ have been similarly constructed, it follows that det $R(z)$ must have simple poles at $z = \pm 1$ in order to remove the zeros introduced by det $Q^T(z) = -\frac{3}{4}[2^{1/2}(1 - z^2)]$. In addition, we prefer to construct $R(z)$ such that eq. (91) yields $\boldsymbol{\Phi}(z) = \boldsymbol{\Phi}_1(z)$, where $\boldsymbol{\Phi}_1(z)$ has the form at infinity specified by equation (21), and hence satisfies $\boldsymbol{\Phi}_{1_*}(z) = \boldsymbol{\Phi}_1(z)$. We find a suitable R-matrix to be

$$R(z) = \tfrac{2}{3}\sqrt{3}\begin{bmatrix} 0 & -[2^{1/2}(1 - z^2)]^{-1} \\ 1 & 2^{-1/2}[(a + bz)/(1 - z^2)] \end{bmatrix} , \tag{93}$$

where the constants a and b are to be chosen such that

$$U(z) = \left(\frac{1}{1 - z^2}\right)[(a + bz)F_2(z) - z^2F_1(z)] \tag{94}$$

remains bounded as $z \to \pm 1$. Clearly, then,

$$a = F_1(1)F_2(-1) + F_1(-1)F_2(1) \tag{95a}$$

and

$$b = F_1(1)F_2(-1) - F_1(-1)F_2(1) . \tag{95b}$$

Having established an appropriate R-matrix, we can now expand equation (92) to find

$$\boldsymbol{\Phi}_1(z) = \begin{bmatrix} F_2(z) & 2^{-1/2}U(z) \\ 0 & -F_1(z) \end{bmatrix} . \tag{96}$$

In terms of the canonical matrix given by equation (96), we can factor $\boldsymbol{\Lambda}(z)$ as

$$\boldsymbol{\Lambda}(z) = \boldsymbol{\Phi}_1(z)L_1(z)L_1^T(-z)\boldsymbol{\Phi}_1^T(-z) , \tag{97}$$

where we may use (Chandrasekhar 1950; Siewert and Fraley 1967) $c = a/b$, $q = -10^{1/2}/b$, and $a^2 + 5 = b^2$ to write

$$L_1(z) = \begin{bmatrix} -1/q & 0 \\ 5^{-1/2}(z - c) & -10^{-1/2}q \end{bmatrix} . \tag{98}$$

Note that we use c here to denote the ratio a/b, not as used in § I. In view of Theorem III we may now consider

$$\boldsymbol{\Phi}_0(z) = \boldsymbol{\Phi}_1(z)L_1(z) \tag{99}$$

to be a more convenient canonical matrix. Since the constants b and c are real, we observe also that $\boldsymbol{\Phi}_{0_*}(z) = \boldsymbol{\Phi}_0(z)$. Equation (97) thus may be written as

$$\boldsymbol{\Lambda}(z) = \boldsymbol{\Phi}_0(z)\boldsymbol{\Phi}_0^T(-z) . \tag{100}$$

If we denote the (ij)th element of $\boldsymbol{\Phi}_0(z)$ by $\phi_{0ij}(z)$, we find that equations (96) and (99) yield

$$\phi_{011}(z) = -\left(\frac{1}{1 - z^2}\right)\left[\tfrac{1}{2}qF_2(z) + \frac{\sqrt{10}}{10}(z - c)z^2F_1(z)\right] , \tag{101a}$$

$$\phi_{012}(z) = \frac{\sqrt{2}}{2}\left(\frac{1}{1 - z^2}\right)\left[(c + z)F_2(z) + \frac{\sqrt{10}}{10}qz^2F_1(z)\right] , \tag{101b}$$

$$\phi_{021}(z) = \frac{\sqrt{5}}{5}(c - z)F_1(z) , \tag{101c}$$

and

$$\phi_{022}(z) = \frac{\sqrt{10}}{10} qF_1(z) . \tag{101d}$$

We note also that

$$\boldsymbol{\Phi}_0(0) = \begin{bmatrix} -2^{-1/2}q & c \\ c & 2^{-1/2}q \end{bmatrix} \tag{102}$$

and

$$\boldsymbol{\Phi}_0(z) \sim \begin{bmatrix} -\frac{\sqrt{10}}{10} + \ldots & \left(\frac{\sqrt{5}}{10} q - \frac{\sqrt{2}}{2}\right)\frac{1}{z} + \ldots \\ \frac{\sqrt{5}}{5} + \ldots & -\frac{\sqrt{10}}{10} q \frac{1}{z} + \ldots \end{bmatrix} , \quad \text{as} \quad |z| \to \infty . \tag{103}$$

Finally we should like to introduce Chandrasekhar's (1950) scalar functions

$$H_l(z) = 5^{1/2}/F_1(-z) \quad \text{and} \quad H_r(z) = 2^{1/2}/F_2(-z) \tag{104}$$

so that

$$\boldsymbol{H}(z) = [\boldsymbol{\Phi}_0^T(-z)]^{-1}\boldsymbol{\Phi}_0^T(0) \tag{105}$$

may be evaluated to yield

$$H_{11}(z) = (1 + cz)H_r(z) , \tag{106a}$$

$$H_{12}(z) = \frac{\sqrt{2}}{2} qzH_r(z) , \tag{106b}$$

$$H_{21}(z) = \frac{\sqrt{2}}{2}\left(\frac{1}{1 - z^2}\right)[qzH_l(z) - z^2(1 + cz)H_r(z)] \tag{106c}$$

and

$$H_{22}(z) = \left(\frac{1}{1 - z^2}\right)[(1 - cz)H_l(z) - \tfrac{1}{2}qz^3H_r(z)] , \tag{106d}$$

where $H_{ij}(z)$ denotes the (ij)th element of $\boldsymbol{H}(z)$.

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