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AN EXPLICIT CLOSED-FORM RESULT FOR THE DISCRETE EIGENVALUE IN STUDIES OF POLARIZED LIGHT

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ABSTRACT

An exact closed-form result for the discrete eigenvalue pertinent to the combination of Rayleigh and isotropic scattering of polarized light is given.

We consider the equation of transfer relevant to a mixture of Rayleigh- and isotropic-scattering laws (Bond and Siewert 1971):

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}(\tau, \mu) + \boldsymbol{I}(\tau, \mu) = \tfrac{1}{2}\omega \boldsymbol{Q}(\mu) \int_{-1}^{1} \boldsymbol{Q}^T(\mu')\boldsymbol{I}(\tau, \mu')d\mu' , \tag{1}$$

where $\boldsymbol{Q}^T(\mu)$ denotes the transpose of

$$\boldsymbol{Q}(\mu) = \frac{3}{2}\frac{(c+2)^{1/2}}{(c+2)} \begin{vmatrix} c\mu^2 + \frac{2}{3}(1-c) & (2c)^{1/2}(1-\mu^2) \\ \frac{1}{3}(c+2) & 0 \end{vmatrix} . \tag{2}$$

Here $\boldsymbol{I}(\tau, \mu)$, with elements $I_l(\tau, \mu)$ and $I_r(\tau, \mu)$, is the intensity two-vector, τ is the optical variable, and μ is the direction cosine (as measured from the positive τ-axis) of the propagating radiation. Further, $\omega \in [0, 1]$ is the single-scattering albedo, and $c \in [0, 1]$ is a measure of the Rayleigh component of the scattering law (Chandrasekhar 1950).

Kriese and Siewert (1971) have reported an expedient method for computing the $\boldsymbol{H}$-matrix required in the solution to half-space problems based on equation (1), and Siewert and Burniston (1972) have proved the required existence and uniqueness theorems concerning the $\boldsymbol{H}$-matrix. Basic to $\boldsymbol{H}$-matrix computations or any discussion of the elementary solutions of equation (1) is the discrete eigenvalue η_0 defined (Bond and Siewert 1971) as the positive zero, in the complex plane cut from -1 to 1 along the real axis, of the dispersion function

$$\Lambda(z) = \tfrac{1}{8}c\Lambda_1(z)\Lambda_2(z) + [1 - c + \tfrac{3}{2}c(1-\omega)z^2]\Lambda_0(z) , \tag{3}$$

where

$$\Lambda_\alpha(z) = (-1)^\alpha + 3(1-z^2)\Lambda_0(z) - (-1)^\alpha 3(1-\omega)z^2 , \quad \alpha = 1 \text{ and } 2 , \tag{4}$$

and

$$\Lambda_0(z) = 1 + \tfrac{1}{2}\omega z \int_{-1}^{1} \frac{d\mu}{\mu - z} . \tag{5}$$

Previously (Kriese and Siewert 1971; Bond and Siewert 1971) η_0 was obtained numerically by solving iteratively the transcendental equation

$$\tfrac{1}{8}c\Lambda_1(\eta_0)\Lambda_2(\eta_0) + [1 - c + \tfrac{3}{2}c(1-\omega)\eta_0^2]\Lambda_0(\eta_0) = 0 . \tag{6}$$

In this note we wish to demonstrate that an exact closed-form solution to the transcendental equation (6) can be established. We find

$$\eta_0 = [(1-\omega)(1-\tfrac{7}{10}\omega c)]^{-1/2} \exp\left[-\frac{1}{\pi}\int_0^1 \Theta(\mu)\frac{d\mu}{\mu}\right], \tag{7}$$

where

$$\Theta(\mu) = \tan^{-1}\left[\frac{A(\mu)}{B(\mu)}\right], \tag{8}$$

$$A(\mu) = \tfrac{1}{8}\omega\pi\mu[9c(1-\mu^2)^2\lambda_0(\mu) + 6c\mu^2(1-\omega) + 4(1-c)], \tag{9}$$

$$B(\mu) = \tfrac{1}{8}c\{-1 + 9(1-\mu^2)^2[\lambda_0^2(\mu) - \tfrac{1}{4}\pi^2\omega^2\mu^2] + 3\mu^2(1-\omega)[4\lambda_0(\mu) - 3\mu^2(1-\omega) + 2]\} + (1-c)\lambda_0(\mu), \tag{10}$$

and

$$\lambda_0(\mu) = 1 - \omega\mu \tanh^{-1}\mu . \tag{11}$$

We note that $\Theta(\mu) \in [0, \pi]$ for $\mu \in [0, 1]$. It is perhaps surprising that a closed-form solution to the transcendental equation (6) exists; however, our result, equation (7), follows immediately from setting $z = 0$ in the factorization

$$\Lambda(z) = \frac{1}{H(z)H(-z)} \tag{12}$$

and from using Bond and Siewert's (1971) analytical solution for $H(z) = \det \boldsymbol{H}(z)$:

$$H(z) = [(1-\omega)(1-\tfrac{7}{10}\omega c)]^{-1/2}\frac{(1+z)}{\eta_0+z}\exp\left\{-\frac{1}{\pi}\int_0^1 \Theta(\mu)\frac{d\mu}{\mu+z}\right\}. \tag{13}$$

It may be observed that the discrete eigenvalue for the gray, nonconservative equation of transfer for isotropic scattering is also given by equation (7) in the limit $c \to 0$.

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